

Decentralized Trading with Private Information*

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Abstract

The paper studies how asset prices are determined in decentralized markets with asymmetric information. We consider an economy in which a large number of agents trade two assets in bilateral meetings and a fraction of the agents has private information on the assets' values. We show that, over time, uninformed agents can elicit information from their trading partners by making small offers. This form of experimentation allows the uninformed agents to learn all the useful information about the assets' values in the long run. As a consequence the economy converges to a Pareto efficient allocation.

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1 Introduction

This paper studies trading and information diffusion in decentralized markets with private information. These markets have two key frictions: (i) trading takes place in decentralized, bilateral meetings; and (ii) some agents have private information on the value of the assets traded. Duffie, Garleanu, and Pedersen (2005) have started a research agenda of providing a theory of asset pricing in decentralized environments, starting from the case in which all information is publicly observable.¹ They argue that many important markets such as over-the-counter markets are decentralized. Examples of such markets include markets for mortgage-backed securities, swaps and many other derivatives. In this paper, we study decentralized environments in which some market participants have private information on the value of the assets traded. We analyze how their information gradually spreads in bilateral meetings. In particular, we ask whether all relevant information is revealed over time and whether the allocation converges to a Pareto efficient allocation.

Our environment is as follows. Agents start with different endowments of two risky assets, match randomly, and trade in bilateral meetings. In each bilateral meeting, one of the agents makes a take-it-or-leave-it offer to the other, who can accept or reject. Therefore, apart from the presence of private information, our game is a trading game in the tradition of Gale (1986, 1987).² Agents are risk averse so there is potential for mutually beneficial trades of the two assets. Before trading begins, a proportion of agents—the informed agents—receive some information about the value of the assets traded. Namely, they observe a binary signal that determines which one of the two assets is more valuable. In each period, with some probability the game ends, assets' payoffs are revealed, and the agents consume; otherwise, the game continues to the next period. The only information observed by the agents during trading is the history of their own individual trades, but not the portfolios or the trades made by the other agents. Uninformed agents form beliefs about the value of the two assets based on their trading history. This environment is challenging to analyze because the joint distribution of agents' asset holdings and beliefs are endogenous and change over time. The evolution of this distribution is crucial in determining the agents' willingness to trade, since it determines their future trading opportunities.

Our objective is to characterize the efficiency properties of the allocation and the value of information in the long run. Our main result is that the equilibrium converges to an ex post Pareto efficient allocation. Our argument is as follows. First, we focus on the informed agents and prove that their marginal rates of substitution converge to the same value. The intuition for this result is similar to the proof of Pareto efficiency in decentralized environments with

¹See also Duffie, Garleanu and Pedersen (2007), Lagos (2007), Lagos and Rocheteau (2007), Lagos, Rocheteau, and Weill (2007), Vayanos (1998), Vayanos and Weill (2007), and Weill (2007).

²See Gale (2000) for a general treatment of matching and bargaining games with symmetric information and Lauerman (2010) for a recent characterization.

common information. If two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. As their utilities converge to their long-run levels, all the potential gains from trade must be exhausted, so marginal rates of substitution must converge. We then show that the marginal rate of substitution for the uninformed agents also converge. Our argument is based on finding strategies that allow the uninformed agents to learn the signal received by the informed agents at an arbitrarily small cost. The existence of such strategies implies that either agents eventually learn the signal or the benefit of learning the signal goes to zero. Having shown that the marginal rates of substitution of all agents are equalized, we can then show that equilibrium allocations converge to ex post Pareto efficient allocations in the long run. Given risk aversion, an ex post Pareto efficient allocation involves all agents holding a fully diversified portfolio of the two assets. The common marginal rate of substitution of all agents converges to the fully revealing Walrasian prices that would arise in a centralized market.

Our work is related to Wolinsky's (1990) seminal article on information revelation in pairwise matching environments. Wolinsky (1990) considers a game with decentralized, bilateral trading in which agents have the option to trade an indivisible good of uncertain quality, at given prices. In his game, a fraction of traders exits in each period and is replaced by new traders. He shows that steady state equilibria are possible in which some trades that would be Pareto improving under symmetric information do not take place. That is, he obtains an inefficiency result. Blouin and Serrano (2001) show that this inefficiency result survives in a version of Wolinsky's model with a fixed population of traders (which is thus closer to our environment). The crucial differences between our setup and the models in Wolinsky (1990) and Blouin and Serrano (2001) are that in our setting the good is perfectly divisible and that agents can choose at what price to trade. These assumptions lead to very different implications in terms of efficiency, leading to equilibria in which information is fully revealed and allocations are ex post efficient in the long run. Intuitively, divisibility allows uninformed agents to strategically experiment by making small, potentially unprofitable trades to learn valuable information.

In this strand of literature, an early paper that explores the potential for uninformed agents to learn through trading is Green (1992). Green's (1992) objective is to find sufficient conditions on equilibrium strategies and on the span of traded assets that ensure that uninformed agents can perfectly elicit the information of their trading partners in equilibrium. Although our goal here is different—to prove long run efficiency—we share his interest in characterizing the learning strategies of uninformed agents.

In the literature on asset pricing in decentralized markets, following Duffie, Garleanu, and Pedersen (2005), a set of recent papers deals with information transmission through bilateral meetings: Duffie and Manso (2007), Duffie, Giroux, and Manso (2009), Duffie, Malamud, and Manso (2009a, 2009b). These papers use powerful analytical tools to characterize in closed

form the dynamics of beliefs and actions in models where agents get to observe directly or indirectly the information of the agents they meet. The main difference with our work is that these papers focus on models in which bilateral meetings lead to the immediate transmission of private information between the two participants, while we focus on the problem of strategic information revelation in bilateral trades. In other words, in our environment the speed at which information is transmitted in bilateral meetings is endogenous and the ability of uninformed traders to elicit this information is at the center of our analysis. Another related contribution focusing on information transmission in financial markets is Ostrovsky (2009), which focuses on the incentives of large strategic traders in dynamic (centralized) markets and shows that information gets aggregated in equilibrium for a broad class of securities.

The paper is structured as follows. Section 2 describes the environment. Section 3 provides the long-run characterization of equilibria and contains our main efficiency result. Section 4 concludes. The Appendix contains most of the proofs which are sketched in the body of the paper.

2 Setup and trading game

In this section, we introduce the model and define an equilibrium.

2.1 Setup

There are two states of the world $S \in \{S_1, S_2\}$ and two assets $j \in \{1, 2\}$. Asset j is an Arrow security that pays one unit of consumption if and only if state S_j is realized. There is a continuum of agents with von Neumann-Morgenstern expected utility $E[u(c)]$, where E is the expectation operator.

At date 0, each agent is randomly assigned a type i , which determines his initial portfolio of the two assets, denoted by the vector $x_{i,0} \equiv (x_{i,0}^1, x_{i,0}^2)$. There is a finite set of types N and each type $i \in N$ is assigned to a fraction f_i of agents. The aggregate endowment of both assets is 1:

$$\sum_{i \in N} f_i x_{i,0}^j = 1 \text{ for } j = 1, 2. \quad (1)$$

We make the following assumptions on preferences and endowments. The first assumption is symmetry. For each agent with a given initial endowment, there is another agent with an endowment that is a mirror image of the first.

Assumption 1. (Symmetry) *For each type $i \in N$ there exists a type $j \in N$ such that the fraction of agents is equal, $f_i = f_j$, and the endowments are such that $(x_{i,0}^1, x_{i,0}^2) = (x_{j,0}^2, x_{j,0}^1)$.*

This is an assumption that allows us to simplify some formal arguments based on market clearing. The role of this assumption is discussed in detail in Section 3.5.

The second assumption imposes usual properties on the utility function. In addition, it requires boundedness from above and a condition ruling out zero consumption in either state.

Assumption 2. *The utility function $u(\cdot)$ is increasing, strictly concave, twice continuously differentiable on R_{++}^2 , bounded above, and satisfies $\lim_{c \rightarrow 0} u(c) = -\infty$.*

Finally, we assume that the initial endowments are interior.

Assumption 3. *The initial endowment $x_{i,0}$ is in the interior of R_+^2 for all types $i \in N$.*

At date 0, nature draws a binary signal s , which takes the values s_1 and s_2 with equal probabilities. The posterior probability of S_1 conditional on s is denoted by $\phi(s)$. We assume that signal s_1 is favorable to state S_1 and that the signals are symmetric: $\phi(s_1) > 1/2$ and $\phi(s_2) = 1 - \phi(s_1)$. After s is realized, a fraction α of agents of each type privately observes the realization of s . The agents who observe s are the *informed agents*. Those who do not observe the signal are the *uninformed agents*.

2.2 Trading

After the realization of the signal s , but before the state S is revealed, all agents engage in a trading game set in discrete time. Apart from the presence of asymmetric information, our game is a trading game with bilateral bargaining in the tradition of Gale (1986, 1987).

At the beginning of each period $t \geq 1$, the game continues with probability $\gamma < 1$ and ends with probability $1 - \gamma$. If the game ends, the state S is publicly revealed and the agents consume the payoffs of their assets.³ If the game does not end, all agents are randomly matched in pairs, and a round of trading takes place. One of the two agents is selected as the *proposer* with probability 1/2. The proposer makes a take it or leave it offer $z = (z^1, z^2) \in R^2$ to the other agent, the *responder*. That is, the proposer offers to exchange z^1 of asset 1 for $-z^2$ of asset 2. The responder can accept or reject the offer. Suppose an agent with the portfolio $x = (x^1, x^2)$ offers a trade z to an agent with the portfolio $\tilde{x} = (\tilde{x}^1, \tilde{x}^2)$. If the offer is accepted, the proposer's portfolio becomes $x - z$, and the responder's portfolio becomes $\tilde{x} + z$. We assume that the proposer can only make feasible offers, $x - z \geq 0$. The responder can only accept an offer if $\tilde{x} + z \geq 0$.⁴ If an offer is rejected, both agents keep their portfolios x and \tilde{x} . This concludes the trading round.⁵

An agent does not observe the portfolio of his opponent or whether his opponent is informed or not. Moreover, an agent only observes the trading round he is involved in but not the trading

³Allowing for further rounds of trading after the revelation of S would not change our results, given that at that point only one asset has positive value and no trade will occur. Instead of assuming that the game ends, we can alternatively assume that with probability γ the private signal becomes public information, and interpret utilities of the agents as the equilibrium payoffs from a continuation game with full information. All the results will remain unchanged.

⁴The proposer only observes if the offer is accepted or rejected. In particular, if an offer is rejected the proposer does not know whether it was infeasible for the responder or the responder just chose to reject.

⁵See Section 3.5 for a discussion of alternative trading mechanisms.

rounds of other agents. Therefore, both trading and information revelation take place through decentralized, bilateral meetings.

2.3 Equilibrium definition

We now define a perfect Bayesian equilibrium of the trading game.

We begin by describing individual histories. At date 0, each agent is assigned a type $i \in N$, which determines his initial portfolio. Then, with probability α the agent becomes informed and observes the signal s and with probability $1 - \alpha$ remains uninformed. The initial history $h_0 \in N \times \{U, I_1, I_2\}$ captures the realization of these initial conditions: U stands for uninformed, I_1 and I_2 stand for informed with signal, respectively, s_1 and s_2 . In each period $t \geq 1$, the event $h_t = (\iota_t, z_t, r_t)$ is a vector including the following elements: the indicator variable $\iota_t \in \{0, 1\}$, equal to 1 if the agent is selected as the proposer and 0 otherwise; the offer made by the proposer $z_t \in R^2$; the indicator $r_t \in \{0, 1\}$, equal to 1 if the offer is accepted and 0 otherwise. The sequence $h^t = \{h_0, h_1, \dots, h_t\}$ denotes the history of play up to period t for an individual agent. Let H^t denote the space of all possible histories of length t . Let H^∞ denote the space of all infinite histories, that is, histories along which the game never ends, and let $\Omega = \{s_1, s_2\} \times H^\infty$. A point in Ω describes the whole potential history of play for an individual agent, if the game continues forever. We will use (s, h^t) to denote the subset of Ω given by all the points $\omega = (s, h^\infty) \in \Omega$ such that the truncation of h^∞ at time t is equal to h^t .

We now describe the strategy of an agent. If the agent is selected as the proposer at time t , his actions are given by the map:

$$\sigma_t^p : H^{t-1} \rightarrow \mathcal{P},$$

where \mathcal{P} denotes the space of probability distributions over R^2 with finite support. That is, we allow for mixed strategies and let the proposer choose the probability distribution $\sigma_t^p(\cdot | h^{t-1})$ from which he draws the offer z .⁶ If the agent is selected as the responder, his behavior is described by:

$$\sigma_t^r : H^{t-1} \times R^2 \rightarrow [0, 1],$$

which denotes the probability that the agent accepts the offer $z \in R^2$ for each history h^{t-1} . The strategies are restricted to be feasible for both players. A strategy is fully described by the sequence $\sigma = \{\sigma_t^p, \sigma_t^r\}_{t=1}^\infty$.

We focus on symmetric equilibria where all agents play the same strategy σ . Given this strategy, we define a probability measure P on Ω which will be used both to represent ex ante uncertainty from the point of view of the single agent and to capture the evolution of the cross sectional distribution of individual histories in the economy. We say that the probability

⁶We restrict agents to mix over a finite set of offers to simplify the measure-theoretic apparatus. None of the arguments in our proofs require this restriction.

measure P is *consistent with* σ , if, for all s and h^{t-1} , $P(s, h^{t-1})$ is the probability that the signal is s and the agent reaches history h^{t-1} , conditional on the game ending at time t and all agents playing σ . The unconditional probability that the signal is s and the game ends at time t with history h^{t-1} is equal to $(1 - \gamma)\gamma^{t-1}P(s, h^{t-1})$. The sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \dots \subset \mathcal{F}$ denotes the filtration generated by the information sets of the agent at the beginning of each period t . The measure P also characterizes the cross sectional distribution of individual histories in a symmetric equilibrium: at the beginning of time t , the mass of agents with history h^{t-1} is equal to $P(h^{t-1}|s)$.

The probability measure P is constructed recursively in the following way. Start from period 0. The probability of the event (s, h^0) for each $h^0 \in H^0$ is determined by the exogenous assignment of initial portfolios and information at date 0. For example, $P(s_1, h^0) = (1/2)\alpha f_i$ if $h_0 = (i, I_1)$, given that the probability of s_1 is $1/2$, and the agent receives the information I_1 with probability α and the portfolio i with probability f_i . Next, take any period $t \geq 1$. Given that agents are randomly matched, the probability of receiving offer $z \in R^2$ in period t for the responding agent is

$$\psi_t(z|s) = \int_{h^{t-1}} \sigma_t^p(z|h^{t-1}) dP(h^{t-1}|s),$$

and the probability that offer $z \in R^2$ is accepted for the proposing agent is

$$\chi_t(z|s) = \int_{h^{t-1}} \sigma_t^r(h^{t-1}, z) dP(h^{t-1}|s).$$

Given $P(s, h^{t-1})$, $\psi_t(\cdot|s)$ and $\chi_t(\cdot|s)$, we can then construct $P(s, h^t)$ as follows. Take an agent with history h^{t-1} and let us compute the probability of reaching history $h^t = (h^{t-1}, (0, z, 1))$ at the beginning of next period. That is, the probability that in period t he is the responder, $\iota_t = 0$, receives offer $z_t = z$, and accepts it, $r_t = 1$. This probability is

$$P(s, h^t) = \frac{1}{2}\sigma^r(h^{t-1}, z)\psi_t(z|s)P(s, h^{t-1}),$$

since the probability of being selected as the responder is $1/2$, the probability of receiving offer z is $\psi_t(z|s)$, and the probability of accepting it is $\sigma^r(h^{t-1}, z)$. In a similar way, we have

$$\begin{aligned} P(s, h^t) &= \frac{1}{2}(1 - \sigma^r(h^{t-1}, z))\psi_t(z|s)P(s, h^{t-1}) \text{ if } h_t = (0, z, 0), \\ P(s, h^t) &= \frac{1}{2}\chi_t(z|s)\sigma^p(z|h^{t-1})P(s, h^{t-1}) \text{ if } h_t = (1, z, 1), \\ P(s, h^t) &= \frac{1}{2}(1 - \chi_t(z|s))\sigma^p(z|h^{t-1})P(s, h^{t-1}) \text{ if } h_t = (1, z, 0). \end{aligned}$$

Notice that, having restricted agents to randomize over a finite set of offers, $P(s, h^t)$ assigns positive probability to a finite set of histories h^t .

To assess whether the strategy σ is individually optimal, an agent has to form expectations

about his opponents' behavior. The agent's beliefs are described by two functions:

$$\begin{aligned}\delta_t & : H^{t-1} \rightarrow [0, 1], \\ \delta_t^r & : H^{t-1} \times R^2 \rightarrow [0, 1],\end{aligned}$$

which represent, respectively, the probability assigned to signal s_1 after history h^{t-1} , at the beginning of the period, and the probability assigned to signal s_1 after history h^{t-1} , if the agent is the responder and receives offer z . The agent's beliefs are denoted compactly by $\boldsymbol{\delta} = \{\delta_t, \delta_t^r\}_{t=1}^\infty$. At each history h^{t-1} , an agent expects that in each period $\tau \geq t$, he will face an opponent with history $\tilde{h}^{\tau-1}$ randomly drawn from the probability distribution $P(\tilde{h}^{\tau-1}|s)$, conditional on s , and he expects his opponent to play the strategy $\boldsymbol{\sigma}$. This completely describes the agent's expectations about the current and future behavior of other players. For example, the probability distribution of offers expected at time $\tau \geq t$ by an agent at h^{t-1} is equal to

$$\psi_\tau(z|s_1)\delta_t(h^{t-1}) + \psi_\tau(z|s_2)(1 - \delta_t(h^{t-1})).$$

The beliefs δ_t are required to be consistent with Bayesian updating on the equilibrium path. This implies that

$$\delta_t(h^{t-1}) = \frac{P(s_1, h^{t-1})}{\sum_s P(s, h^{t-1})},$$

for all histories h^{t-1} such that $\sum_s P(s, h^{t-1}) > 0$. The same requirement is imposed on the beliefs δ_t^r , which implies that

$$\delta_t^r(h^{t-1}, z) = \frac{\psi_t(z|s_1)P(s_1, h^{t-1})}{\sum_s \psi_t(z|s)P(s, h^{t-1})}$$

for all histories h^{t-1} and offers z such that $\sum_s \psi_t(z|s)P(s, h^{t-1}) > 0$.

