

Lecture #4

Real Analysis

Real analysis starts from the simple idea of closeness of numbers and of N-vectors of numbers to each other. The set of points close to a given point is taken to be points in an N-dimensional ball of small radius about the point.

Definition: An open ball of radius ε about a point y in \mathbb{R}^N is $\{x \in \mathbb{R}^N \mid \|x - y\| < \varepsilon\}$. This set is denoted $B_\varepsilon(y)$.

Definition: A subset U of \mathbb{R}^N is said to be open if for every $y \in U$, there is an $\varepsilon > 0$ such that $B_\varepsilon(y) \subset U$.

In other words, a set is open if every point in it is surrounded by other points in the set.

Examples:

- 1) \mathbb{R}^N is open.
- 2) The empty set is open in \mathbb{R}^N , because any assertion about the properties of nothing is true, except the assertion that it has something in it.
- 3) $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is open in \mathbb{R} .
- 4) $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is not open in \mathbb{R} .
- 5) $\{(x_0, 0) \in \mathbb{R}^2 \mid 0 < x_0 < 1\}$ is not open in \mathbb{R}^2 .
- 6) If $\varepsilon > 0$ and $x \in \mathbb{R}^N$, $B_\varepsilon(x)$ is open in \mathbb{R}^N .

Definitions: 1) If A is a subset of X , the set theoretic complement of A in X is $X \setminus A = \{x \in X \mid x \notin A\}$.

2) If A and B are sets, the union of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

3) If A and B are sets, the intersection of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Remark: The following equations are referred to as De Morgan's laws. If A and B are subsets of the set X , then $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. Similarly if \mathcal{A} is a set of subsets of X , then

$$X \setminus \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (X \setminus A) \text{ and}$$
$$X \setminus \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (X \setminus A).$$

Definition: A subset A of \mathbb{R}^N is closed in \mathbb{R}^N if its complement, $\mathbb{R}^N \setminus A$, is open.

Examples:

- 1) \mathbb{R}^N and the empty set are closed.
- 2) $[0, 1]$ is closed in \mathbb{R} .
- 3) $(0, 1)$ is not closed in \mathbb{R} .
- 4) $\{(x_0, 0) \in \mathbb{R}^2 \mid 0 \leq x_0 \leq 1\}$ is closed in \mathbb{R}^2 .

Real analysis makes heavy use of the vague idea of being arbitrarily close to a point. One way to capture this idea uses a sequence of points converging to a given point.

Definition: A sequence in a set X is a function $x: \{1, 2, \dots\} \rightarrow X$. It is denoted by x_n or x_1, x_2, \dots .

Example: $x_n = \sin(n^2)$, $n = 1, 2, \dots$, is a sequence.

Definition: A sequence x_n in \mathbb{R}^N converges to x in \mathbb{R}^N if for every $\varepsilon > 0$, there exists an integer M such that $\|x - x_n\| < \varepsilon$, if $n \geq M$. To indicate that x_n converges to x , we write $\lim_{n \rightarrow \infty} x_n = x$, to be read as "the limit of x_n is x ."

Examples: $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, and $\lim_{n \rightarrow \infty} \frac{n-1}{n+1} = 1$. The sequence $x_n = \sin(n^2)$ does not converge.

Theorem 4.1: A subset A of \mathbb{R}^N is closed if and only if every sequence in A that converges, converges to a point in A .

Proof: Suppose that A is closed and that x_n is a sequence in A that converges to x . I must show that $x \in A$. Suppose that $x \notin A$. Because A is closed, $\mathbb{R}^N \setminus A$ is open, so that for some positive number ε , $B_\varepsilon(x) \subset \mathbb{R}^N \setminus A$. That is, $B_\varepsilon(x) \cap A$ is empty. Since $\lim_{n \rightarrow \infty} x_n = x$, there is a positive integer M such that $x_n \in B_\varepsilon(x)$, if $n \geq M$. But then $x_n \notin A$, if $n \geq M$, which is impossible since x_n is a sequence in A .

