August 2011 Truman Bewley

Math Camp
Homework #14

1) A measure on the Borel subsets $\mathcal{B}$ of R is a function $\mu: \mathcal{B} \to [0, \infty)$ such that $\mu(\emptyset) = 0$, and if $A_1, A_2, \ldots$ is a sequence of mutually disjoint sets in $\mathcal{B}$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$. Lebesgue measure on $\mathcal{B}$ is the measure such that $\mu([a, b]) = b - a$, for all numbers a and b such that $a < b$. Let Q be the set of rational numbers. That is, $Q = \{n/m \mid n \text{ and } m \text{ are integers}\}$.

Show that $\mu(Q) = 0$. In order to show this, you should know that Q is countable. That is, there is a one to one and onto function $f: \{1, 2, \ldots\} \to Q$.

2) The Riemann integral of a function $f: [a, b] \to R$, where $a < b$, is defined as follows. Call (T, C) a tagged partition of [a, b], if $T = (t_1, t_2, \ldots, t_{N+1})$, where N is a positive integer and $a = t_1 < t_2 < \ldots < t_{N+1} = b$, and if $C = (c_1, \ldots, c_N)$, where $t_n \le c_n \le t_{n+1}$, for all n. For any tagged partition (T, C), let

$$S_{(T, C)} f = \sum_{n=1}^{N} f(c_n)(t_{n+1} - t_n).$$

Then f is Riemann integrable if for all sequences of tagged partitions, (T_k, C_k), $k = 1, 2, \ldots$, where $T_k = (t_{k1}, \ldots, t_{k, N_k+1})$ and such that $\lim_{k \to \infty} \max_{n = 1, \ldots, N_k} (t_{k, n+1} - t_{kn}) = 0$, $\lim_{k \to \infty} S_{(T_k, C_k)} f$ converges and has the same limit. If f is Riemann integrable, the limit is called $\int_a^b f(t)\,dt$ and is the Riemann integral of f.

Let $f: [0, 1] \to R$ be defined as

$$f(t) = \begin{cases} 1, & \text{if } t \text{ is rational and} \\ 0, & \text{if } t \text{ is irrational.} \end{cases}$$

Show that f is not Riemann integrable but that f is Lebesgue integrable and that $\int_a^b f(t)\,dt = 0$. The Lebesgue integral is the integral defined in class using simple functions and with respect to Lebesgue measure on R.