This representation of the agents' beliefs embeds an important assumption: an agent who observes his opponent play an off-the-equilibrium-path action can change his beliefs about s , but maintains that the behavior of all other agents, conditional on s , is unchanged. That is, he believes that all other agents will continue to play $\boldsymbol{\sigma}$ in the future. This is a reasonable restriction on off-the-equilibrium-path beliefs in a game with atomistic agents and allows us to focus on the agent's beliefs about s , given that s is a sufficient statistic for the future behavior of the agent's opponents.

Moreover, the beliefs of informed agents are required to always assign probability 1 to the signal observed at date 0:

$$\begin{aligned}\delta_t(h^{t-1}) & = \delta_t^r(h^{t-1}, z) = 1 \text{ if } h_0 = (i, I_1), \\ \delta_t(h^{t-1}) & = \delta_t^r(h^{t-1}, z) = 0 \text{ if } h_0 = (i, I_2).\end{aligned}$$

That is, informed agents do not change their beliefs on signal s , even after observing off-the-equilibrium-path behavior from their opponents. This fact will play a crucial role in the equilibrium analysis, since it will allow us to characterize the behavior of informed agents after any possible offer.

We are now ready to define an equilibrium.

Definition 1 *A perfect Bayesian equilibrium is given by a strategy σ , beliefs δ , and a probability space (Ω, \mathcal{F}, P) , such that:*

- (i) *the strategy σ is individually optimal at each history h^{t-1} given the beliefs δ and given that agents expect that at each round $\tau \geq t$ they will face an opponent with history $\tilde{h}^{\tau-1}$ randomly drawn from $P(\tilde{h}^{\tau-1}|s)$ who plays σ ;*
- (ii) *the beliefs δ are consistent with Bayes' rule whenever possible;*
- (iii) *the probability measure P is consistent with σ .*

Notice that the cross sectional behavior of the economy in equilibrium is purely determined by the signal s . In other words, s is the only relevant aggregate state variable for our trading game, and, for this reason, we will call it interchangeably signal s or state s .

To establish our results, we restrict attention to equilibria that satisfy two properties, which we call *symmetry across states* and *uniform market clearing*. Let us first state these two properties and then discuss their role in the analysis.

Informally, symmetry across states means that strategies and beliefs are the same if we switch the labels of assets 1 and 2 and those of signals 1 and 2. To define this property formally, let us define the *complement* of history h^t . The history \tilde{h}^t is the complement of history h^t if the following are true: (i) if (x^1, x^2) is the initial endowment in h_0 , then (x^2, x^1) is the initial endowment in \tilde{h}_0 ; (ii) if the agent is informed and observes s_j in h_0 , he is informed and observes s_{-j} in \tilde{h}_0 ; (iii) if the offer $z = (z^1, z^2)$ was made/received in h_t , offer $z = (z^2, z^1)$ is made/received in \tilde{h}_t ; (iv) the responses are the same in \tilde{h}_t and h_t . We can then define symmetry across state as follows.

Definition 2 *An equilibrium satisfies symmetry across states if the strategy σ and the beliefs δ satisfy the following: (a) $\sigma_{t+1}^p((z^1, z^2) | h^t) = \sigma_{t+1}^p((z^2, z^1) | \tilde{h}^t)$ and $\sigma_{t+1}^r(h^t, (z^1, z^2)) = \sigma_{t+1}^r(\tilde{h}^t, (z^2, z^1))$; (b) $\delta(h^t) = 1 - \delta(\tilde{h}^t)$ and $\delta^r(h^t, (z^1, z^2)) = 1 - \delta^r(\tilde{h}^t, (z^2, z^1))$ for all h^t and (z^1, z^2) , where \tilde{h}^t is the complement of h^t .*

This restriction is more stringent than the standard symmetry requirement that all agents follow the same strategy, which we also assume.

Focusing on equilibria that satisfy symmetry across states greatly helps in two important steps of our proofs: in the proof of Lemma 7 in the Appendix, which is needed to prove Proposition 2, and in the proof of our main theorem, Theorem 1. We will discuss its role in detail when we present these results. For now, let us anticipate that it helps in the following

argument. At any equilibrium allocation, we can look at the marginal rate of substitution between the two assets for each agent. In the course of the analysis, we want to rule out allocations in which the marginal rates of substitution of all agents converge to the same value *independent of signal* s . Notice that the supply of the two assets is fixed and independent of s . So, by market clearing, agents must hold on average the two assets in the same proportion whether s is s_1 or s_2 . At the same time, informed agents observe signal s and thus, for given asset holdings, should value one of the two assets more at the margin. In other words, one expects the marginal valuation of the more valuable asset in state s to be higher for at least some agents in that state. Although this argument is intuitive, it is not easy to develop formally. Symmetry across states helps us in this task.

Uniform market clearing is a more technical restriction, which requires that market clearing approximately holds for agents with asset holdings in an interval $[0, M]$, for M large enough. Formally, it is defined as follows.

Definition 3 *A symmetric equilibrium satisfies uniform market clearing if for all $\varepsilon > 0$ there is an M such that*

$$\int_{x_t^j(\omega) \leq M} x_t^j(\omega) dP(\omega|s) \geq 1 - \varepsilon,$$

for all t and for all j .

For a given t , this property is just an implication of market clearing and of the dominated convergence theorem. The additional restriction comes from imposing that the property holds uniformly over t . Notice that all equilibria in which the portfolios x_t converge almost surely satisfy uniform market clearing.⁷

We will derive our main result for equilibria that satisfy both symmetry across states and uniform market clearing. Unfortunately, we do not have a general existence proof of equilibria that satisfy these properties. In a companion paper (Goloso, Lorenzoni and Tsyvinski, 2011), we take a computational approach and, under parametric assumptions, we compute equilibria that satisfy them.

2.4 Rational Expectations Equilibrium

Before turning to the analysis of our decentralized trading game, let us briefly discuss a version of our economy with centralized markets, as, for example, in Grossman (1989). The rational

⁷Use Theorem 16.14 in Billingsley (1995). This assumption would not be required in a trading game with a large but *finite* number of agents, as in that case there would be a natural upper bound on the assets holdings of each agent, given by the aggregate endowment. However, to extend the model to a finite number of agents is not trivial since a law-of-large-numbers argument cannot be invoked, so the aggregate state of the game is not just s . Moreover, to derive limit theorems in trading games with large but finite number of agents one usually needs to impose further restrictions on strategies, as has been shown in symmetric information environments (Rubinstein and Wolinsky, 1990, and Gale, 2000, Chapter 3).

expectations equilibrium of this economy will provide a useful benchmark for our long-run results.

Consider an economy with the same endowments, preferences and information as described above but where, at date 0, agents trade the two assets on a centralized Walrasian market. Let $\delta^I(s)$ denote the belief of an informed agent in state s : $\delta^I(s_1) = 1$ and $\delta^I(s_2) = 0$ (recall that δ is the probability assigned to s_1). Let $\delta^U = 1/2$ denote the belief of an uninformed agent. A *rational expectations equilibrium* consists of prices $q(s) \in R_+^2$ and allocations $\{x_i^*(s, \delta)\}_{i=1}^I$ such that agents optimize:

$$x_i^*(s, \delta) = \arg \max_{q(s) \cdot x \leq q(s) \cdot x_{i,0}} E \{u(x) | q(s), \delta\} \text{ for } s \in \{s_1, s_2\} \text{ and } \delta \in \{\delta^I(s), \delta^U\}, \quad (2)$$

and markets clear:

$$\sum_{i \in N} f_i \left[\alpha x_i^{j*}(s, \delta^I(s)) + (1 - \alpha) x_i^{j*}(s, \delta^U) \right] = 1 \text{ for } j = 1, 2 \text{ and } s \in \{s_1, s_2\}.$$

In a fully revealing equilibrium $q(s_1) \neq q(s_2)$, so the prices perfectly reveal the signal s to the uninformed agents. This implies that the optimal quantities in (2) are the same for all δ , that is, informed and uninformed with the same endowments get the same equilibrium allocations. To satisfy market clearing it must then be true that the relative price is equal to the ratio of the probabilities:

$$\frac{q^1(s)}{q^2(s)} = \frac{\phi(s)}{1 - \phi(s)}.$$

It is easy to verify that a fully revealing equilibrium exists and is ex post Pareto efficient. The second welfare theorem also holds. For any Pareto efficient allocation, there is some initial allocation for which the efficient allocation constitutes a rational expectations equilibrium.

It is more difficult to rule out the existence of non fully revealing equilibria. The logic of the argument will be useful in the analysis of the economy with decentralized trading (Proposition 2). Suppose there is a non fully revealing equilibrium. Then the relative price $q^1(s)/q^2(s)$ must be the same for both signals s_1 and s_2 . The optimality condition for problem (2) gives:

$$\frac{\pi(\delta)u'(x_i^{1*}(s, \delta))}{(1 - \pi(\delta))u'(x_i^{2*}(s, \delta))} = \frac{q^1(s)}{q^2(s)} \quad (3)$$

for $\delta \in \{\delta^U, \delta^I(s_1)\}$, where $\pi(\delta)$ is the probability that an agent with belief δ assigns to state S_1 :

$$\pi(\delta) \equiv \delta\phi(s_1) + (1 - \delta)\phi(s_2). \quad (4)$$

Now suppose that the relative price satisfies $q^1(s)/q^2(s) \leq 1$. Then condition (3) implies that, conditional on s_1 , informed agents will hold more of the first asset than of the second, $x_i^{1*}(s_1, \delta^I(s_1)) > x_i^{2*}(s_1, \delta^I(s_1))$, given that they attach a higher probability to state

S^1 , i.e., given that $\pi(\delta^I(s_1))/(1 - \pi(\delta^I(s_1))) > 1$. Similarly, uninformed agents will hold (weakly) more of the first asset than of the second, $x_i^{1*}(s_1, \delta^U(s_1)) \geq x_i^{2*}(s_1, \delta^U(s_1))$, given that

$$\pi(\delta^U(s_1))/(1 - \pi(\delta^U(s_1))) = 1.$$

Therefore, conditional on s_1 , total holdings of asset 1 must be greater than total holdings of asset 2. This violates market clearing and gives a contradiction. By a symmetric argument, if the relative price $q^1(s)/q^2(s)$ is greater than 1 total holdings of asset 1 must be smaller than total holdings of asset 2, a contradiction again. Therefore, a non fully revealing equilibrium does not exist.

3 Long-run efficiency

We now provide a characterization of the equilibrium in the long run—i.e., along the path where the game does not end. Our main result is that the equilibrium allocation converges to an ex post Pareto efficient allocation. By ex post Pareto efficient we mean Pareto efficient when the expected utility of each agent is computed conditional on the signal s , that is, after s is publicly revealed but before the state S is revealed.⁸

Notice that if agents play a decentralized trading game for finitely many periods, the allocation will not be, in general, Pareto efficient, due to the matching friction. For example, with positive probability an agent could meet only agents with his own same endowment and would not be able to trade. However if the agents keep playing the game, they will eventually meet other agents with whom profitable trades are possible. Absent informational frictions, if we make the horizon long enough all potential gains from trade will eventually be realized. In other words, in a decentralized game with symmetric information the allocation converges to efficiency in the long run. Different versions of this result under symmetric information are discussed in Gale (2000).

Once we introduce asymmetric information, however, it is not clear how to extend this argument. The problem is that, with asymmetric information, when two agents meet, there may be a Pareto improving trade between them conditional on s , and yet, since s is not common information, they might not be able to credibly signal to each other the presence of this trade. For example, take an uninformed agent with a relatively small amount of asset 1 who meets an informed agent with a relatively large amount of asset 1. Suppose the state is s_1 and the informed agent offers to sell some amount of asset 1 at a price that is mutually beneficial, conditional on s_1 . The uninformed responder may reject the offer because he is afraid that the proposer has observed signal s_2 and is trying to sell asset 1 just because it is

⁸This is the standard notion of ex post efficiency as in Holmstrom and Myerson (1983). The same efficiency notion was used above in discussing the Rational Expectations Equilibrium. Notice that after S is revealed all allocations are trivially ex post efficient.

less valuable. Such reasoning may lead to an equilibrium in which long-run allocations are inefficient. Importantly, in environments with asymmetric information, Wolinsky (1990) and Blouin and Serrano (2001) conclude that there are equilibria in which equilibrium allocations are *not* efficient. Our contribution is to show that the long-run allocation is efficient in our environment, in which uninformed agents have the possibility of experimenting by trading repeatedly. Our line of reasoning is as follows. We begin by considering the behavior of informed agents and show that they equalize their marginal rates of substitution in the long run. Then we show that the uninformed agents' marginal rates of substitution also converge to the same value. This is where we introduce experimentation, showing that in equilibrium the uninformed agents can always construct small trades that allow them to learn the signal s arbitrarily well. We finally combine these results to prove our efficiency result in Theorem 1.

It is important to clarify that throughout the paper we keep fixed the central trading friction in the model: the frequency of trading captured by the parameter γ . This parameter determines the average number of trades before the game ends and information is revealed. All our long-run results feature implicit dependence on γ . That is, we show that for a given γ there exists a large enough $T(\gamma)$ such that for all periods $t \geq T(\gamma)$ the relevant property holds with sufficiently high probability (conditional on reaching period t).

Another point to notice is that typically our environment features multiple equilibria. Our long-run results apply to all equilibria.

3.1 Preliminary considerations

We first define and characterize the stochastic process for the agents' expected utility in equilibrium. We use the martingale convergence theorem to show that expected utility converges in the long run, conditionally on the game not ending.

Let us define two stochastic processes, which describe the equilibrium dynamics of individual portfolios and beliefs, conditional on the game not ending. Take the probability space (Ω, \mathcal{F}, P) and let $x_t(\omega)$ and $\delta_t(\omega)$ denote the portfolio and belief of the agent at the beginning of period t , at ω . Since an agent's current portfolio and belief are, by construction, in his information set at time t , $x_t(\omega)$ and $\delta_t(\omega)$ are \mathcal{F}_t -measurable stochastic processes on (Ω, \mathcal{F}, P) .

If the game ends, an agent with the portfolio-belief pair (x, δ) receives the expected payoff:

$$U(x, \delta) \equiv \pi(\delta)u(x^1) + (1 - \pi(\delta))u(x^2),$$

where $\pi(\delta)$ is defined in (4). Using the stochastic processes x_t and δ_t , we obtain a stochastic process u_t for the equilibrium expected utility of an agent if the trading game ends in period t :

$$u_t(\omega) \equiv U(x_t(\omega), \delta_t(\omega)).$$

We can then define the stochastic process v_t as the expected lifetime utility of an agent at the

beginning of period t :

$$v_t \equiv (1 - \gamma)E \left[\sum_{s=t}^{\infty} \gamma^{s-t} u_s \mid \mathcal{F}_t \right].$$

In recursive terms, we have

$$v_t = (1 - \gamma)u_t + \gamma E[v_{t+1} \mid \mathcal{F}_t]. \quad (5)$$

Notice that, since v_t is constructed using the equilibrium stochastic processes for x_t and δ_t , it represents the expected utility from following the equilibrium strategy, which is, by definition, individually optimal. Using this fact, the next lemma establishes that v_t is a bounded martingale and converges in the long run.

Lemma 1 *There exists a random variable $v^\infty(\omega)$ such that*

$$\lim_{t \rightarrow \infty} v_t(\omega) = v^\infty(\omega) \text{ a.s.}$$

Proof. An agent always has the option to keep his time t portfolio x_t and wait for the end of the game, rejecting all offers and offering zero trades in all $t' \geq t$. His expected lifetime utility under this strategy is equal to u_t . Therefore, optimality implies that

$$u_t \leq E[v_{t+1} \mid \mathcal{F}_t],$$

which, combined with equation (5), gives

$$v_t \leq E[v_{t+1} \mid \mathcal{F}_t].$$

This shows that v_t is a submartingale. It is bounded above because the utility function $u(\cdot)$ is bounded above. Therefore, it converges by the martingale convergence theorem. ■

It is useful to introduce an additional stochastic process, \hat{v}_t , which will be used as a reference point to study the behavior of agents who make and receive off-the-equilibrium-path offers. Let \hat{v}_t be the expected lifetime utility of an agent who adopts the following strategy: (i) if selected as the proposer at time t , follow the equilibrium strategy σ ; (ii) if selected as the responder, reject all offers at time t and follow an optimal continuation strategy from $t + 1$ onwards. The expected utility \hat{v}_t is computed at time t immediately after the agent is selected as the proposer or the responder, i.e., it is measurable with respect to (h^{t-1}, ι_t) , and, by definition, satisfies $\hat{v}_t \leq E[v_{t+1} \mid h^{t-1}, \iota_t]$.

Recall from the proof of Lemma 1 that u_t is the expected utility from holding the portfolio x_t until the end of the game. The following lemma shows that, in the long run, an agent is almost as well off keeping his time t portfolio as he is under the strategy leading to \hat{v}_t .

Lemma 2 Both u_t and \hat{v}_t converge almost surely to v^∞ :

$$\lim_{t \rightarrow \infty} u_t(\omega) = \lim_{t \rightarrow \infty} \hat{v}_t(\omega) = v^\infty(\omega) \text{ a.s.}$$

Proof. As argued in Lemma 1, v_t is a bounded submartingale and converges almost surely to v^∞ . Let $y_t \equiv E[v_{t+1} | \mathcal{F}_t]$. Since a bounded martingale is uniformly integrable (see Williams, 1991), we get $y_t - v_t \rightarrow 0$ almost surely. Rewrite equation (5) as

$$(1 - \gamma) u_t = \gamma (v_t - E[v_{t+1} | \mathcal{F}_t]) + (1 - \gamma) v_t.$$

This gives

$$u_t - v_t = \frac{\gamma}{1 - \gamma} (v_t - E[v_{t+1} | \mathcal{F}_t]) = \frac{\gamma}{1 - \gamma} (v_t - y_t),$$

which implies $u_t - v_t \rightarrow 0$ almost surely. The latter implies $u_t \rightarrow v^\infty$ almost surely. Letting $\hat{y}_t \equiv E[v_{t+1} | h^{t-1}, \iota_t]$, notice that $\hat{y}_t \rightarrow v^\infty$ almost surely. Since $u_t \leq \hat{v}_t \leq \hat{y}_t$, it follows that $\hat{v}_t \rightarrow v^\infty$ almost surely. ■

3.2 Informed agents

We first focus on informed agents and show that their marginal rates of substitution converge in probability. In particular, we show that, conditional on each signal s , the marginal rates of substitution of all informed agents converge in probability to the same sequence, which we denote $\kappa_t(s)$. In the following, we will refer to $\kappa_t(s)$ as the “long-run marginal rate of substitution” of the informed agents.