Suppose that every sequence in A that converges, converges to a point in A . If A is not closed, then $\mathbb{R}^N \setminus A$ is not open, so that there exists an $x \in \mathbb{R}^N \setminus A$, such that $B_\varepsilon(x) \cap A$ is not empty, for every $\varepsilon > 0$. Therefore, for every positive integer n , there exists an $x_n \in A$, such that $\|x - x_n\| < 1/n$. Since $\lim_{n \rightarrow \infty} x_n = x$, it follows that $x \in A$, which is impossible. This contradiction proves that A is closed. ■

Theorem 4.2: If A and B are open subsets of \mathbb{R}^N , then $A \cap B$ is open. If \mathcal{U} is a collection of open subset of \mathbb{R}^N , then $\bigcup_{U \in \mathcal{U}} U$ is open.

If A and B are closed subsets of \mathbb{R}^N , then $A \cup B$ is closed. If \mathcal{C} is a collection of closed subsets of \mathbb{R}^N , then $\bigcap_{C \in \mathcal{C}} C$ is closed.

Proof: I show that if A and B are open, then $A \cap B$ is open. If $x \in A \cap B$, there is a positive number ε_A such that $B_{\varepsilon_A}(x) \subset A$ and there exists a positive number ε_B such that $B_{\varepsilon_B}(x) \subset B$. Let $\varepsilon = \min(\varepsilon_A, \varepsilon_B)$. Then $B_{\varepsilon}(x) \subset B_{\varepsilon_A}(x) \subset A$ and $B_{\varepsilon}(x) \subset B_{\varepsilon_B}(x) \subset B$, so that $B_{\varepsilon}(x) \subset A \cap B$. Therefore $A \cap B$ is open.

I show that $\bigcup_{U \in \mathcal{U}} U$ is open. If $x \in \bigcup_{U \in \mathcal{U}} U$, then $x \in U'$, for some $U' \in \mathcal{U}$. Since U' is open, there is a positive number ε such that $B_{\varepsilon}(x) \subset U' \subset \bigcup_{U \in \mathcal{U}} U$. Therefore $\bigcup_{U \in \mathcal{U}} U$ is open.

I show that $A \cup B$ is closed if A and B are closed. By De Morgan's law, $\mathbb{R}^N \setminus (A \cup B) = (\mathbb{R}^N \setminus A) \cap (\mathbb{R}^N \setminus B)$. Since A and B are closed, $\mathbb{R}^N \setminus A$ and $\mathbb{R}^N \setminus B$ are open, so that $(\mathbb{R}^N \setminus A) \cap (\mathbb{R}^N \setminus B)$ is open by what has already been proved. Therefore $\mathbb{R}^N \setminus (A \cup B)$ is open and hence $A \cup B$ is closed.

In order to prove the last part of the theorem, notice that since C is closed, $\mathbb{R}^N \setminus C$ is open, and so $\bigcup_{C \in \mathcal{C}} (\mathbb{R}^N \setminus C)$ is open. By De Morgan's law, it follows that $\mathbb{R}^N \setminus \left(\bigcap_{C \in \mathcal{C}} C \right) = \bigcup_{C \in \mathcal{C}} (\mathbb{R}^N \setminus C)$ is open and so $\bigcap_{C \in \mathcal{C}} C$ is closed. ■

Examples: 1) The intervals $[1/n, 1]$ are closed, for $n = 1, 2, \dots$, yet

$$\bigcup_{n=1}^{\infty} [1/n, 1] = (0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$$

is not closed, so that the union of infinitely many closed sets is not necessarily closed.

2) The intervals $(-1/n, 1 + 1/n)$ are open, for $n = 1, 2, \dots$, yet

$$\bigcap_{n=1}^{\infty} (-1/n, 1 + 1/n) = [0, 1]$$

is not open, so that the intersection of infinitely many opens set is not necessarily open

Definition: If $A \subset B \subset \mathbb{R}^N$, A is open in B, if for every $x \in A$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap B \subset A$. A is closed in B, if $B \setminus A$ is open in B .