The intuition for this result is that if two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. As their utilities converge to their long-run levels, all the potential gains from bilateral trade must be exhausted. This implies that marginal rates of substitution must converge. Since this lemma is about the behavior of the informed agents, our argument is similar to those used to prove Pareto efficiency in decentralized market with full information (Gale, 2000).

Proposition 1 (Convergence of MRS for informed agents) *There exist two sequences $\kappa_t(s_1)$ and $\kappa_t(s_2)$ such that, conditional on each s , the marginal rates of substitution of informed agents converge in probability to $\kappa_t(s)$:*

$$\lim_{t \rightarrow \infty} P \left(\left| \frac{\phi(s) u'(x_t^1)}{(1 - \phi(s)) u'(x_t^2)} - \kappa_t(s) \right| > \varepsilon \mid \delta_t = \delta^I(s), s \right) = 0 \text{ for all } \varepsilon > 0. \quad (6)$$

Proof. We provide a sketch of the proof here and leave the details to the appendix. Without loss of generality, suppose (6) is violated for $s = s_1$. Then, it is always possible to find a period T , arbitrarily large, in which there are two groups, of positive mass, of informed agents with

marginal rates of substitution sufficiently different from each other. In particular, we can find a κ^* such that a positive mass of informed agents have marginal rates of substitution below κ^* :

$$\frac{\phi(s_1)u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} < \kappa^*,$$

and a positive mass of informed agents have marginal rates of substitution above $\kappa^* + \varepsilon$:

$$\frac{\phi(s_1)u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} > \kappa^* + \varepsilon,$$

for some positive ε . An informed agent in the first group can then offer to sell a small quantity ζ^* of asset 1 at the price $p^* = \kappa^* + \varepsilon/2$, that is, he can offer the trade $z^* = (\zeta^*, -p^*\zeta^*)$. Suppose this offer is accepted and the proposer stops trading afterwards. Then his utility can be approximated as follows:

$$\begin{aligned} & \phi(s_1)u(x_T^1 - \zeta^*) + (1 - \phi(s_1))u(x_T^2 + p^*\zeta^*) \\ & \approx u_T + [-\phi(s_1)u'(x_T^1) + (1 - \phi(s_1))p^*u'(x_T^2)]\zeta^* \\ & \approx \hat{v}_T + (1 - \phi(s_1))u'(x_T^2)\zeta^*\varepsilon/2, \end{aligned}$$

where we use a Taylor expansion to approximate the utility gain and we use Lemma 2 to show that the continuation utility \hat{v}_T can be approximated by the current utility u_T . By choosing T sufficiently large and the size of the trade ζ^* sufficiently small we can make the approximation errors in the above equation small enough, so that when this trade is accepted it strictly improves the utility of the proposer. All the informed responders with marginal rate of substitution above $\kappa^* + \varepsilon$ are also better off, by a similar argument. Therefore, they will all accept the offer. Since there is a positive mass of them, the strategy described gives strictly higher utility than the equilibrium strategy to the proposer, and we have a contradiction. ■

Two remarks on the argument above: First, there may be uninformed agents who also potentially accept z^* , but this only increases the probability of acceptance, further improving the utility of the proposer. Second, the deviation described (offer z^* at T and stop trading afterwards) is not necessarily the best deviation for the proposer. However, since our argument is by contradiction, it is enough to focus on a simple deviation of this form. We will take a similar approach in many of the following proofs.

3.3 Uninformed agents

We now turn to the characterization of equilibria for uninformed agents. The main difficulty here is that uninformed agents may change their beliefs upon observing their opponent's behavior. Thus an agent who would be willing to accept a given trade ex ante—before updating his beliefs—might reject it ex post. Additional complexity comes from the fact that updated

beliefs are not determined by Bayes' rule after off-the-equilibrium-path offers, while our objective is to develop a general argument, independent of how off-the-equilibrium-path beliefs are specified. For these reasons, we need a strategy of proof different from the one used for informed agents.

Our argument is based on finding strategies that allow the uninformed agents to learn the signal s at an arbitrarily small cost. This is done in Lemma 3 below. The existence of such strategies implies that either uninformed agents eventually learn the signal or the benefit of learning goes to zero. In Theorem 1 we show that this implies that equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

To build our argument, it is first useful to show that in equilibrium the marginal rates of substitution of all agents cannot converge to the same value *independently of the state s* . Since individual marginal rates of substitution determine the prices at which agents are willing to trade, this rules out equilibria in which agents, in the long run, are all willing to trade at the same price, independent of s . The fact that agents are willing to trade at different prices in the two states s_1 and s_2 will be key in constructing the experimentation strategies below. This fact will allow us to construct small trades that are accepted with different probability in the two states. By offering such trades an uninformed agent will be able to extract information on s and thus acquire the information obtained by the informed agents at date 0.

Remember that $\kappa_t(s)$ denotes the long-run marginal rates of substitution of informed agents in state s (Proposition 1). The next proposition shows that in the long run two cases are possible: either the two values $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently far from each other, or, in each state s , there must be a sufficient mass of agents with marginal rates of substitution far enough from $\kappa_t(s)$. That is, either the informed agents' marginal rates of substitution converge to different values or there must be enough uninformed agents with marginal rates of substitution different from that of the informed.

Proposition 2 *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There exists a period T and a scalar $\bar{\varepsilon} > 0$ such that in all periods $t \geq T$ one of the following holds: (i) the long-run marginal rates of substitution of the informed agents are sufficiently different in the two states:*

$$|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon},$$

or (ii) sufficiently many agents have a marginal rate of substitution different from $\kappa_t(s)$:

$$P \left(\left| \frac{\pi(\delta_t) u'(x_t^1)}{(1 - \pi(\delta_t)) u'(x_t^2)} - \kappa_t(s) \right| \geq 2\bar{\varepsilon} \mid s \right) \geq \bar{\varepsilon}$$

for $s \in \{s_1, s_2\}$.

The proof of this proposition is in the appendix. The proof shows that if both (i) and

(ii) are violated we reach a contradiction. If both (i) and (ii) are violated, we can always find a period t , as large as we like, in which all the agents' marginal rates of substitution are concentrated around some value κ which is *independent of the state s* . The main point of the proof is to show that this leads to a violation of market clearing. The argument is similar in spirit to the one used to rule out non fully revealing equilibria in the Walrasian market of Section 2.4. Recall that in that environment, if prices are independent of s market clearing cannot be satisfied (see p. 11). There is an important difficulty to overcome in order to apply this argument in our environment. In the rational expectation equilibrium of the Walrasian economy of Section 2.4, if prices are not revealing, uninformed agents keep their initial beliefs, while in our economy with decentralized trading uninformed agents typically learn something in the early periods of trading. That is, when they reach period t , they have already observed a number of trades, and their beliefs are no longer $\delta^U = 1/2$. In the proof of Proposition 2, we extend the argument by showing that in equilibrium the distribution of beliefs of the uninformed is always biased in the direction of the true signal. That is, when $s = s_1$ there are more uninformed agents with a belief $\delta_t > 1/2$ than uninformed agents with a belief $\delta_t \leq 1/2$. This allows us to show that market clearing is violated if the marginal rates of substitution of all agents are the same and independent of s .

The proof of this proposition uses both symmetry across states and uniform market clearing (defined in Definitions 2 and 3). Symmetry across states is used in Lemma 7 in the Appendix. That lemma shows that if the state is, say, s_1 , and we want to check market clearing at any time t , instead of checking it on the original population of informed and uninformed agents, we can restrict attention to informed agents plus the subset of uninformed agents with beliefs $\delta_t \geq 1/2$. This construction relies on symmetry across states and allows us to overcome the difficulty mentioned in the previous paragraph. Uniform market clearing is used in Step 3 of the proof of Proposition 2, to show that if the average holdings of asset j exceed 1 when integrating over a large enough fraction of agents, market clearing cannot be satisfied when integrating over the whole population.

3.3.1 Experimentation

We now show how uninformed agents can experiment and acquire information on the state s by making small offers. In the proof of Proposition 3, we will construct a sequence of offers with the following property: given any $\varepsilon > 0$, if the uninformed agent makes the offers $\{\hat{z}_{t+j}\}_{j=0}^{J-1}$ at times $t, t+1, \dots, t+J-1$, and receives the “right” string of responses (e.g., $\{\hat{r}_{t+j}\}_{j=0}^{J-1} = \{0, 1, 1, 0, \dots, 1\}$) then the probability he assigns to s_1 at time $t+J$ will be larger than $1-\varepsilon$. That is, this sequence of offers allows the experimenter to acquire arbitrarily precise information on state s_1 (a similar construction can be done for s_2). Here we make the crucial step in the construction of this sequence of offers. Namely, we find a single offer z such that if the right response is received, the proposer's belief increases by a sufficient amount.

Consider an uninformed agent who assigns probability $\delta \in (0, 1)$ to signal s_1 at the beginning of period t and makes offer z . Recall that the probability of acceptance of z , conditional on s , is $\chi_t(z|s)$. Bayes' rule implies that if the offer is accepted the agent's updated belief δ' satisfies

$$\frac{\delta'}{1 - \delta'} = \frac{\delta}{1 - \delta} \frac{\chi_t(z|s_1)}{\chi_t(z|s_2)},$$

while if the offer is rejected his updated belief satisfies

$$\frac{\delta'}{1 - \delta'} = \frac{\delta}{1 - \delta} \frac{1 - \chi_t(z|s_1)}{1 - \chi_t(z|s_2)}.$$

If $\chi_t(z|s_1) > \chi_t(z|s_2)$ the acceptance of offer z provides a signal in favor of s_1 , if $\chi_t(z|s_1) < \chi_t(z|s_2)$ the rejection of offer z provides a signal in favor of s_1 . Our objective is to find a constant $\rho > 1$, such that we can always find an offer z such that either

$$\frac{\chi_t(z|s_1)}{\chi_t(z|s_2)} > \rho$$

or

$$\frac{1 - \chi_t(z|s_1)}{1 - \chi_t(z|s_2)} > \rho.$$

In this way, if the agent makes offer z and receives the right response (a “yes” in the first case, a “no” in the second) his beliefs satisfy

$$\frac{\delta'}{1 - \delta'} > \rho \frac{\delta}{1 - \delta}.$$

Since $\rho > 1$, this ensures that we can choose a long enough sequence of offers such that, if the right responses are received, the agent's belief will converge to 1.

The following lemma shows how to construct the offer z .

Lemma 3 *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There are two scalars $\beta > 0$ and $\rho > 1$ with the following property: for all $\theta > 0$ there is a time T such that for all $t \geq T$ there exist a trade z with $\|z\| < \theta$ that satisfies either*

$$\chi_t(z|s_1) > \beta, \quad \chi_t(z|s_1) > \rho \chi_t(z|s_2), \quad (7)$$

or

$$1 - \chi_t(z|s_1) > \beta, \quad 1 - \chi_t(z|s_1) > \rho(1 - \chi_t(z|s_2)). \quad (8)$$

Proof. We provide a sketch of the argument here and leave the details to the appendix. We distinguish two cases. By Proposition 2 one of the following must be true in any period t following some period T : (i) either the long-run marginal rates of substitutions of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently different from each other or (ii) there is a sufficiently

large mass of agents with marginal rates of substitution sufficiently different from $\kappa_t(s)$. The proof proceeds differently in the two cases.

Case 1. Suppose that there is a large enough difference between $\kappa_t(s_1)$ and $\kappa_t(s_2)$. Assume without loss of generality that $\kappa_t(s_1) > \kappa_t(s_2)$. Suppose the uninformed agent offers to sell a small quantity ζ of asset 1 at the price $p = (\kappa_t(s_1) + \kappa_t(s_2))/2$, which lies between the two marginal rates of substitutions $\kappa_t(s_1)$ and $\kappa_t(s_2)$. That is, he offers the trade $z = (\zeta, -p\zeta)$. We now make two observations on offer z :

Observation 1. In state s_1 , there is a positive mass of informed agents willing to accept offer z , provided ζ is small enough and t is sufficiently large. Combining Lemma 2 and Proposition 1, we can show that in state s_1 , for t large enough, there is a positive mass of informed agents with marginal rates of substitution sufficiently close to $\kappa_t(s_1)$, who are close enough to their long-run utility. These agents are better off accepting z , as they are buying asset 1 at a price smaller than their marginal valuation.

Observation 2. Conditional on signal s_2 , the offer z cannot be accepted by any agent, informed or uninformed, except possibly by a vanishing mass of agents. Suppose, to the contrary, that a positive fraction of agents accepted z in state s_2 . By an argument symmetric to the one above, informed agents in state s_2 are strictly better off *making* the offer z , if this offer is accepted with positive probability, given that they would be selling asset 1 at a price higher than their marginal valuation (which converges to $\kappa_t(s_2)$ by Proposition 1). But then an optimal deviation on their part is to make such an offer and strictly increase their expected utility above its equilibrium level, leading to a contradiction.

The first observation can be used to show that the probability of acceptance $\chi_t(z|s_1)$ can be bounded from below by a positive number. The second observation can be used to show that the probability of acceptance $\chi_t(z|s_2)$ can be bounded from above by an arbitrarily small number. These two facts imply that we can make $\chi_t(z|s_1) > \beta$ for some $\beta > 0$ and $\chi_t(z|s_1)/\chi_t(z|s_2) > \rho$ for some $\rho > 1$. So in this case we can always find a trade such that (7) is satisfied, i.e., such that the acceptance of z is good news for s_1 . However, when we turn to the next case this will not always be true, and we will need to allow for the alternative condition (8), i.e., rejection of z is good news for s_1 .

Case 2. Consider now the second case where the long-run marginal rates of substitution of the informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are close enough but there is a large enough mass of uninformed agents whose marginal rates of substitution is far from $\kappa_t(s_1)$, conditional on s_1 .

This means that we can find a price p such that the marginal rates of substitution of a group of uninformed agents are on one side of p and the long-run marginal rates of substitution of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are on the other side. Consider the case where the MRS of a group of uninformed agents is greater than p , and $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are smaller than p (the other case is symmetric). Then the uninformed agents in this group can make a small offer to buy asset 1 at a price p and the informed will accept this offer conditional on both

signals s_1 and s_2 . If the probabilities of acceptance conditional on s_1 and s_2 were sufficiently close to each other, this would be a profitable deviation for the uninformed, since then their ex post beliefs would be close to their ex ante beliefs. In other words, the uninformed would not learn from the trade but making the offer would increase their expected utility relative to their equilibrium strategy, leading to a contradiction. It follows that the probabilities of acceptance of this trade must be sufficiently different in the two states s_1 and s_2 . This leads to either (7) or (8), completing the proof. ■

3.3.2 Convergence of marginal rates of substitution

We now characterize the properties of the long-run marginal rates of substitution of uninformed agents. The next proposition shows that the convergence result established for informed agents (Proposition 1) extends to uninformed agents.

In what follows, instead of looking at the ex ante marginal rate of substitution, given by $\pi(\delta_t)u'(x_t^1)/(1 - \pi(\delta_t))u'(x_t^2)$, we establish convergence for the ex post marginal rate of substitution $\phi(s)u'(x_t^1)/(1 - \phi(s))u'(x_t^2)$. This is the marginal rate of substitution at which an agent would be willing to trade asset 2 for asset 1 *if he could observe the signal s* . As we will see, this is the appropriate convergence result given our objective, which is to establish the ex post efficiency of the equilibrium allocation.

Proposition 3 (*Convergence of MRS for uninformed agents*) *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. Conditional on each s , the marginal rate of substitution of any agent, evaluated at the full information probabilities $\phi(s)$ and $1 - \phi(s)$, converges in probability to $\kappa_t(s)$:*

$$\lim_{t \rightarrow \infty} P \left(\left| \frac{\phi(s)u'(x_t^1)}{(1 - \phi(s))u'(x_t^2)} - \kappa_t(s) \right| > \varepsilon \mid s \right) = 0 \text{ for all } \varepsilon > 0. \quad (9)$$

Proof. We provide a sketch of the proof, leaving the details to the appendix. Suppose condition (9) fails to hold. Without loss of generality, we focus on the case where (9) fails for $s = s_1$. This means that there is a period T in which with a positive probability an uninformed agent has ex post marginal rate of substitution sufficiently far from $\kappa_T(s_1)$ and is sufficiently close to his long-run utility. Without loss of generality, suppose his marginal rate of substitution is larger than $\kappa_T(s_1)$. To reach a contradiction, we construct a profitable deviation for this agent.

Before discussing the deviation, it is useful to clarify that, at time T , the uninformed agent has all the necessary information to check whether he should deviate or not. He can observe his own allocation x_T , compute $\phi(s_1)u'(x_T^1)/(1 - \phi(s_1))u'(x_T^2)$, and verify whether this quantity is sufficiently larger than $\kappa_T(s_1)$ (which is known, since it is an equilibrium object).

The deviation then consists of two stages:

Stage 1. This is the experimentation stage, which lasts from period T to period $T + J - 1$. As stated in Lemma 3, the agent can construct a sequence of small offers $\{\hat{z}_j\}_{j=0}^{J-1}$ such that, if these offers are followed by the appropriate responses, the agent's ex post belief on signal s_1 will converge to 1. To be precise, for this to be true it must be the case that the agent does not start his deviation with a belief δ_T too close to 0. Otherwise, a sequence of J signals favorable to s_1 is not enough to bring δ_{T+J} sufficiently close to 1. Therefore, when an agent starts deviating we also require δ_T to be larger than some positive lower bound $\underline{\delta}$, appropriately defined.

Stage 2. At date $T + J$, if the agent has been able to make the whole sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and has received the appropriate responses (that is, the responses which bring the probability of s_1 close to 1), he then makes one final offer z^* . This is an offer to buy a small quantity ζ^* of asset 1 at a price p^* , which is in between the agent's own marginal rate of substitution and $\kappa_T(s_1)$. By choosing T large enough, we can ensure that there is a positive mass of informed agents close enough to their long-run marginal rate of substitution, who are willing to sell asset 1 at that price.⁹ Therefore, the offer is accepted with a positive probability. The utility gain for the uninformed agent, conditional on reaching Stage 2 and conditional on z^* being accepted, can be approximated by

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1),$$

given that, after the experimentation stage the agent's ex post belief approaches 1. Moreover, by making the final offer z^* and the experimenting offers \hat{z}_j sufficiently small, this utility gain can be approximated by

$$U(x_T + z^*, 1) - U(x_T, 1) \approx \phi(s)u'(x_T^1)\zeta^* - (1 - \phi(s))u'(x_T^2)p^*\zeta^* > 0.$$

The last expression is positive because p^* was chosen smaller than the marginal rate of substitution $\phi(s)u'(x_T^1)/(1 - \phi(s))u'(x_T^2)$. In the appendix we show that this utility gain is large enough that the deviation described is ex ante profitable, i.e., it is profitable from the point of view of period T . To do so, we must ensure that the utility losses that may happen along the deviating path (e.g., when some of the experimenting offers do not generate a response favorable to s_1 or when the agent is not selected as the proposer) are small enough. To establish this, we use again the fact that the experimenting offers are small. The argument in the appendix makes use of the convergence of utility levels in Lemma 2, to show that a utility gain relative to the current utility u_t , leads to a profitable deviation relative to the expected

⁹Notice that the uninformed agent is using $\kappa_T(s_1)$ as a reference point for the informed agents' marginal rate of substitution, while offer z^* is made in period $T + J$. Lemma 9 in the Appendix ensures that $\kappa_T(s_1)$ and $\kappa_{T+J}(s_1)$ are sufficiently close, so that at $T + J$ enough informed agents have marginal rate of substitution near $\kappa_T(s_1)$.

utility \hat{v}_t . Since we found a profitable deviation for the uninformed agents, a contradiction is obtained which completes the argument. ■

3.4 Main result

Having characterized the portfolios of informed and uninformed agents in the long run, we can finally derive our efficiency result.

Theorem 1 *All symmetric equilibrium allocations which satisfy symmetry across states and uniform market clearing converge to ex post efficient allocations in the long run, i.e.,*

$$\lim_{t \rightarrow \infty} P(|x_t^1 - x_t^2| > \varepsilon) = 0 \text{ for all } \varepsilon > 0. \quad (10)$$

The long-run marginal rates of substitution $\kappa_t(s)$ converge to the ratios of the conditional probabilities of states S_1 and S_2 :

$$\lim_{t \rightarrow \infty} \kappa_t(s) = \phi(s)/(1 - \phi(s)) \text{ for all } s \in \{s_1, s_2\}. \quad (11)$$

Proof. We provide a sketch of the proof and leave the formal details to the appendix. First, suppose that $\kappa_t(s) > (1 + \varepsilon) \phi(s)/(1 - \phi(s))$ for some $\varepsilon > 0$, for infinitely many periods. Then Proposition 3 can be used to show that the agents' holdings of asset 1 will be larger than their holdings of asset 2. This, however, violates market clearing. In a similar way, we rule out the case in which $\kappa_t(s) < (1 - \varepsilon) \phi(s)/(1 - \phi(s))$ for some $\varepsilon > 0$, for infinitely many periods. This proves (11). Then, using this result and Proposition 3, we can show that $u'(x_t^1)/u'(x_t^2)$ converges in probability to 1, which implies (10). ■

This theorem establishes that, equilibrium allocations converge to ex post Pareto efficient allocations. This implies that the long-run equilibrium allocation coincides with a rational expectation equilibrium defined in Section 2.4 for *some* initial allocations. It does not show that starting with the *same* initial allocations our decentralized trading game converges to the *same* long-run outcomes as the rational expectation equilibrium of the centralized Walrasian market. In fact, this is typically not the case. Remember that in the centralized environment informed and uninformed agents with the same initial endowment reach the same equilibrium allocation. This is not the case in the decentralized environment, where informed agents can reach, on average, a higher expected utility. In Golosov, Lorenzoni and Tsyvinski (2011), we take a computational approach to simulate equilibria of our model and we show that indeed informed agents in equilibrium achieve higher utility.

It is useful to notice that our results do not necessarily imply that uninformed agents fully learn the value of the signal s in the long run. The proof of Proposition 3 shows that in the long run uninformed agents are able to learn the value of s with arbitrary precision at an arbitrarily low cost. However, as uninformed agents' asset holdings converge to a perfectly diversified

portfolio, with equal holdings of the two assets, incentives to acquire this information also go to zero. The reason is that uninformed agents know that asset holdings of all other agents are also converging to a perfectly diversified portfolio, and no further trades are possible. Therefore, even though the allocation converges to an efficient allocation, our argument does not require uninformed agents to perfectly learn the signal s . What it implies is that either uninformed agents perfectly learn the signal or the value of learning converges to zero.

3.5 Further remarks

We now provide further discussion on how our result compares with existing results and on the role of our assumptions.

First, it is useful to compare our result with Wolinsky (1990). In our model, in the long run all agents are only willing to trade at a single price (conditional on the state s) which is the same as the Walrasian REE price $\phi(s) / (1 - \phi(s))$. Wolinsky (1990) also analyzes a dynamic trading game with asymmetric information and shows that in steady state different trades can occur at different prices, so a fraction of trades can occur at a price different from the Walrasian REE price. That paper studies a game where a fraction of traders enter and exit at each point of time, focuses on steady-state equilibria, and takes limits as discounting goes to zero. We consider a game with a fixed set of participants and a fixed probability of ending the game γ and study long-run outcomes. The key difference is that the model in Wolinsky (1990) features an indivisible good which can only be traded once. Our environment features perfectly divisible goods (assets) which are traded repeatedly. This makes the process of experimentation by market participants very different in the two environments. In Wolinsky (1990) agents only learn if their offers are rejected. Once the offer is accepted they trade and exit the market. In our environment, agents keep learning and trading along the equilibrium play. In particular, they can learn by making small trades (as shown in Lemma 3) and then use the information acquired to make Pareto improving trades with informed agents (as shown in Proposition 3).

Next, let us discuss how our results can be generalized. We derived Theorem 1 in a simple environment with two states and two signals and we focused on equilibria that satisfy a symmetry property (defined in Definition 2), so it is useful to discuss how the logic of our arguments can be extended to richer environments. The experimentation and deviation arguments in Proposition 3, which shows that the marginal rates of substitution of all agents converge to the same value, can be easily extended to the case of finitely many states and signals, as long as markets are complete and there is an Arrow security for each state S . Take a signal s that carries information about two Arrow securities which pay in states S and S' . Partition the signal space in two subsets: a singleton that includes only s and the subset of all the other signals. Then the arguments in our binary environment can be adapted so as to prove that the marginal rates of substitution between assets S and S' must converge to the

same value for all agents, informed and uninformed, conditional on signal s .

It is more difficult to generalize some other arguments used to arrive to Theorem 1. In particular, in Lemma 7 in the Appendix (which is needed to prove Proposition 2) we use the following type of argument by contradiction: if the marginal rate of substitution of all agents converge to the same value independent of s , then market clearing is violated. The general idea behind this result, which applies with any number of states and signals, is the following. Take two signals s and s' , which carry information about two assets S and S' . If signal s is relatively more favorable to asset S , and the informed agents' marginal rate of substitution between assets S and S' is the same conditional on s and s' , then informed agents must hold more of asset S relative to asset S' , conditional on s . If we can show that, at the same time, uninformed agents do not hold on average more of asset S' relative to asset S in state s , then it means that the aggregate holdings of asset S relative to asset S' are larger in state s . This violates market clearing since the total supply of assets is independent of the signal s . Although this argument is intuitive, the hard step is the characterization of the uninformed agents' relative holdings of the two assets, given that uninformed agents hold, in general, a range of beliefs about the signal. The proof of Lemma 7 deals with this step in our environment. However, our approach in that proof makes use of the symmetry of our environment (Assumption 1) and of the symmetry of our equilibrium notion (Definition 2). How to extend that argument to more general non-symmetric environments is an interesting open issue.

Finally, an assumption we made is that agents can only make take-it-or-leave-it offers, following Wolinsky (1990) and Gale (2000). One could enrich the trading protocol, by assuming, for example, that the proposer can offer a menu of possible trades. By enriching the trading protocol more information can be transmitted in each bilateral meeting. Duffie, Malamud and Manso (2010) consider a model where agents engage in a dual auction, which, in equilibrium, fully reveals the information of both agents. To do so they assume that when two agents meet they perfectly observe whether there is a mutually beneficial trade from a hedging point of view, so that the double auction only serves to reveal information about the asset quality. The analogous assumption in our environment would be to assume that agents perfectly observe each other's holdings of the assets when they meet. This eliminates a crucial source of uncertainty in our environment, namely the uncertainty whether your trading partner wants to trade for hedging/insurance reasons or for speculative reasons. The focus of our paper is to explore how, in spite of this uncertainty, repeated rounds of trading allow agents to reach efficiency. Therefore, it is important for our objective to choose a trading protocol in which, at each meeting, agents can only acquire a limited amount of information. In particular, in our trading protocol agents only observe whether a trade took place and, if it did, the quantities exchanged. The surprising result is that despite the limited amount of information transmitted at each trading round, agents learn all the useful information in the long run.

4 Conclusion

We provide a theory of asset pricing in an environment characterized by two frictions: private information and decentralized trade. These frictions often go hand in hand—as it is reasonable that in decentralized markets, such as, for example, over the counter markets, agents also receive different pieces of information on the value of the assets traded. In this paper we ask whether the information held by informed agents is eventually absorbed by the market through repeated trading and show how, through small trades, uninformed agents are able to acquire this information whenever ex post gains from trade are available.

Notice that the arguments on learning and experimentation in this paper are essentially arguments by contradiction, aimed at establishing ex post efficiency. In a companion paper (Goloso, Lorenzoni, and Tsyvinski 2011), we explore computationally how learning occurs on the equilibrium path. In particular, we show that in equilibrium uninformed agents learn from trading with “rich” informed agents, that is, with informed agents that hold a portfolio biased towards assets that are more valued after the information is revealed. This type of agents have the incentive to reveal information as they are trying to sell the more valuable asset.

Throughout the paper we have kept fixed the central trading friction in the model, the frequency of trading captured by the parameter γ . This parameter determines the average number of trades before the game ends. A question that remains for future research is what happens in the limit when the trading friction vanishes—i.e., as γ goes to 1—and, in particular, if convergence to efficiency is fast enough that the probability of reaching an efficient allocation goes to 1 and whether the value of information goes to zero at time 0.

5 Appendix

5.1 Preliminary results

Here, we introduce two technical results that will be useful throughout the appendix.

Lemma 4 is an elementary probability result (stated without proof) that will be useful whenever we need to establish joint convergence in probability for two or more events.

Lemma 4 *Take two sets $A, B \subset \Omega$ such that $P(A|s) \geq 1 - \varepsilon$ and $P(B|s) > 1 - \eta$ for some positive scalars ε and η . Then, $P(A \cap B|s) > 1 - \varepsilon - \eta$.*

Lemma 5 shows that the portfolios x_t converge to a compact set X in the interior of R_+^2 with probability arbitrarily close to one. This type of set will be used to ensure that several optimization problems used in the proofs are well defined.

Lemma 5 *For any $\varepsilon > 0$ and any state s , there are a compact set $X \subset R_{++}^2$ and a time T such that $P(x_t \in X | s) \geq 1 - \varepsilon$ for all $t \geq T$.*

Proof. To prove the lemma, we will find two scalars \bar{x} and \underline{u} such that the set

$$X = \{x : x \in (0, \bar{x}]^2, U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1]\}$$

satisfies the desired properties. The proof combines two ideas: use market clearing to put an upper bound on the holdings of the two assets, that is, to show that with probability close to 1 agents have portfolios in $(0, \bar{x}]^2$; use optimality to bound their holdings away from zero, by imposing the inequality $U(x, \delta) \geq \underline{u}$.

First, let us prove that X is a compact subset of R_{++}^2 . The following two equalities follow from the fact that $U(x, \delta)$ is continuous, non-decreasing in δ if $x^1 \geq x^2$, and non-increasing if $x^1 \leq x^2$:

$$\begin{aligned} \{x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1], x^1 \geq x^2\} &= \{x : U(x, 1) \geq \underline{u}, x^1 \geq x^2\}, \\ \{x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1], x^1 \leq x^2\} &= \{x : U(x, 0) \geq \underline{u}, x^1 \leq x^2\}. \end{aligned}$$

The sets on the right-hand sides of these equalities are closed sets. Then X can be written as the union of two closed sets, intersected with a bounded set:

$$X = (\{x : U(x, 1) \geq \underline{u}, x^1 \geq x^2\} \cup \{x : U(x, 0) \geq \underline{u}, x^1 \leq x^2\}) \cap (0, \bar{x}]^2,$$

and thus is compact. Notice that $x \notin X$ if $x^j = 0$ for some j because of Assumption 2 and $\underline{u} > -\infty$. Therefore, X is a compact subset of R_{++}^2 .

Next, let us define \bar{x} and \underline{u} and the time period T . Given any $\varepsilon > 0$, set $\bar{x} = 4/\varepsilon$. Goods market clearing implies that

$$P(x_t^j > \bar{x} | s) \leq \varepsilon/4 \text{ for all } t, \text{ for } j = 1, 2. \quad (12)$$

To prove this, notice that

$$1 = \int x_t^j(\omega) dP(\omega | s) \geq \int_{x_t^j(\omega) > 4/\varepsilon} x_t^j(\omega) dP(\omega | s) \geq (4/\varepsilon) P(x_t^j > 4/\varepsilon | s),$$

which gives the desired inequality. Let \bar{u} be an upper bound for the agents' utility function $u(\cdot)$ (from Assumption 2). Choose a scalar $u < \bar{u}$ such that

$$\frac{\bar{u} - U(x_0, \delta_0)}{\bar{u} - u} \leq \frac{\varepsilon}{8},$$

for all initial endowments x_0 and initial beliefs δ_0 . Such a u exists because $U(x_0, \delta_0) > -\infty$, as initial endowments are strictly positive by Assumption 3, and there is a finite number of types. Then notice that $U(x_0, \delta_0) \leq E[v_t | h^0]$ for all initial histories h^0 , because an agent always has the option to refuse any trade. Moreover

$$E[v_t | h^0] \leq P(v_t < u | h^0) u + P(v_t \geq u | h^0) \bar{u}.$$

Combining these inequalities and rearranging gives

$$P(v_t < u | h^0) \leq \frac{\bar{u} - U(x_0, \delta_0)}{\bar{u} - u} \leq \frac{\varepsilon}{8}.$$

Taking unconditional expectations shows that $P(v_t < u) \leq \varepsilon/8$. Since $P(s) = 1/2$ it follows that

$$P(v_t < u | s) \leq \varepsilon/4 \text{ for all } t, \text{ for all } s. \quad (13)$$

Choose T so that

$$P(|u_t - v_t| > u/2 | s) \leq \varepsilon/4 \text{ for all } t \geq T. \quad (14)$$

This can be done by Lemma 2, given that almost sure convergence implies convergence in probability. We can then set $\underline{u} = u/2$.

Finally, we check that $P(x_t \in X | s) \geq 1 - \varepsilon$ for all $t \geq T$, using the following chain of

inequalities:

$$\begin{aligned}
P(x_t \in X \mid s) &\geq P(x_t \in (0, \bar{x}]^2, U(x_t, \delta_t) \geq \underline{u} \mid s) \geq \\
&P(x_t \in (0, \bar{x}]^2, v_t \geq u, |u_t - v_t| \leq u/2 \mid s) \geq \\
&1 - \sum_j P(x_t^j > \bar{x} \mid s) - P(v_t < u \mid s) - P(|u_t - v_t| > u/2 \mid s) \geq 1 - \varepsilon.
\end{aligned}$$

The first inequality follows because $U(x_t(\omega), \delta_t(\omega)) \geq \underline{u}$ implies $U(x_t(\omega), \delta) \geq \underline{u}$ for some $\delta \in [0, 1]$. The second follows because $v_t(\omega) \geq u$ and $|u_t(\omega) - v_t(\omega)| \leq u/2$ imply $u_t(\omega) = U(x_t(\omega), \delta_t(\omega)) \geq u/2 = \underline{u}$. The third follows from repeatedly applying Lemma 4. The fourth combines (12), (13), and (14). ■

5.2 Proof of Proposition 1

We start by proving a lemma which shows that given any two agents with portfolios in some compact set X , whose marginal rates of substitution differ by at least ε , there is a trade z that achieves a gain in current utility of at least Δ , for some positive Δ .

The lemma is stated in a more general form than what is required to prove Proposition 1. The generality is threefold. First, it applies not just to informed agents but also to agents with any (possibly different) beliefs. Second, we show that the utility gain Δ can be achieved with small trades, i.e., trades such that $\|z\| < \theta$ for some $\theta > 0$. (Throughout the paper, $\|\cdot\|$ is the Euclidean norm). Finally, since an uninformed proposer making offer z can change his beliefs depending on whether his offer is accepted or rejected, we bound the potential utility losses of the proposer under all possible beliefs. These extensions will be useful for later results when we analyze the behavior of uninformed agents.

It will be useful for the rest of the appendix to define the function

$$\mathcal{M}(x, \delta) \equiv \frac{\pi(\delta) u'(x^1)}{(1 - \pi(\delta)) u'(x^2)},$$

which gives the *ex ante* marginal rate of substitution between the two assets for an agent with the portfolio x and belief δ .

Lemma 6 *Let X be a compact subset of R_{++}^2 . For any $\varepsilon > 0$ and $\theta > 0$ there is a minimal utility gain $\Delta > 0$ and an amount of asset 1 traded, $\zeta > 0$, with the following property. Take any two agents with portfolios $x_A, x_B \in X$ and beliefs $\delta_A, \delta_B \in [0, 1]$ with marginal rates of substitution that differ by more than ε , $\mathcal{M}(x_B, \delta_B) - \mathcal{M}(x_A, \delta_A) > \varepsilon$. Choose any price sufficiently close to the middle of the interval between the two marginal rates of substitution:*

$$p \in [\mathcal{M}(x_A, \delta_A) + \varepsilon/2, \mathcal{M}(x_B, \delta_B) - \varepsilon/2].$$

Then the trade $z = (\zeta, -p\zeta)$ is sufficiently small, $\|z\| < \theta$, and the gain in current utility associated with the trade is larger than or equal to Δ :

$$U(x_A - z, \delta_A) - U(x_A, \delta_A) \geq \Delta, \quad (15)$$

$$U(x_B + z, \delta_B) - U(x_B, \delta_B) \geq \Delta. \quad (16)$$

Moreover, there is a constant $\lambda > 0$, which depends on the set X and on the difference between the marginal rates of substitution ε , but not on the size of the trade θ , such that the potential loss in current utility associated with the trade z is bounded below by $-\lambda\Delta$ for all beliefs δ :

$$U(x_A - z, \delta) - U(x_A, \delta) \geq -\lambda\Delta \text{ for all } \delta \in [0, 1]. \quad (17)$$

Proof. The idea of the proof is as follows. We construct a Taylor expansion to compute the utility gains for any trade. Then we define the traded amount ζ and the utility gain Δ satisfying (15) and (16).