Examples:

- 1) $(0, 1] = \{x \mid 0 < x \leq 1\}$ is open in $[0, 1] = \{x \mid 0 \leq x \leq 1\}$, but is not open in \mathbb{R} .
- 2) $(0, 1]$ is closed in $(0, 2)$, but is not closed in \mathbb{R} .
- 3) $\{(x_0, 0) \mid 0 < x_0 < 1\}$ is open in $\{(x_0, 0) \mid -\infty < x_0 < \infty\}$, though it is not open in \mathbb{R}^2 .

Theorem 4.3: If $A \subset B \subset \mathbb{R}^N$, A is open in B if and only if $A = B \cap U$, where U is open in \mathbb{R}^N .

Proof: If $A = B \cap U$, where U is open in \mathbb{R}^N , then it should be clear that A is open in B .

Suppose that A is open in B . For each $x \in A$, let $\varepsilon(x)$ be a positive number such that $B_{\varepsilon(x)}(x) \cap B \subset A$. Let $U = \bigcup_{x \in A} B_{\varepsilon(x)}(x)$. Then U is open, because it is the union of open sets. Since $B_{\varepsilon(x)}(x) \cap B \subset A$, for all $x \in A$, it follows that $U \cap B \subset A$. Since $A \subset U$ and $A \subset B$, it follows that $A \subset U \cap B$. Therefore $A = B \cap U$. ■

Remarks: 1) It follows from theorem 4.3 that if $A \subset B \subset \mathbb{R}^N$, then A is closed in B if and only if $A = B \cap C$, where C is closed in \mathbb{R}^N .

2) Consequently the subset A of B is closed in B if and only if whenever a sequence a_n in A converges to a point b in B , then $b \in A$.

Recall that corresponding to the linear structure on vector spaces, there are functions, called linear, that respect the linear structure on the domain and codomain. Similarly there is a class of functions, called continuous, that respect the structure of open sets on the domain and codomain.

Definition: If $f: A \rightarrow B$ and C is a subset of B , then $f^{-1}(C) = \{x \in A \mid f(x) \in C\}$.

Definition: Let $A \subset \mathbb{R}^N$, $B \subset \mathbb{R}^M$, and $f: A \rightarrow B$. Then f is continuous if for every subset U of B that is open in B , $f^{-1}(U)$ is open in A .

Theorem 4.4: The function $f: A \rightarrow B$ is continuous if and only if for every subset C of B that is closed in B , $f^{-1}(C)$ is closed in A .

Proof: It should be clear that $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$. If f is continuous and C is closed in B , then $B \setminus C$ is open in B and hence by the definition of continuity $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$ is open in A , and so $f^{-1}(C)$ is closed in A .

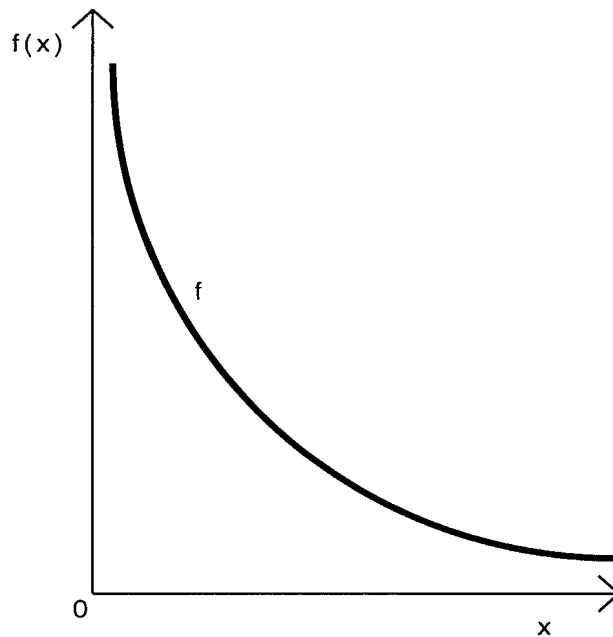
Suppose that $f^{-1}(C)$ is closed in A whenever C is closed in B . If U is a subset of B that is

open in B, then $B \setminus U$ is closed in B, so that $f^{-1}(B \setminus U) = A \setminus f^{-1}(U)$ is closed in A and hence $f^{-1}(U)$ is open in A, and so f is continuous. ■

Theorem 4.5: The function $f: A \rightarrow B$ is continuous, if and only if $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$, whenever x_n is a sequence in A that converges to a point in A.