Choose any two portfolios $x_A, x_B \in X$ and any two beliefs $\delta_A, \delta_B \in [0, 1]$ such that $\mathcal{M}(x_B, \delta_B) - \mathcal{M}(x_A, \delta_A) > \varepsilon$. Pick a price p sufficiently close to the middle of the interval between the marginal rates of substitution:

$$p \in [\mathcal{M}(x_A, \delta_A) + \varepsilon/2, \mathcal{M}(x_B, \delta_B) - \varepsilon/2].$$

This price is chosen so that both agents will make positive gains. Consider agent A and a traded amount $\tilde{\zeta} \leq \bar{\zeta}$ (for some $\bar{\zeta}$ which we will properly choose below). The current utility gain associated with the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ can be written as a Taylor expansion:

$$\begin{aligned} & U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \\ &= -\pi(\delta_A)u'(x_A^1)\tilde{\zeta} + (1 - \pi(\delta_A))u'(x_A^2)p\tilde{\zeta} + \frac{1}{2} (\pi(\delta_A)u''(y^1) + (1 - \pi(\delta_A))u''(y^2)p^2) \tilde{\zeta}^2 \\ &\geq (1 - \pi(\delta_A))u'(x_A^2) (\varepsilon/2) \tilde{\zeta} + \frac{1}{2} [\pi(\delta_A)u''(y^1) + (1 - \pi(\delta_A))u''(y^2)p^2] \tilde{\zeta}^2, \end{aligned} \quad (18)$$

for some $(y^1, y^2) \in [x_A^1, x_A^1 - \bar{\zeta}] \times [x_A^2 + p\bar{\zeta}, x_A^2]$. The inequality above follows because $p \geq \mathcal{M}(x_A, \delta_A) + \varepsilon/2$. An analogous expansion can be done for agent B .

Now we want to bound the last line in (18). To do so we first define the minimal and the maximal prices for agents with any belief in $[0, 1]$ and any portfolio in X :

$$\begin{aligned} \underline{p} &= \min_{x \in X, \delta \in [0, 1]} \{\mathcal{M}(x, \delta) + \varepsilon/2\}, \\ \bar{p} &= \max_{x \in X, \delta \in [0, 1]} \{\mathcal{M}(x, \delta) - \varepsilon/2\}. \end{aligned}$$

These prices are well-defined as X is a compact subset of R_{++}^2 and $u(\cdot)$ has continuous first

derivative on R_{++}^2 . Then, choose $\bar{\zeta} > 0$ such that for all $\tilde{\zeta} \leq \bar{\zeta}$ and all $p \in [\underline{p}, \bar{p}]$, the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ satisfies $\|\tilde{z}\| < \theta$ and $x + \tilde{z}$ and $x - \tilde{z}$ are in R_+^2 for all $x \in X$. This means that the trade is small enough. Next, we separately bound from below the two terms in the last line of the Taylor expansion (18). Let

$$\begin{aligned} D'_A &= \min_{x \in X, \delta \in [0, 1]} (1 - \pi(\delta))u'(x^2)\varepsilon/2, \\ D''_A &= \min_{\substack{x \in X, \delta \in [0, 1], \tilde{p} \in [\underline{p}, \bar{p}], \\ y \in [x^1, x^1 + \tilde{\zeta}] \times [x^2 - \tilde{p}\tilde{\zeta}, x^2]}} \frac{1}{2} [\pi(\delta)u''(y^1) + (1 - \pi(\delta))u''(y^2)\tilde{p}^2]. \end{aligned}$$

Note that D'_A is positive, D''_A is negative but $D''_A\tilde{\zeta}^2$ is of second order. Then, there exist some $\zeta_A \in (0, \bar{\zeta})$ such that, for all $\tilde{\zeta} \leq \zeta_A$,

$$D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2 > 0$$

and, by construction,

$$U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \geq D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2.$$

Analogously, we can find D'_B, D''_B , and ζ_B such that for all $\tilde{\zeta} \leq \zeta_B$ the utility gain for agent B is bounded from below:

$$U(x_B + \tilde{z}, \delta_B) - U(x_B, \delta_B) \geq D'_B\tilde{\zeta} + D''_B\tilde{\zeta}^2 > 0.$$

We are finally ready to define ζ and Δ . Let $\zeta = \min\{\zeta_A, \zeta_B\}$ and

$$\Delta = \min\{D'_A\zeta + D''_A\zeta^2, D'_B\zeta + D''_B\zeta^2\}.$$

By construction Δ and ζ satisfy the inequalities (15) and (16).

To prove the last part of the lemma, let

$$\lambda = \frac{1}{2} \frac{\pi(1) \min_{x \in X} \{u'(x^1)\}}{\min\{D'_A, D'_B\}},$$

which, as stated in the lemma, only depends on X and ε . Using a second-order expansion similar to the one above, the utility gain associated to $z = (\zeta, -p\zeta)$ for an agent with portfolio x_A and any belief $\delta \in [0, 1]$, can be bounded below:

$$U(x_A - \tilde{z}, \delta) - U(x_A, \delta) \geq -\pi(1) \min_{x \in X} \{u'(x^1)\} \zeta + D''_A\zeta^2.$$

Therefore, to ensure that (17) is satisfied, we need to slightly modify the construction above,

by choosing ζ so that the following holds

$$\frac{-\pi(1) \min_{x \in X} \{u'(x^1)\} \zeta + D_A'' \zeta^2}{\Delta} > \lambda.$$

The definitions of Δ and λ and a continuity argument show that this inequality holds for some positive $\zeta \leq \min\{\zeta_A, \zeta_B\}$, completing the proof. ■

Proof of Proposition 1. Proceeding by contradiction suppose (6) does not hold. Without loss of generality, let us focus on state s_1 . If (6) is violated in s_1 then there exist an $\varepsilon > 0$ and an $\eta \in (0, 1)$ such that the following holds for infinitely many periods t :

$$P(|\mathcal{M}(x_t, \delta_t) - \kappa| > \varepsilon, \delta_t = 1 \mid s_1) > \eta P(\delta_t = 1 \mid s_1) \text{ for all } \kappa. \quad (19)$$

Equation (19) means that the distribution of the informed agents' marginal rates of substitution is such that there is a sufficiently large mass (more than η) which is sufficiently far (more than ε) from any possible value κ .

We want to show that (19) implies that there is a profitable deviation for informed agents. The deviation takes the following form. The informed agent starts deviating at some date T (to be defined) if three conditions are satisfied:

- (a) his marginal rate of substitution is below some level κ^* (to be defined): $\mathcal{M}(x_T, \delta_T) < \kappa^*$;
- (b) his utility is close enough to its long-run level: $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ (for some $\Delta > 0$ to be defined);
- (c) his portfolio x_T is in some compact set X (to be defined).

We will show that when (a)-(c) hold the agent can make an offer z^* which is accepted with probability $\chi_T(z^*|s_1) \geq \alpha\eta/4$ and gives him a utility gain of at least Δ . The expected payoff of this strategy at time T is

$$u_T + \chi_T(z^*|s_1) (U(x_T - z^*, \delta_T) - u_T) > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T.$$

Since \hat{v}_T is, by definition, the expected payoff of a proposer who follows an optimal strategy, this leads to a contradiction.

To complete the proof we need to define the scalars κ^* and Δ , the set X , the deviating period T and the offer z^* . In the process, we will check that conditions (a)-(c) are satisfied with positive probability, that offer z^* gives a utility gain of at least Δ to the agents who satisfy (a)-(c), and that offer z^* is accepted with probability $\chi_T(z^*|s_1) \geq \alpha\eta/2$.

Define X to be a compact subset of R_{++}^2 such that the portfolios of sufficiently many agents are eventually in this set, i.e., for some T' we have $P(x_t \in X \mid s_1) \geq 1 - \alpha\eta/4$ for all $t \geq T'$. Such a set exists by Lemma 5.

Define $\Delta > 0$ to be the minimal gain from trade for two agents with marginal rates of substitution that differ by at least ε with portfolios in X . Such a Δ exists by Lemma 6.

We can now find a time T large enough that condition (19) also holds if we restrict attention to agents close to their long-run utility, with portfolios in X , i.e., agents who satisfy (b)-(c). Applying Lemmas 2 and 4, choose a $T'' \geq T'$ such that

$$P(u_t \geq \hat{v}_t - \alpha\eta\Delta/4, x_t \in X \mid s_1) > 1 - \alpha\eta/2 \text{ for all } t \geq T''.$$

Then, using (19) and the fact that there is at least a mass α of informed agents in each period t , we can find a $T \geq T''$ such that there are enough agents (at least $\alpha\eta$) whose marginal rates of substitution are far (at least ε) from any κ :

$$P(|\mathcal{M}(x_T, \delta_T) - \kappa| > \varepsilon, \delta_T = 1 \mid s_1) > \eta P(\delta_t = 1 \mid s_1) \geq \alpha\eta \text{ for all } \kappa.$$

Using Lemma 4, it follows that at time T there are at least $\alpha\eta/2$ informed agents who satisfy $|\mathcal{M}(x_T, \delta_T) - \kappa| > \varepsilon$ for any κ and conditions (b)-(c):

$$P(|\mathcal{M}(x_T, \delta_T) - \kappa| > \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) > \alpha\eta/2 \text{ for all } \kappa.$$

It will be useful to rewrite this equation as

$$\begin{aligned} P(|\mathcal{M}(x_T, \delta_T) - \kappa| \leq \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) < \\ < P(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) - \alpha\eta/2 \text{ for all } \kappa, \end{aligned} \quad (20)$$

To define κ^* , the idea is to use condition (19)—which states that there are not too many informed agents around any κ —to find a κ^* such that enough agents have marginal rate of substitution below κ^* and enough agents have marginal rate of substitution above $\kappa^* + \varepsilon$. The first group of agents will make the offer z^* , the second group will accept it. Define

$$\kappa^* = \sup \{ \kappa : P(\mathcal{M}(x_T, \delta_T) > \kappa + (3/2)\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) \geq \alpha\eta/4 \}.$$

The definition of κ^* implies that there are less than $\alpha\eta/4$ informed agents with marginal rate of substitution above $\kappa^* + 2\varepsilon$ who satisfy (b)-(c),

$$P(\mathcal{M}(x_T, \delta_T) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) < \alpha\eta/4, \quad (21)$$

given that $\kappa^* + \varepsilon/2 > \kappa^*$. Consider the following chain of equalities and inequalities:

$$\begin{aligned}
& P(\mathcal{M}(x_T, \delta_T) \geq \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) = \\
& P(\kappa^* \leq \mathcal{M}(x_T, \delta_T) \leq \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) \\
& P(\mathcal{M}(x_T, \delta_T) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) < \\
& < P(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) - \alpha\eta/4,
\end{aligned}$$

where the equalities are immediate and the inequality follows from (20) (with $\kappa = \kappa^* + \varepsilon$) and (21). This implies

$$P(\mathcal{M}(x_T, \delta_T) < \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) > 0, \quad (22)$$

which shows that conditions (a)-(c) are met with positive probability.

To define the deviating offer z^* , notice that, by the definition of Δ , there exists an offer $z^* = (\zeta^*, -p^*\zeta^*)$, with price $p^* = \kappa^* + \varepsilon/2$, such that

$$U(x - z^*, \delta) \geq U(x, \delta) + \Delta \text{ if } \mathcal{M}(x, \delta) < \kappa^* \text{ and } x \in X, \quad (23)$$

$$U(x + z^*, \delta) \geq U(x, \delta) + \Delta \text{ if } \mathcal{M}(x, \delta) > \kappa^* + \varepsilon \text{ and } x \in X. \quad (24)$$

Condition (23) shows that an informed proposer who satisfies (a)-(c) gains at least Δ if offer z^* is accepted.

Finally, the definition of κ^* implies that there must be at least $\alpha\eta/4$ agents with marginal rate of substitution above $\kappa^* + \varepsilon$,

$$P(\mathcal{M}(x_T, \delta_T) > \kappa^* + \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/2, x_T \in X, \delta_T = 1 \mid s_1) \geq \alpha\eta/4, \quad (25)$$

given that $\kappa^* - \varepsilon/2 < \kappa^*$. Recall that \hat{v}_t represents, by definition, the maximal expected utility the responder can get from rejecting all offers and behaving optimally in the future. A responder who receives z^* has the option to accept it and stop trading from then on, which yields expected utility $U(x_T + z^*, \delta_T)$. For all informed agents who satisfy $\mathcal{M}(x_T, \delta_T) \geq \kappa^* + \varepsilon$, $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ and $x_T \in X$, we have the chain of inequalities

$$U(x_T + z^*, \delta_T) \geq u_T + \Delta > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T,$$

where the first inequality follows from (24). This shows that rejecting z^* at time T is a strictly dominated strategy for these informed agents. Since there are at least $\alpha\eta/4$ of them, by (25), the probability that z^* is accepted must then satisfy $\chi_T(z \mid s_1) \geq \alpha\eta/4$. ■

5.3 Proof of Proposition 2

The proof proceeds by contradiction, supposing that for all $\varepsilon > 0$ there are infinitely many periods t in which: (a) the long-run marginal rates of substitution of informed agents in states s_1 and s_2 are similar, $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$, and (b) almost all informed and uninformed agents have marginal rates of substitution close to $\kappa_t(s)$, i.e.,

$$P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < 2\varepsilon \mid s) > 1 - \varepsilon$$

in some state s .

First of all, let us manipulate this expression, to derive a version that is easier to contradict. Specifically, let us show that (a) and (b) imply that there are infinitely many periods in which almost all agents have marginal rates of substitution near 1. By symmetry, the long-run marginal rates of substitutions of informed agents in states s_1 and s_2 are one the inverse of the other:

$$\kappa_t(s_1) = 1/\kappa_t(s_2).$$

Then, some algebra shows that $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$ implies $|\kappa_t(s_1) - 1| < \varepsilon$. Moreover, by the triangle inequality, $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < 2\varepsilon$ and $|\kappa_t(s_1) - 1| < \varepsilon$ imply $|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon$. Therefore, (a) and (b) imply that for all $\varepsilon > 0$ there are infinitely many periods t in which almost all agents have marginal rates of substitution close to 1, i.e., $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s) > 1 - \varepsilon$ in some state s . Without loss of generality, we focus on the case where this condition holds for infinitely many periods in state s_1 ,

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1) > 1 - \varepsilon. \quad (26)$$

The idea of the proof is to show that when almost all agents have marginal rates of substitution close to 1 in state s_1 , agents will hold on average more of asset 1 than of asset 2. This contradicts market clearing, which requires the average holdings of the two assets to be equal:

$$\int (x_t^1(\omega) - x_t^2(\omega)) dP(\omega \mid s_1) = 0. \quad (27)$$

Informed agents have belief $\delta_t = 1$ in state 1, so if their marginal rates of substitution are close enough to 1 they will clearly have larger holdings of asset 1 than of asset 2. The main difficulty of the proof is to make sure that there aren't too many uninformed agents with larger holdings of asset 2 than of asset 1. Since also uninformed agents have marginal rates of substitution near 1, they can hold more asset 2 than asset 1 only if their beliefs are biased towards state s_2 , i.e., if they have $\delta_t < 1/2$. However, we will argue that if the true state is s_1 there are always more uninformed agents biased towards s_1 than uninformed agents biased towards s_2 . Therefore, the agents in the economy will hold, on average, more of asset 2 than

asset 1. The main formal step for this argument is given by the following lemma. We will complete the proof of Proposition 2 after stating and proving the lemma.

The lemma shows that we can start from the equilibrium distribution of portfolios and beliefs implied by $P(\omega|s_1)$ and construct an auxiliary distribution of portfolios and beliefs, G_t , with the following three properties: (i) it only includes agents with beliefs greater than $1/2$; (ii) it includes the same mass of informed agents as the original distribution; (iii) the average holdings of assets 1 and 2 are equalized. In particular, G_t is constructed by eliminating symmetric masses of agents with $\delta_t > 1/2$ and $\delta_t < 1/2$ and with symmetric holdings of the two assets: if the agents in the first group hold (x^1, x^2) and have belief δ , we find a group of agents with holdings (x^2, x^1) and belief $1 - \delta$, and reduce the masses of both groups by an equal amount. Since $(x^1 - x^2) = -(x^2 - x^1)$, this procedure ensures that the average holdings of the two assets are still equalized under the new measure. Moreover, Bayesian reasoning implies that the first group is always larger than the second, so we can construct G_t leaving only a positive mass of agents in the first group. By this process, we end up with a distribution where every agent has $\delta_t(\omega) > 1/2$ and the average portfolios of goods 1 and 2 are equal.

Notice that the lemma is stated using a modified version of condition (27). That is, instead of showing the equality of average asset holdings of assets 1 and 2, we truncate the portfolio distribution, imposing $x_t^2 \leq m$ for some arbitrarily large m , and show that the average holdings of asset 1 can only exceed the average holdings of asset 2 by an arbitrarily small $\varepsilon > 0$. Here is where we exploit the assumption of uniform market clearing. This property will be useful when completing the proof of Proposition 2.

Lemma 7 *For all $\varepsilon > 0$, there are a scalar M and a sequence of (discrete) measures G_t on the space of portfolios and beliefs $R_+^2 \times [0, 1]$ such that the following properties are satisfied: (i) the measure is zero for all beliefs smaller than or equal to $1/2$:*

$$G_t(x, \delta) = 0 \text{ if } \delta \leq 1/2;$$

(ii) G_t corresponds to the distribution generated by the measure P conditional on s_1 for informed agents:

$$G_t(x, 1) = P(\omega : x_t(\omega) = x, \delta_t(\omega) = 1 \mid s_1) \text{ for all } x \text{ and } t;$$

(iii) the average holdings of asset 1 exceed the average holdings of asset 2, truncated at any $m \geq M$, by less than ε :

$$\int_{x^2 \leq m} (x^1 - x^2) dG_t(x, \delta) \leq \varepsilon \text{ for all } m \geq M \text{ and all } t. \quad (28)$$

Proof. For all $x \in R^2$ and all $\delta \in [0, 1]$ define the measure G_t as follows

$$G_t(x, \delta) \equiv \begin{cases} P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_1) - P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_2) & \text{if } \delta > 1/2 \\ 0 & \text{if } \delta \leq 1/2 \end{cases}.$$

We first prove that G_t is a well defined measure and next we prove properties (i)-(iii).

Since P generates a discrete distribution over x and δ for each t , to prove that G_t is a well defined measure we only need to check that

$$P(x_t = x, \delta_t = \delta \mid s_2) \leq P(x_t = x, \delta_t = \delta \mid s_1)$$

so that G_t is non-negative. Take any $\delta > 1/2$. Bayesian rationality implies that a consumer who knows his belief is δ must assign probability δ to s_1 :

$$\delta = P(s_1 \mid x_t = x, \delta_t = \delta).$$

Moreover, Bayes' rule implies that

$$\frac{P(s_2 \mid x_t = x, \delta_t = \delta)}{P(s_1 \mid x_t = x, \delta_t = \delta)} = \frac{P(x_t = x, \delta_t = \delta \mid s_2) P(s_2)}{P(x_t = x, \delta_t = \delta \mid s_1) P(s_1)}.$$

Rearranging and using $P(s_1) = P(s_2)$ and $\delta > 1/2$, yields

$$\frac{P(x_t = x, \delta_t = \delta \mid s_2)}{P(x_t = x, \delta_t = \delta \mid s_1)} = \frac{1 - \delta}{\delta} < 1,$$

which gives the desired inequality.

Property (i) is immediately satisfied by construction. Property (ii) follows because $P(x_t = x, \delta_t = 1 \mid s_2) = 0$ for all x , given that $\delta_t = 1$ requires that we are at a history which arises with zero probability conditional on s_2 . The proof of property (iii) is longer and involves the manipulation of market clearing relations and the use of our symmetry assumption. Using the assumption of uniform market clearing, find an M such that

$$\int_{x_t^2(\omega) \leq m} x_t^2(\omega) dP(\omega \mid s_1) \geq 1 - \varepsilon \text{ for all } m \geq M. \quad (29)$$

Notice that

$$\int_{x_t^2(\omega) \leq m} x_t^1(\omega) dP(\omega \mid s_1) \leq \int x_t^1(\omega) dP(\omega \mid s_1) = 1.$$

Which combined with (29) implies that

$$\int_{x_t^2(\omega) \leq m} (x_t^1(\omega) - x_t^2(\omega)) dP(\omega \mid s_1) \leq \varepsilon \text{ for all } m \geq M.$$

Decomposing the integral on the left-hand side gives

$$\begin{aligned} & \int_{\substack{x_t^1 > x_t^2 \\ x_t \in [0, m]^2}} (x_t^1 - x_t^2) dP(\omega|s_1) + \int_{\substack{x_t^2 = x_t^1 \\ x_t \in [0, m]^2}} (x_t^1 - x_t^2) dP(\omega|s_1) + \int_{\substack{x_t^1 < x_t^2 \\ x_t \in [0, m]^2}} (x_t^1 - x_t^2) dP(\omega|s_1) \\ & + \int_{\substack{x_t^1 > m \\ x_t^2 \leq m}} (x_t^1 - x_t^2) dP(\omega|s_1) \leq \varepsilon. \end{aligned} \quad (30)$$

Let us first focus on the first three terms on the left-hand side of this expression. The second term is zero. Using symmetry to replace the third term, the sum of the first three terms can then be rewritten as

$$\int_{\substack{x_t^1 > x_t^2 \\ x_t \in [0, m]^2}} (x_t^1 - x_t^2) dP(\omega|s_1) + \int_{\substack{x_t^1 > x_t^2 \\ x_t \in [0, m]^2}} (x_t^2 - x_t^1) dP(\omega|s_2). \quad (31)$$

These two integrals are equal to the sums of a finite number of non-zero terms, one for each value of x and δ with positive mass. Summing the corresponding terms in each integral, we have three cases: (a) terms with $\delta_t = \delta > 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) > P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which can be written as

$$\begin{aligned} (x^1 - x^2) P(x_t = x, \delta_t = \delta|s_1) - (x^1 - x^2) P(x_t = x, \delta_t = \delta|s_2) \\ = (x^1 - x^2) G_t(x); \end{aligned}$$

(b) terms with $\delta_t = \delta = 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which are equal to zero,

$$(x^1 - x^2) P(x_t = x, \delta_t = \delta|s_1) - (x^1 - x^2) P(x_t = x, \delta_t = \delta|s_2) = 0;$$

(c) terms with $\delta_t = \delta < 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (once more, by Bayes' rule), which can be rewritten as follows, exploiting symmetry,

$$\begin{aligned} & (x^1 - x^2) P(x_t = (x^1, x^2), \delta_t = \delta|s_1) - (x^1 - x^2) P(x_t = (x^1, x^2), \delta_t = \delta|s_2) \\ & = (x^1 - x^2) [P(x_t = (x^2, x^1), \delta_t = 1 - \delta|s_2) - P(x_t = (x^2, x^1), \delta_t = 1 - \delta|s_1)] \\ & = (x^2 - x^1) G_t((x^2, x^1), 1 - \delta). \end{aligned}$$

Combining all these terms, the integral (31) is equal to

$$\begin{aligned}
& \int_{\substack{x^1 > x^2, \delta > 1/2 \\ x \in [0, m]^2}} (x^1 - x^2) dG_t(x, \delta) + \int_{\substack{x^1 > x^2, \delta < 1/2 \\ x \in [0, m]^2}} (x^2 - x^1) dG_t((x^2, x^1), 1 - \delta) \\
&= \int_{\substack{x^1 > x^2, \delta > 1/2 \\ x \in [0, m]^2}} (x^1 - x^2) dG_t(x, \delta) + \int_{\substack{x^2 > x^1, \delta > 1/2 \\ x \in [0, m]^2}} (x^1 - x^2) dG_t(x, \delta) = \\
&= \int_{x \in [0, m]^2} (x^1 - x^2) dG_t(x, \delta),
\end{aligned}$$

where the first equality follows from a change of variables and the second from the fact that G_t is zero for all $\delta \leq 1/2$. We can now go back to the integral on the right-hand side of (30), and notice that the integrand $(x_t^1 - x_t^2)$ in the fourth term is positive, so replacing the measure P with the measure G_t , which is smaller or equal than P , reduces the value of that term. Therefore the inequality (30) in terms of the measure P , leads to the following inequality in terms of the measure G_t

$$\int_{x^2 \leq m} (x^1 - x^2) dG_t \leq \varepsilon,$$

completing the proof of property (iii). ■

We can now complete the proof of the proposition.

Proof of Proposition 2. We now proceed to use the two conditions (26) and (28) introduced above to reach a contradiction. We first show that agents with marginal rates of substitution near 1 and beliefs greater than 1/2 must hold more of asset 1 than of asset 2. Then we show that such asset holdings violate market clearing.

Formally, our objective is to show that, for some appropriately chosen positive scalars m and ζ , the following inequality holds for some t^*

$$\int_{x^2 \leq m} (x^1 - x^2) dG_{t^*} > \zeta, \tag{32}$$

and then showing that this contradicts (28). To evaluate the integral in (32) we will divide the agents into three groups.

Group 1: Informed agents with marginal rate of substitution sufficiently close to 1 and portfolios in some compact set X . We will prove that for these agents the difference $x^1 - x^2$ is bounded below by some positive number.

Group 2: Uninformed agents with marginal rate of substitution sufficiently close to 1 and informed agents with marginal rate of substitution sufficiently close to 1 but portfolios not in X . We will prove that, for all such agents the difference $x^1 - x^2$ is bounded below by some small negative number.

Group 3: Agents with marginal rate of substitution far from 1 at time t . We will prove

that the measure of such agents goes to zero.

In the rest of the proof, we construct the three groups above, we define the constants m and ζ , we find period t^* , and, finally, we prove inequality (32).

Step 1 (Group 1). Since there is at least a mass α of informed agents, using Lemmas 4 and 5, we can find a compact set $X \subset R_{++}^2$ and a time T such that for all $\varepsilon > 0$ there is a large enough mass of informed agents with (a) marginal rate of substitution close to 1 (within 3ε) and (b) portfolios in the set X , that is,

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1) > (5/6)\alpha - \varepsilon \quad (33)$$

for all periods $t \geq T$ in which $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1) > 1 - \varepsilon$.

Consider the following minimization problem

$$\begin{aligned} d_I(\varepsilon) &= \min_{x \in X} (x^1 - x^2) \\ \text{s.t. } &|\mathcal{M}(x, 1) - 1| \leq 3\varepsilon. \end{aligned}$$

The value $d_I(\varepsilon)$ is the minimal difference between the holdings of the two assets, for informed agents who satisfy (a) and (b). For future reference, notice that $d_I(\varepsilon)$ is continuous, from the theorem of the maximum.

Consider this problem with $\varepsilon = 0$. Let us prove that $d_I(0) > 0$. If $x^1 \leq x^2$, then $u'(x^1) \geq u'(x^2)$ and, therefore, the marginal rate of substitution

$$\mathcal{M}(x, 1) = \frac{\pi(1)u'(x^1)}{(1 - \pi(1))u'(x^2)} \geq \frac{\pi(1)}{1 - \pi(1)} > 1.$$

Therefore, all x that satisfy $|\mathcal{M}(x, 1) - 1| \leq 0$ must also satisfy $x^1 > x^2$. In other words, given that informed agents have a signal favorable to state 1, if their marginal rate of substitution is exactly 1 they must hold strictly more of asset 1.

We can now define the constant ζ —the lower bound for the average difference in the holdings of assets 1 and 2 in expression (32)—as

$$\zeta = \frac{\alpha}{6}d_I(0).$$

Next, we define the quantity m . Applying uniform market clearing and Lemma 7, we can find an $m \geq d_I(0)$ such that the following inequalities hold for all t :

$$\int_{x_t^2 > m} x_t^2 dP(\omega \mid s_1) \leq \zeta \quad (34)$$

and

$$\int_{x^2 \leq m} (x^1 - x^2) dG_t \leq \zeta. \quad (35)$$

From (34), we have

$$mP(x_t^2(\omega) > m) \leq \int_{x_t^2(\omega) > m} x_t^2(\omega) dP(\omega|s_1) \leq \zeta \text{ for all } t,$$

which, given the definition of ζ and the fact that $m \geq d_I(0)$, implies

$$P(x_t^2(\omega) > m) \leq \frac{\alpha d_I(0)}{6m} \leq \frac{\alpha}{6} \text{ for all } t.$$

We then obtain the following chain of equalities and inequalities,

$$\begin{aligned} & P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X | s_1) = \\ & P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X | s_1) \\ & \quad + P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 > m, x_t \in X | s_1) \\ \leq & P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X | s_1) + \alpha/6, \end{aligned}$$

and combine it with (33) to conclude that

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X | s_1) > (2/3)\alpha - \varepsilon \quad (36)$$

for all $t \geq T$ in which $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon | s_1) > 1 - \varepsilon$.

We are almost ready to construct group 1 as the set of informed agents that satisfy (a) and (b) above, plus the additional restriction $x_t^2 \leq m$, for appropriately chosen values of t and ε . The last step of this construction is to choose t and ε , but we will only be able to do so after constructing group 2 in the next step.

Step 2 (Groups 2 and 3). Consider the problem

$$\begin{aligned} d_U(\varepsilon) &= \min_{\substack{x^2 \leq m \\ \delta \geq 1/2}} (x^1 - x^2) \\ &\text{s.t. } |\mathcal{M}(x, \delta) - 1| \leq 3\varepsilon. \end{aligned}$$

The value $d_U(\varepsilon)$ is the minimum difference between the holdings of the two assets for all agents: (a) with marginal rates of substitution sufficiently close to 1, (b) holdings of asset 2 less than or equal to m , and (c) beliefs above 1/2. The theorem of the maximum implies that $d_U(\varepsilon)$ is continuous. Moreover, $d_U(\varepsilon)$ is negative for all $\varepsilon > 0$ and $d_U(0) = 0$.

Recall from Step 1 that $d_I(\varepsilon)$ is continuous and $d_I(0) > 0$. It is then possible to find a

positive ε^* , smaller than both $\alpha/6$ and ζ/m , such that

$$\frac{\alpha}{2}d_I(\varepsilon^*) + d_U(\varepsilon^*) > \frac{\alpha}{3}d_I(0) = 2\zeta, \quad (37)$$

(the second equality comes from the definition of ζ).

Since, by construction $\varepsilon^* < \alpha/6$, it follows from (36) that the mass of informed agents with marginal rates of substitution near 1 (within $3\varepsilon^*$) and a portfolio that satisfies $x_t^2 \leq m$ and $x_t \in X$ is sufficiently high:

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^*, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1) > \alpha/2 \quad (38)$$

for all $t \geq T$ in which $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$.

Moreover, by Lemma 4, in all periods $t \geq T$ in which almost all agents have marginal rate of substitution close to 1, i.e., $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$, almost all agents with beliefs higher than $1/2$ and portfolios satisfying $x_t^2 \leq m$ also have a marginal rate of substitution close to 1:

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^*, \delta_t > 1/2, x_t^2 \leq m \mid s_1) > P(\delta_t > 1/2, x_t^2 \leq m \mid s_1) - \varepsilon^*. \quad (39)$$

By hypothesis, i.e., by (26), we can choose a $t^* \geq T$ such that

$$P(|\mathcal{M}(x_{t^*}, \delta_{t^*}) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$$

so that both (38) and (39) are satisfied.

We can finally define groups 1, 2 and 3 as follows

$$\begin{aligned} A_1 &= \{(x, \delta) : |\mathcal{M}(x, \delta) - 1| < 3\varepsilon^*, \delta = 1, x^2 \leq m, x \in X\}, \\ A_2 &= \{(x, \delta) \notin A_1 : |\mathcal{M}(x, \delta) - 1| < 3\varepsilon^*, \delta > 1/2, x^2 \leq m\}, \\ A_3 &= \{(x, \delta) \notin A_1 \cup A_2 : \delta > 1/2, x^2 \leq m\}. \end{aligned}$$

Step 3. Now we split the integral (32) in three parts, corresponding to groups 1, 2, and 3, and determine a lower bound for each of them. First, we have

$$\int_{A_1} (x^1 - x^2) dG_{t^*} = \int_{(x_{t^*}, \delta_{t^*}) \in A_1} (x_{t^*}^1(\omega) - x_{t^*}^2(\omega)) dP(\omega \mid s_1) \geq \frac{\alpha}{2}d_I(\varepsilon^*), \quad (40)$$

where the equality follows from property (ii) of the distribution G_t (in Lemma 7) and the inequality follows from the definition of $d_I(\varepsilon^*)$ and condition (38). The definition of $d_U(\varepsilon^*)$ implies that

$$\int_{A_2} (x^1 - x^2) dG_{t^*} \geq d_U(\varepsilon^*) P(A_2) \geq d_U(\varepsilon^*), \quad (41)$$

since $d_U(\varepsilon^*) < 0$ and $P(A_2) \leq 1$. Finally, the definition of the measure G_t and condition (39) imply that

$$G_{t^*}(A_3) \leq P((x_{t^*}, \delta_{t^*}) \in A_3 | s_1) \leq P(\delta_{t^*} > 1/2, x_{t^*}^2 \leq m | s_1) - P((x_{t^*}, \delta_{t^*}) \in A_1 \cup A_2 | s_1) \leq \varepsilon^* < \zeta/m,$$

where the last inequality follows from the definition of ε^* . We then have the following lower bound

$$\int_{A_3} (x^1 - x^2) dG_{t^*} \geq -mG_{t^*}(A_3) \geq -\zeta. \quad (42)$$

We can now combine (40), (41) and (42) and use inequality (37) to obtain a lower bound for the whole integral (32):

$$\int_{x^2 \leq m} (x^1 - x^2) dG_{t^*} \geq \frac{\alpha}{2} d_I(\varepsilon^*) + d_U(\varepsilon^*) - \zeta > \zeta.$$

Comparing this inequality and (35) leads to the desired contradiction. ■

5.4 Proof of Lemma 3

Proof. We start with the usual convergence properties. Since the marginal rates of substitution of informed agents converge, by Proposition 1, and there is at least a mass α of informed agents, using Lemmas 4 and 5 we can find a compact set $X \subset R_{++}^2$ and a time T' such that there is a sufficiently large mass of informed agents with marginal rates of substitution sufficiently close to $\kappa_t(s)$ (within $\bar{\varepsilon}/2$) and portfolios in X :

$$P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \bar{\varepsilon}/2, \delta_t = \delta^I(s), x_t \in X | s) > (3/4)\alpha$$

for all $t \geq T'$ and for all s , where $\bar{\varepsilon}$ is defined as in Proposition 2.

Now we provide an important concept. We want to focus on the utility gains that can be achieved by small trades (of norm less than θ), by agents with marginal rates of substitution sufficiently different from each other (by at least $\bar{\varepsilon}/2$). Formally, we proceed as follows. Take any $\theta > 0$. Using Lemma 6, we can then find a lower bound for the utility gain $\Delta > 0$ from trade between two agents with marginal rates of substitution differing by at least $\bar{\varepsilon}/2$ and with portfolios in X , making trades of norm less than θ . It is important to notice that this is the gain achieved if the agents trade but do not change their beliefs. Therefore, it is also important to bound from below the gains that can be achieved by such trades if beliefs are updated in the most pessimistic way. This bound is also given by Lemma 6, which ensures that $-\lambda\Delta$ is a lower bound for the gains of the agent offering z at any possible ex post belief (where λ is a positive scalar independent of θ).

Next we want to restrict attention to agents who are close to their long-run expected utility. Per period utility u_t converges to the long-run value \hat{v}_t , by Lemma 2. We can then

apply Lemma 4 and find a time period $T \geq T'$ such that, for all $t \geq T$ and for all s :

$$P(u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s) > 1 - \bar{\varepsilon}/2, \quad (43)$$

and

$$P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/8, \delta_t = \delta^I(s), x_t \in X \mid s) > \alpha/2. \quad (44)$$

Equation (43) states that there are enough agents, both informed and uninformed, close to their long-run utility. Equation (44) states that there are enough informed agents close to both their long-run utility and to their long-run marginal rates of substitution.