Proof: Suppose that f is continuous and that x_n is a sequence in A that converges to a point x in A. If ϵ is a positive number, then $B_\epsilon(f(x)) \cap B$ is open in B, so that $f^{-1}(B_\epsilon(f(x)))$ is open in A. Since $x \in f^{-1}(B_\epsilon(f(x)))$, there is a positive number δ such that $A \cap B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Since $\lim_{n \rightarrow \infty} x_n = x$, there is a positive integer K such that $x_n \in B_\delta(x)$, if $n \geq K$. Therefore if $n \geq K$, $x_n \in f^{-1}(B_\epsilon(f(x)))$, and hence $\|f(x_n) - f(x)\| < \epsilon$. Hence $\lim_{n \rightarrow \infty} f(x_n) = f(x)$.

Suppose that if for any sequence x_n in A that converges to a point x in A it is true that $\lim_{n \rightarrow \infty} f(x_n) = f(x)$. In order to show that f is continuous, let C be a subset of B that is closed in B. I must show that $f^{-1}(C)$ is closed in A, that is, that $A \setminus f^{-1}(C)$ is open in A. If $A \setminus f^{-1}(C)$ is not open in A, there is an $x \in A \setminus f^{-1}(C)$ such that for every positive number ϵ , $B_\epsilon(x) \cap f^{-1}(C)$ is not empty. Therefore, for every positive integer n, there exists an $x_n \in f^{-1}(C)$ such that $\|x_n - x\| < 1/n$. Hence $\lim_{n \rightarrow \infty} x_n = x$, so that $\lim_{n \rightarrow \infty} f(x_n) = f(x)$. Since $f(x_n) \in C$, for all n, and C is closed in B, $f(x) \in C$. Thus $x \in f^{-1}(C)$, contrary to hypothesis. Therefore f is continuous. ■

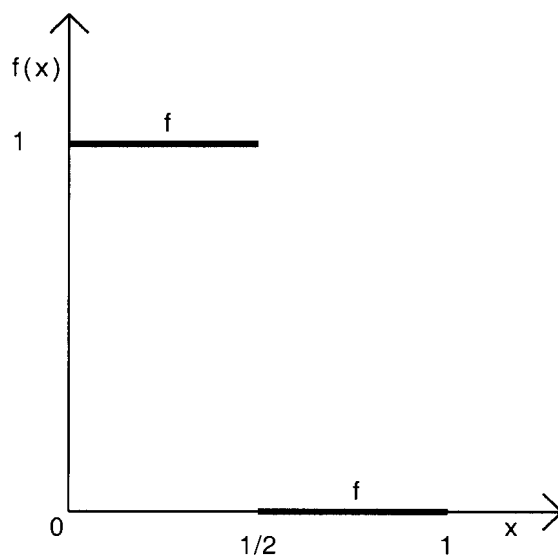
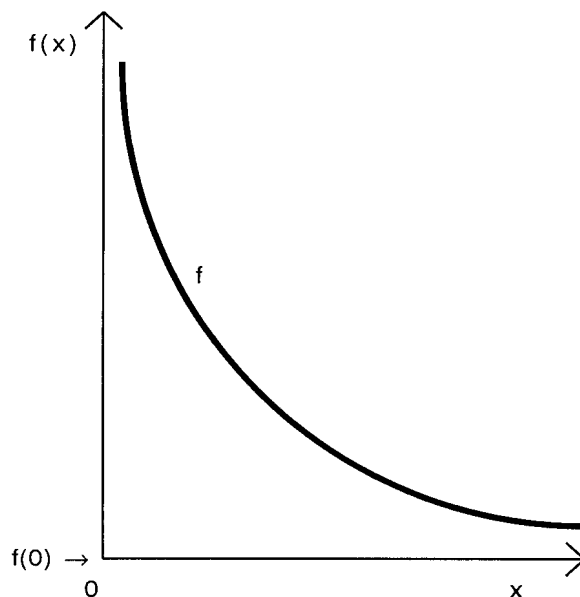