We are now done with the preliminary steps ensuring proper convergence and can proceed to the body of the argument.

Choose any $t \geq T$. By Proposition 2, two cases are possible: (i) either the informed agents' long-run marginal rates of substitution are far enough from each other, $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$; or (ii) they are close to each other, $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$, but there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents, $P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| \geq 2\bar{\varepsilon} \mid s) \geq \bar{\varepsilon}$ for all s .

In the next two steps, we construct the desired trade z for each of these two cases, and then complete the argument in step 3.

Step 1. Consider the first case, in which $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$. In this case, an uninformed agent can exploit the difference between the informed agents' marginal rates of substitution in states s_1 and s_2 , making an offer at an intermediate price. This offer will be accepted with higher probability in the state in which the informed agents' marginal rate of substitution is higher. In particular, suppose

$$\kappa_t(s_2) + \bar{\varepsilon} \leq \kappa_t(s_1)$$

(the opposite case is treated symmetrically). Lemma 6 and the definition of the utility gain Δ imply that there is a trade $z = (\zeta, -p\zeta)$, with price $p = (\kappa_t(s_1) + \kappa_t(s_2))/2$ and size $\|z\| < \theta$, that satisfies the following inequalities:

$$U(x_t + z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}(x_t, \delta_t) > \kappa_t(s_1) - \bar{\varepsilon}/4 \text{ and } x_t \in X, \quad (45)$$

$$U(x_t - z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}(x_t, \delta_t) < \kappa_t(s_2) + \bar{\varepsilon}/4 \text{ and } x_t \in X. \quad (46)$$

Equation (45) states that all (informed and uninformed) agents with marginal rate of substitution above $(\kappa_t(s_1) - \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade z , in terms of current utility. Equation (46) states that all (informed and uninformed) agents with marginal rate of substitution below $(\kappa_t(s_2) + \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade $-z$, in terms of current utility.

Combining conditions (44) and (45) shows that in state s_1 there is at least $\alpha/2$ informed agents with after-trade utility above the long-run utility, $U(x_t + z, \delta_t) > \hat{v}_t$. Since all these

agents would accept the trade z , this implies that the probability of acceptance of the trade is $\chi_t(z|s_1) > \alpha/2$.

Next, we want to show that the trade z is accepted with sufficiently low probability conditional on s_2 . In particular, we want to show that $\chi_t(z|s_2) < \alpha/4$. The key step here is to make sure that the trade is rejected not only by informed but also by uninformed agents. The argument is that if this trade were to be accepted by uninformed agents, then informed agents should be offering z and gaining in utility. Formally, proceeding by contradiction, suppose that the probability of z being accepted in state s_2 is large: $\chi_t(z|s_2) \geq \alpha/4$. Condition (44) implies that there is a positive mass of informed agents with $\mathcal{M}(x_t, \delta_t) < \kappa_t(s_2) + \bar{\varepsilon}/2$, $x_t \in X$, and close enough to the long-run utility $u_t \geq \hat{v}_t - \alpha\Delta/8$. By (46), these agents would be strictly better off making the offer z and consuming $x_t - z$ if the offer is accepted and consuming x_t if it is rejected, since

$$(1 - \chi_t(z|s_2))U(x_t, \delta_t) + \chi_t(z|s_2)U(x_t - z, \delta_t) > u_t + \alpha\Delta/4 > \hat{v}_t.$$

Since this strategy dominates the equilibrium payoff, this is a contradiction, proving that $\chi_t(z|s_2) < \alpha/4$.

Step 2. Consider the second case, in which the long-run marginal rates of substitution of the informed agents are close to each other and there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents.

The argument is as follows: with positive probability we can reach a point where it is possible to separate the marginal rates of substitution of a group of uninformed agents from the marginal rates of substitution of a group of informed agents. This means that the uninformed agents in the first group can make an offer z to the informed agents in the second group and they will accept the offer in *both* states s_1 and s_2 . If the probabilities of acceptance $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ are sufficiently close to each other, this would be a profitable deviation for the uninformed, since their ex post beliefs after the offer is accepted would be close to their ex ante beliefs. In other words, in contrast to the previous case they would gain utility but not learn from the trade. It follows that the probabilities $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ must be sufficiently different in the two states, which leads to either (7) or to (8).

To formalize this argument, consider the expected utility of an uninformed agent with portfolio x_t and belief δ_t , who offers a trade z and stops trading from then on:

$$u_t + \delta_t \chi_t(z|s_1) (U(x_t - z, 1) - U(x_t, 1)) + (1 - \delta_t) \chi_t(z|s_2) (U(x_t - z, 0) - U(x_t, 0)),$$

where u_t is the expected utility if the offer is rejected and the following two terms are the expected gains if the offer is accepted, respectively, in states s_1 and s_2 . This expected utility

can be rewritten as

$$u_t + \chi_t(z|s_1) (U(x_t - z, \delta_t) - U(x_t, \delta_t)) + (1 - \delta_t) (\chi_t(z|s_2) - \chi_t(z|s_1)) (U(x_t - z, 0) - U(x_t, 0)), \quad (47)$$

using the fact that $U(x_t, \delta_t) = \delta_t U(x_t, 1) + (1 - \delta_t) U(x_t, 0)$ (by the definition of U). To interpret (47) notice that, if the probability of acceptance was independent of the signal, $\chi_t(z|s_1) = \chi_t(z|s_2)$, then the expected gain from making offer z would be equal to the second term: $\chi_t(z|s_1) (U(x_t - z, \delta_t) - U(x_t, \delta_t))$. The third term takes into account that the probability of acceptance may be different in two states, i.e., $\chi_t(z|s_2) - \chi_t(z|s_1)$ may be different from zero. An alternative way of rearranging the same expression yields:

$$u_t + \chi_t(z|s_2) (U(x_t - z, \delta_t) - U(x_t, \delta_t)) + (1 - \delta_t) (\chi_t(z|s_1) - \chi_t(z|s_2)) (U(x_t - z, 1) - U(x_t, 1)). \quad (48)$$

In the rest of the argument, we will use both (47) and (48).

Suppose that there exists a trade z and a period t which satisfy the following properties:

(a) the probability that z is accepted in state 1 is large enough,

$$\chi_t(z|s_1) > \alpha/4,$$

and (b) there is a positive mass of uninformed agents with portfolios and beliefs that satisfy

$$u_t \geq \hat{v}_t - (\alpha/4) \Delta, \quad (49)$$

$$U(x_t - z, \delta_t) - U(x_t, \delta_t) \geq \Delta, \quad (50)$$

$$U(x_t - z, \delta) - U(x_t, \delta) \geq -\lambda \Delta \text{ for all } \delta \in [0, 1], \quad (51)$$

for some $\Delta > 0$ and $\lambda > 0$. In words, the uninformed agents are sufficiently close to their long-run utility, their gains from trade at *fixed* beliefs have a positive lower bound Δ , and their gains from trade at *arbitrary* beliefs have a lower bound $-\lambda \Delta$.

Now we distinguish two cases. Suppose first that $\chi_t(z|s_2) \geq \chi_t(z|s_1)$. Then, for the uninformed agents who satisfy (49)-(51) the expected utility (47) is greater or equal than

$$\hat{v}_t - (\alpha/4) \Delta + \chi_t(z|s_1) \Delta - (\chi_t(z|s_2) - \chi_t(z|s_1)) \lambda \Delta.$$

From individual optimality, this expression cannot be larger than \hat{v}_t , since \hat{v}_t is the maximum expected utility for a proposer in period t . We then obtain the following restriction on the acceptance probabilities $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$:

$$\chi_t(z|s_1) (1 + \lambda) \Delta \leq \alpha \Delta / 4 + \chi_t(z|s_2) \lambda \Delta.$$

Since $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_1) \geq \chi_t(z|s_2)$ it follows that $\alpha/4 < (1/2)\chi_t(z|s_2)$ and we obtain

$$\chi_t(z|s_1)(1 + \lambda) \leq \chi_t(z|s_2)(1/2 + \lambda),$$

which is equivalent to

$$\chi_t(z|s_1) \geq \frac{1 + \lambda}{1/2 + \lambda} \chi_t(z|s_2). \quad (52)$$

This shows that the probability of acceptance in state s_1 is larger than the probability of acceptance in state s_2 by a factor $(1 + \lambda) / (1/2 + \lambda)$ greater than 1.

Consider next the case $\chi_t(z|s_2) < \chi_t(z|s_1)$. Then, for the uninformed agents who satisfy (49)-(51) the expected utility (48) is greater or equal than

$$\hat{v}_t - \alpha\Delta/4 + \chi_t(z|s_2)\Delta - (\chi_t(z|s_1) - \chi_t(z|s_2))\lambda\Delta.$$

An argument similar to the one above shows that optimality requires

$$\chi_t(z|s_2) \geq \frac{1 + \lambda}{1/2 + \lambda} \chi_t(z|s_1).$$

Some algebra shows that this inequality and $\chi_t(z|s_1) > \alpha/2$ imply

$$1 - \chi_t(z|s_1) > 1 - \frac{\alpha}{2} \frac{1/2 + \lambda}{1 + \lambda}, \quad (53)$$

$$1 - \chi_t(z|s_1) > \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4} (1 - \chi_t(z|s_2)), \quad (54)$$

giving us a positive lower bound for the probability of rejection $1 - \chi_t(z|s_1)$ and showing that $1 - \chi_t(z|s_1)$ exceeds $1 - \chi_t(z|s_2)$ by a factor greater than 1.

To complete this step, we show that there exists a trade z and a period t which satisfy properties (a) and (b).

Notice that $P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| \geq 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}$ requires that either $P(\mathcal{M}(x_t, \delta_t) \leq \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2$ holds or $P(\mathcal{M}(x_t, \delta_t) \geq \kappa_t(s_1) + 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2$. We concentrate on the first case, as the second is treated symmetrically. Set the trading price at $p = \min\{\kappa(s_1), \kappa(s_2)\} - \bar{\varepsilon}/2$. Lemma 6 implies that there are positive scalars Δ and λ and a trade $z = (\zeta, -p\zeta)$ with $\|z\| < \theta$ that satisfies the following inequalities:

$$U(x_t - z, \delta_t) \geq u_t + \Delta, \quad U(x_t - z, \delta) \geq u_t - \lambda\Delta \text{ for all } \delta \in [0, 1], \quad (55)$$

if $\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4$ and $x_t \in X$,

and

$$U(x_t + z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}(x_t, \delta_t) > p + \bar{\varepsilon}/4 \text{ and } x_t \in X. \quad (56)$$

Since $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < \bar{\varepsilon}/4$ implies $\mathcal{M}(x_t, \delta_t) > \kappa_t(s_1) - \bar{\varepsilon}/4$ and $\kappa_t(s_1) - \bar{\varepsilon}/4$ is larger than $p + \bar{\varepsilon}/4$ by construction, conditions (44) and (56) guarantee that there is a positive mass of informed agents who accept z , ensuring that $\chi_t(z|s_1) > \alpha/2$, showing that z satisfies property (a).

Next, we want to prove that there is a positive mass of uninformed agents who gain from making offer z . To do so, notice that $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$ implies

$$p - \bar{\varepsilon}/4 = \min\{\kappa_t(s_1), \kappa_t(s_2)\} - (3/4)\bar{\varepsilon} \geq \kappa_t(s_1) - (7/4)\bar{\varepsilon} > \kappa_t(s_1) - 2\bar{\varepsilon},$$

which implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4 \mid s_1) \geq P(\mathcal{M}(x_t, \delta_t) \leq \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2.$$

This, using Lemma 4 and condition (43), implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s_1) > 0,$$

which, combined with (55), shows that the trade z satisfies property (b).

Step 3. Here we put together the bounds established above and define the scalars β and ρ in the lemma's statement. Consider the case treated in Step 1. In this case, we can find a trade z such that the probability of acceptance conditional on each signal satisfies: $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_2) < \alpha/4$. Therefore, in this case condition (7) is true as long as β and ρ satisfy

$$\beta \leq \alpha/2 \text{ and } \rho \leq 2.$$

Consider the case treated in Step 2. In this case, we can find a trade z such that either $\chi_t(z|s_1) > \alpha/2$ and (52) hold or (53) and (54) hold. This implies that either condition (7) or condition (8) hold, as long as β and ρ satisfy

$$\beta \leq 1 - \frac{\alpha}{2} \frac{1/2 + \lambda}{1 + \lambda}, \rho \leq \frac{1 + \lambda}{1/2 + \lambda}, \rho \leq \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}.$$

Setting

$$\begin{aligned} \beta &= \min\left\{\alpha/2, 1 - \frac{\alpha}{2} \frac{1/2 + \lambda}{1 + \lambda}\right\} > 0, \\ \rho &= \min\left\{2, \frac{1 + \lambda}{1/2 + \lambda}, \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}\right\} > 1, \end{aligned}$$

ensures that all the conditions above are satisfied, completing the proof. ■

5.5 Proof of Proposition 3

Proposition 3 states that marginal rates of substitution converge for uninformed agents. The proof is by contradiction and relies on constructing a deviation that yields a positive utility gain for the uninformed agents if marginal rates of substitution fail to converge. This deviation consists of making a sequence of offers in periods T to $T + J$. The first $T + J - 1$ offers, denoted by the sequence $\{\hat{z}_j\}_{j=0}^{J-1}$, allow the agent to learn the signal with arbitrary precision. This is the experimentation stage. If the agent receives the appropriate sequence of responses to these $J - 1$ offers, the agent makes one final offer, denoted z^* , which gives him a positive utility gain by trading with the informed agents.

Two preliminary results need to be established first: Lemma 8 and Lemma 9. Lemma 8 shows that the beliefs of uninformed agents δ_t stay away from zero when the signal is s_1 . That is, the uninformed agents can only be very wrong with a small probability. This result is used to ensure that when the uninformed agent deviates and enters the experimentation phase to learn signal s_1 , his ex post beliefs will converge to 1 with positive probability.

Lemma 8 *For all $\varepsilon > 0$ the probability that the belief δ_t is above the threshold $\varepsilon/(1 + \varepsilon)$ conditional on signal s_1 is bounded below for all t :*

$$P(\delta_t \geq \varepsilon/(1 + \varepsilon) \mid s_1) > 1 - \varepsilon.$$

Proof. Since $\delta_t(\omega)$ are equilibrium beliefs, Bayesian rationality requires $P(s_1 \mid \delta_t < \varepsilon/(1 + \varepsilon)) < \varepsilon/(1 + \varepsilon)$ for all $\varepsilon > 0$. The latter condition implies $P(s_2 \mid \delta_t < \varepsilon/(1 + \varepsilon)) > 1 - \varepsilon/(1 + \varepsilon)$ and thus

$$\frac{P(s_1 \mid \delta_t < \varepsilon/(1 + \varepsilon))}{P(s_2 \mid \delta_t < \varepsilon/(1 + \varepsilon))} < \varepsilon,$$

for all $\varepsilon > 0$. Bayes' rule implies that

$$\frac{P(s_1 \mid \delta_t < \varepsilon/(1 + \varepsilon))}{P(s_2 \mid \delta_t < \varepsilon/(1 + \varepsilon))} = \frac{P(\delta_t < \varepsilon/(1 + \varepsilon) \mid s_1) P(s_1)}{P(\delta_t < \varepsilon/(1 + \varepsilon) \mid s_2) P(s_2)}.$$

Combining the last two equations and using $P(s_1) = P(s_2) = 1/2$ yields

$$P(\delta_t < \varepsilon/(1 + \varepsilon) \mid s_1) < \varepsilon P(\delta_t < \varepsilon/(1 + \varepsilon) \mid s_2) \leq \varepsilon,$$

which gives the desired inequality.

Lemma 9 is a stronger version of the convergence result for the marginal rates of substitution of informed agents (Proposition 1). It shows that the series $\kappa_t(s)$ (the long-run marginal rate of substitution of informed agents) is approximately constant over fixed intervals of length J , for any choice of the length J . This implies that the marginal rates of substitution of informed agents at time $t + J$ are close to the value $\kappa_t(s)$, if we choose t large enough. This property

will be useful when constructing the final offer z^* made by the deviating uninformed agent in the proof of Proposition 3. ■

Lemma 9 *For any integer J , the sequence $\kappa_t(s_1)$ satisfies the property:*

$$\lim_{t \rightarrow \infty} |\kappa_{t+J}(s_1) - \kappa_t(s_1)| = 0.$$

For all $\varepsilon > 0$ and all integers J it is possible to find a T such that

$$P(|\mathcal{M}(x_{t+J}, \delta_{t+J}) - \kappa_t(s_1)| < \varepsilon, \delta_{t+J} = 1 \mid s) > \alpha - \varepsilon \text{ for all } t \geq T.$$

Proof. Let us begin from the first part of the lemma. Suppose, by contradiction, that

$$|\kappa_{t+J}(s_1) - \kappa_t(s_1)| > \varepsilon$$

for some $\varepsilon > 0$ for infinitely many periods. Then, at some date t , an informed agent with marginal rate of substitution close to $\kappa_t(s)$ can find a profitable deviation by holding on to his portfolio x_t for J periods and then trade with other informed agents at $t + J$. Let us formalize this argument. Suppose, without loss of generality, that

$$\kappa_{t+J}(s_1) > \kappa_t(s_1) + \varepsilon$$

for infinitely many periods (the other case is treated in a symmetric way). Next, using our usual steps and Proposition 1, it is possible to find a compact set X , a time T , and a utility gain $\Delta > 0$ such that the following two properties are satisfied: (i) in all periods $t \geq T$ there is at least a measure $\alpha/2$ of informed agents with marginal rate of substitution sufficiently close to $\kappa_t(s)$, utility close to its long-run level, and portfolio x_t in X , that is,

$$P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \varepsilon/3, u_t \geq \hat{v}_t - \gamma^J \alpha \Delta/2, x_t \in X, \delta_t = 1 \mid s) > \alpha/2, \quad (57)$$

and (ii) in all periods $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ there is a trade z such that

$$U(x - z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) < \kappa_t(s) + \varepsilon/3 \text{ and } x \in X, \quad (58)$$

and

$$U(x + z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) > \kappa_{t+J}(s) - \varepsilon/3 \text{ and } x \in X. \quad (59)$$

Pick a time $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ and consider the following deviation. Whenever an informed agent reaches time t and his portfolio x_t satisfies $\mathcal{M}(x_t, 1) < \kappa_t(s) + \varepsilon/3$ and $x_t \in X$, he stops trading for J periods and then makes an offer z that satisfies (58) and (59). If the offer is rejected he stops trading from then on. The probability that this offer is

accepted at time $t + J$ must satisfy $\chi_{t+J}(z|s_1) > \alpha/2$, because of conditions (57) and (59). Therefore, the expected utility from this strategy, from the point of view of time t is

$$u_t + \gamma^J \chi_{t+J}(z|s_1) (U(x_t - z, 1) - u_t) > u_t + \gamma^J \alpha \Delta / 2 \geq \hat{v}_t,$$

so this strategy is a profitable deviation and we have a contradiction.

The second part of the lemma follows from the first part, using Proposition 1 and the triangle inequality. ■

Proof of Proposition 3. Suppose, by contradiction, that there exist an $\varepsilon > 0$ such that for some state $s \in \{s_1, s_2\}$ the following condition holds for infinitely many t :

$$P(|\mathcal{M}(x_t, \delta^I(s)) - \kappa_t(s)| > \varepsilon | s) > \varepsilon,$$

where $\mathcal{M}(x_t, \delta^I(s))$ is the marginal rate of substitution of an agent (informed or uninformed) evaluated at the belief of the informed agents $\delta^I(s)$. In other words, it is the marginal rate of substitution evaluated as if the agent knew the true signal.

Without loss of generality, let us focus on state s_1 and suppose

$$P(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon | s_1) > \varepsilon \tag{60}$$

for infinitely many t . The other case is treated in a symmetric way.

We want to show that if (60) holds, we can find a profitable deviation for the uninformed agent. In particular, we consider a deviation of this form:

- (i) The player follows the equilibrium strategy σ up to period T .
- (ii) At time T , if his portfolio satisfies $\mathcal{M}(x_T, 1) > \kappa_t(s) + \varepsilon$ and his beliefs δ_T is above some positive lower bound $\underline{\delta}$ (and some other technical conditions are satisfied) he goes on to the experimentation stage (iii), otherwise, he keeps playing σ .
- (iii) The experimentation stage lasts between T and $T + J - 1$. An agent makes the sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ as long as he is selected as the proposer. The “favorable” responses to the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ are given by the binary sequence $\{\hat{r}_j\}_{j=0}^{J-1}$. If at any point during the experimentation stage the agent is not selected as the proposer or fails to receive response \hat{r}_j after offer \hat{z}_j , he stops trading. Otherwise, he goes to (iv).
- (iv) At time $T + J$, after making all the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and receiving responses equal to $\{\hat{r}_j\}_{j=0}^{J-1}$, if the player is selected as the proposer one more time he offers z^* and stops trading at $T + J + 1$. Otherwise, he stops trading right away.

The expected payoff of this strategy, from the point of view of a deviating agent at time T , is

$$\begin{aligned}
w = & u_T - \hat{L} + \delta_T \gamma^J 2^{-J-1} \xi_1 \chi_{T+J}(z^* | s_1) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) \right] + \\
& + (1 - \delta_T) \gamma^J 2^{-J-1} \xi_2 \chi_{T+J}(z^* | s_2) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) \right], \quad (61)
\end{aligned}$$

where the term \hat{L} captures the expected utility losses if the player makes some or all of the offers in $\{\hat{z}_j\}_{j=0}^{J-1}$ but not the last offer z^* and the following two terms capture the expected utility gains in states s_1 and s_2 , if all the deviating offers, including z^* , are accepted. The factors ξ_1 and ξ_2 denote the probabilities in states s_1 and s_2 , that player receives the sequence of responses $\{\hat{r}_j\}_{j=0}^{J-1}$. Notice that γ^J is the probability that the game does not end between periods T and $T+J$ and 2^{-J-1} is the probability of being selected as the proposer in all these periods.

In order to show that the strategy above is a profitable deviation, we need to show that the utility gain in the first square brackets is large enough, by choosing z^* to be a profitable trade with informed agents in s_1 , and that the remaining terms are sufficiently small. In the rest of the proof, we choose the time T , the lower bound $\underline{\delta}$, and the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and z^* to achieve this goal.

Step 1. (Bounds on gains and losses for the final trade) Following steps similar to the ones in the proof of Proposition 1, we can use Lemmas 5 and 8 to find a compact set $X \subset R_{++}^2$ and a period T' such that

$$P(\delta_t \geq \underline{\delta}, x_t \in X \mid s_1) > 1 - \varepsilon/2 \quad (62)$$

for all $t \geq T'$, where $\underline{\delta} = (\varepsilon/2) / (1 + \varepsilon/2) > 0$. Pick a scalar $\theta^* > 0$ such that $x + z > 0$ when $x \in X$ and $\|z\| < \theta^*$. Using Lemma 6, we can then find a $\Delta^* > 0$ which is a lower bound for the gains from trade between two agents with marginal rates of substitution differing by at least $\varepsilon/2$ and portfolios in X , making trades of norm smaller than θ^* . This will be used as a lower bound for the gains from trading in state s_1 . Define an upper bound for the potential losses of an uninformed agent who makes a trade of norm smaller than or equal to θ^* in the other state, s_2 :

$$L^* \equiv - \min_{x \in X, \|z\| \leq \theta^*} \{U(x+z, 0) - U(x, 0)\}.$$

Next, choose J to be an integer large enough that

$$\underline{\delta} (\alpha/2) \Delta^* - (1 - \underline{\delta}) \rho^{-J} L^* > 0,$$

where ρ is the scalar defined in Lemma 3. This choice of J ensures that the experimentation

phase is long enough that, when offering the last trade, the agent assigns sufficiently high probability to state s_1 , so that the potential gain Δ^* dominates the potential loss L^* .

Step 2. (*Bound on losses from experimentation*) To simplify notation, let

$$\tilde{\Delta} = \gamma^J 2^{-J-1} \beta^J (\underline{\delta} (\alpha/2) \Delta^* - (1 - \underline{\delta}) \rho^{-J} L^*),$$

where β is the positive scalar defined in Lemma 3. Choose a scalar $\hat{\theta} > 0$ such that for all $x \in X$, all $\|z_1\| < J\hat{\theta}$, all $\|z_2\| \leq \theta^*$, and any $\delta \in [0, 1]$ the following inequality holds

$$|U(x + z_1 + z_2, \delta) - U(x + z_2, \delta)| < \tilde{\Delta}/3. \quad (63)$$

Next, applying Lemma 3 we can find a time $T'' \geq T'$ such that in all $t \geq T''$ there is a trade of norm smaller than $\hat{\theta}$ that satisfies either (7) or (8). Before using this property to define the offers $\{\hat{z}_j\}_{j=0}^{J-1}$, we need to define the time period T where the deviation occurs. To do so, using our starting hypothesis (60), condition (62), and applying Lemma 2, we can find a $T''' \geq T''$ such that for infinitely many periods $t \geq T'''$ there is a positive mass of uninformed agents who have: marginal rate of substitution sufficiently above $\kappa_t(s_1)$, utility near its long-run level, beliefs sufficiently favorable to s_1 , and portfolio in X ; that is,

$$P\left(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon, \delta_t \geq \underline{\delta}, u_t > \hat{v}_t - \tilde{\Delta}/3, x_t \in X \mid s_1\right) > 0. \quad (64)$$

Finally, applying Lemma 9, we pick a $T \geq T'''$ so that (64) holds at $t = T$ and, at time $T + J$, there is a sufficiently large mass of informed agents who have: marginal rate of substitution sufficiently near $\kappa_T(s_1)$, utility near its long-run level, and portfolio in X ; that is,

$$P(|\mathcal{M}(x_{T+J}, 1) - \kappa_T(s_1)| < \varepsilon/2, \delta_{T+J} = 1, u_{T+J} > \hat{v}_{T+J} - \Delta/2, x_{T+J} \in X \mid s_1) > \alpha/2. \quad (65)$$

Having defined T , we can apply Lemma 3 to find the desired sequence of trades $\{\hat{z}_j\}_{j=0}^{J-1}$ of norm smaller than $\hat{\theta}$, that satisfy either (7) or (8). For each trade \hat{z}_j , if (7) holds we set $\hat{r}_j = 1$ (accept). In this way the probability of observing \hat{r}_j is $\chi_{T+j}(\hat{z}_j | s_1) > \beta$ in state s_1 and $\chi_{T+j}(\hat{z}_j | s_2) < \rho^{-1} \chi_{T+j}(\hat{z}_j | s_1)$ in state s_2 . Otherwise, if (8) holds, we set $\hat{r}_j = 0$ and obtain analogous inequalities. This implies that the factors ξ_1 and ξ_2 in (61) satisfy

$$\xi_1 > \beta^J \text{ and } \xi_2 < \xi_1 \rho^{-J}. \quad (66)$$

Step 3. (*Define z^* and check profitable deviation*) We can now define the final trade z^* to be a trade of norm smaller than θ^* , such that

$$\begin{aligned} U(x - z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) > \kappa_T(s_1) + \varepsilon \text{ and } x \in X, \\ U(x + z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) < \kappa_T(s_1) + \varepsilon/2 \text{ and } x \in X, \end{aligned}$$

which is possible given the definition of Δ^* . Finally, we check that we have constructed a profitable deviation. Let uninformed agents start deviating whenever the following conditions are satisfied at date T :

$$\mathcal{M}(x_T, 1) > \kappa_T(s_1) + \varepsilon, \delta_T \geq \underline{\delta}, u_T > \hat{v}_T - \tilde{\Delta}/3, x_T \in X.$$

Equation (64) shows that this happens with positive probability. Let us evaluate the deviating strategy payoff (61), beginning with the last two terms. The triangle inequality implies $\left\| \sum_{j=0}^{J-1} \hat{z}_j \right\| < J\hat{\theta}$. Then the definition of z^* and (63) imply that the gain from trade of the uninformed agent, conditional on s_1 , is bounded below:

$$\begin{aligned} U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) &\geq U(x_T + z^*, 1) - U(x_T, 1) - \left| U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T + z^*, 1) \right| \\ &> \Delta^* - \tilde{\Delta}/3. \end{aligned}$$

The definition of L^* implies that the gain conditional on s_2 is also bounded:

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) > -L^* - \tilde{\Delta}/3.$$

Moreover, condition (65) shows that the probability that informed agents accept z^* at $T + J$ satisfies $\chi_{T+J}(z^*|s_1) > \alpha/2$. These results, together with the inequalities (66) and the fact that $\chi_{T+J}(z^*|s_2) \leq 1$, imply that the last two terms in (61) are bounded below by

$$\gamma^J 2^{-J-1} \beta^J \left[\underline{\delta}(\alpha/2) \left(\Delta^* - \tilde{\Delta}/3 \right) - (1 - \underline{\delta}) \rho^{-J} \left(L^* + \tilde{\Delta}/3 \right) \right],$$

which, by the definition of $\tilde{\Delta}$, is greater than $(2/3)\tilde{\Delta}$. Finally, all the expected losses in \hat{L} in (61) are bounded above by $\tilde{\Delta}/3$, thanks to (63). Therefore, $w > u_T + \tilde{\Delta}/3$. Since $u_T > \hat{v}_T - \tilde{\Delta}/3$, we conclude that $w > \hat{v}_T$ and we have found a profitable deviation. ■

5.6 Proof of Theorem 1

We begin from the second part of the theorem, proving (11), which characterizes the limit behavior of $\kappa_t(s)$.

Without loss of generality, let $s = s_1$. Suppose first that for infinitely many periods the long-run marginal rate of substitution $\kappa_t(s_1)$ is larger than the ratio of the probabilities $\phi(s_1)/(1 - \phi(s_1))$ by a factor larger than $1 + \varepsilon$:

$$\kappa_t(s_1) > (1 + \varepsilon) \phi(s_1)/(1 - \phi(s_1)) \text{ for some } \varepsilon > 0.$$

Proposition 3 then implies that for all $\eta > 0$ and T there is a t such that almost all agents

have portfolios that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$:

$$P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2 \mid s_1) > 1 - \eta. \quad (67)$$

We want to show that this property violates uniform market clearing, since it implies that almost all agents hold more of asset 2 than of asset 1.

Uniform market clearing implies that for any $\zeta > 0$ we can find an M such that

$$\int_{x_t^1(\omega) \leq m} x_t^1(\omega) dP(\omega \mid s_1) \geq 1 - \zeta \text{ for all } m \geq M \text{ and all } t. \quad (68)$$

Moreover, since $\int x_t^2(\omega) dP(\omega \mid s_1) = 1$, this implies that

$$\int_{x_t^1(\omega) \leq m} (x_t^2(\omega) - x_t^1(\omega)) dP(\omega \mid s_1) \leq \zeta \text{ for all } m \geq M \text{ and all } t. \quad (69)$$

The idea of the proof is to reach a contradiction by splitting the integral on the left-hand side of (69) in three pieces: a group of agents with a strictly positive difference $x_t^2 - x_t^1$, a group of agents with a non-negative difference $x_t^2 - x_t^1$, and a small residual group. The argument here follows a similar logic as the proof of Proposition 2.

Using Lemma 5, find a compact set X and a period T such that for all $t \geq T$ at least half of the agents have portfolios in X :

$$P(x_t \in X \mid s_1) \geq 1/2 \text{ for all } t \geq T. \quad (70)$$

Let us then find a lower bound for the difference between the holdings of asset 1 and 2 for agents with portfolios in X that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$. We do so by solving the problem

$$\begin{aligned} d &= \min_{x \in X} (x^2 - x^1) \\ &\text{s.t. } u'(x^1)/u'(x^2) \geq 1 + \varepsilon/2, \end{aligned}$$

which gives a $d > 0$.

Let us pick $\zeta = d/5$ and find an M such that (68) and (69) hold. Condition (69) (with $\zeta = d/5$) is the market clearing condition that we will contradict below. Condition (68) is also useful, because it gives us a lower bound for $P(x_t^1 \leq m)$:

$$P(x_t^1 \leq m) \geq 1 - \zeta/m \text{ for all } m \geq M \text{ and all } t, \quad (71)$$

which follows from the chain of inequalities

$$mP(x_t^1 > m) \leq \int_{x_t^1(\omega) > m} x_t^1(\omega) dP(\omega|s_1) \leq \zeta.$$

Using our hypothesis (67) we know that for any $\eta > 0$ we can find a period $t \geq T$ in which more than $1 - \eta$ agents satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$. Combining this with (70) and (71) (applying Lemma 4), we can always find a $t \geq T$ in which almost all agents satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$ and $x_t \leq m$:

$$P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \leq m | s_1) > 1 - \eta - \zeta/m, \quad (72)$$

and almost half of them satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$ and $x_t \leq m$, and have portfolios in X :

$$P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \leq m, x_t \in X | s_1) > 1/2 - \eta - \zeta/m. \quad (73)$$

Define the three disjoint sets

$$\begin{aligned} A_1 &= \{\omega : u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \in X, x_t^1 \leq m\}, \\ A_2 &= \{\omega : u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t^1 \leq m\} / A_1, \\ A_3 &= \{\omega : u'(x_t^1)/u'(x_t^2) < 1 + \varepsilon/2, x_t^1 \leq m\}, \end{aligned}$$

which satisfy $A_1 \cup A_2 \cup A_3 = \{\omega : x_t^1 \leq m\}$. We can then bound from below the following three integrals:

$$\begin{aligned} \int_{A_1} (x_t^2 - x_t^1) dP(\omega|s_1) &\geq d \cdot (1/2 - \eta - \zeta/m), \\ \int_{A_2} (x_t^2 - x_t^1) dP(\omega|s_1) &\geq 0, \\ \int_{A_3} (x_t^2 - x_t^1) dP(\omega|s_1) &\geq -m \cdot (\eta + \zeta/m). \end{aligned}$$

The first inequality follows from the definitions of d and A_1 and the fact that $P(A_1|s_1) > 1/2 - \eta - \zeta/m$ from (73). The second follows from the definition of A_2 and the fact that $u'(x_t^1)/u'(x_t^2) > 1$ implies $x_t^2 > x_t^1$. The third follows from the definition of A_3 (which implies $x_t^2 - x_t^1 \geq -m$) and the fact that $P(A_3|s_1) < \eta + \zeta/m$ from (72). Summing term by term, we then obtain

$$\int_{x_t^1(\omega) \leq m} (x_t^2 - x_t^1) dP(\omega|s_1) \geq d \cdot (1/2 - \eta - \zeta/m) - m \cdot (\eta + \zeta/m).$$

Since we can choose an m arbitrarily large and an η arbitrarily close to 0 (in that order), we

can make this expression as close as we want to $d/2 - \zeta$ which is strictly greater than ζ , given that $\zeta = d/5 < d/4$. This contradicts the market clearing condition (69).

In a similar way we can rule out the case in which $\kappa_t(s_1) < (1 - \varepsilon)\phi(s_1)/(1 - \phi(s_1))$ for infinitely many periods. This completes the argument for $\lim_{t \rightarrow \infty} \kappa_t(s_1) = \phi(s_1)/(1 - \phi(s_1))$. An analogous argument can be applied to s_2 .

To complete the proof, we need to prove the long-run efficiency of equilibrium portfolios, i.e., property (10). Proposition 3 and $\lim \kappa_t(s) = \phi(s)/(1 - \phi(s))$, imply, by the properties of convergence in probability, that

$$\lim_{t \rightarrow \infty} P(|u'(x_t^1)/u'(x_t^2) - 1| > \varepsilon) = 0. \quad (74)$$

We want to show that negating (10) leads to a contradiction of (74).

Suppose that for some $\varepsilon > 0$

$$P(|x_t^1 - x_t^2| > \varepsilon) > \varepsilon$$

holds for infinitely many periods. Then, as usual, we can use Lemmas 4 and 5 to find a compact set X such that the following condition holds for infinitely many periods:

$$P(|x_t^1 - x_t^2| > \varepsilon, x_t \in X) > \varepsilon/2.$$

But then the continuity of $u'(\cdot)$ implies that there is a $\delta > 0$ such that

$$|u'(x^1)/u'(x^2) - 1| > \delta \implies |x^1 - x^2| > \varepsilon \text{ for all } x \in X$$

which implies

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta, x_t \in X) \geq P(|x_t^1 - x_t^2| > \varepsilon, x_t \in X) > \varepsilon/2.$$

Given that

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta) \geq P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta, x_t \in X)$$

we conclude that there are $\varepsilon, \delta > 0$ such that

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta) > \varepsilon/2,$$

contradicting (74) and completing the proof.

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