Examples: 1) Let $f: (0, \infty) \rightarrow (0, \infty)$ be defined by $f(x) = 1/x$. Then f is continuous.

2) Let $f: [0, \infty) \rightarrow [0, \infty)$ be defined by

$$f(x) = \begin{cases} 1/x, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then f is not continuous.



3) Let $f: [0, 1] \rightarrow [0, 1]$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1/2, \\ 0, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Then f is not continuous.

People speak of continuity at a single point in the domain of a function.

Definition: The function $f: A \rightarrow B$ is continuous at $x \in A$ if for every positive number ϵ there is a positive number δ such that $\|f(x) - f(y)\| < \epsilon$, whenever $\|x - y\| < \delta$.

Theorem 4.6: The function $f: A \rightarrow B$ is continuous at x if and only if for every sequence x_1, x_2, \dots in A that converges to $x \in A$, $\lim_{n \rightarrow \infty} f(x_n) = f(x)$.

Proof: The argument should be clear, given what was presented earlier. ■

Theorem 4.7: The function $f: A \rightarrow B$ is continuous on A if and only if it is continuous at every point in A .

Proof: This is so, because by theorem 4.5 f is continuous if and only if $f(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} f(x_n)$, for every sequence x_n in A converging to a point in A . ■

The Completeness Property of the Real Numbers

A crucial property of the real numbers is, in figurative language, that they contain no gaps. More accurately, the property is that every convergent sequence of numbers converges to some number. In order to make sense of this last statement, we need a definition of what it means for a sequence to converge without necessarily converging to something.

Definition: A sequence of numbers x_1, x_2, \dots is said to be Cauchy if for every positive number ϵ , there exists a positive integer M such that $|x_n - x_m| < \epsilon$, if $n > M$ and $m > M$.

The Completeness Property of the Real Numbers: If x_1, x_2, \dots is a Cauchy sequence of numbers, then there is a number x such that $\lim_{n \rightarrow \infty} x_n = x$.

The completeness property applies not only to numbers but to vectors of numbers.

Definition: A sequence of vectors x_1, x_2, \dots in \mathbb{R}^N is said to be Cauchy if for every positive number ϵ , there exists a positive integer M such that $\|x_n - x_m\| < \epsilon$, if $n > M$ and $m > M$.

Lemma 4.8: If x_1, x_2, \dots is a Cauchy sequence of vectors in \mathbb{R}^N , then there is a vector $y \in \mathbb{R}^N$, such that $\lim_{n \rightarrow \infty} x_n = y$.

Proof: For each k , the vector x_k may be written as $x_k = (x_{k1}, \dots, x_{kN})$. Since the sequence x_n is Cauchy, it follows that for each $n = 1, \dots, N$, the sequence x_{kn} is Cauchy, since $|x_{kn} - x_{mn}| \leq \|x_k - x_m\|$, for all k and m . By the completeness of the real numbers, for each n , there is a number y_n such that $\lim_{k \rightarrow \infty} x_{kn} = y_n$. Let $y = (y_1, \dots, y_N)$. Then $\lim_{k \rightarrow \infty} x_k = y$. ■

An alternative way to express the completeness property of the real numbers is to say that every set of numbers with an upper bound has a least upper bound.

Definitions: If X is a set of numbers, the number b is an upper bound for X if $x \leq b$, for all $x \in X$. If X has an upper bound, X is said to be bounded from above. The number c is said to be a least upper bound for X , if c is an upper bound for X and $c \leq b$, for any upper bound b for X .

The least upper bound for X is denoted $\text{lub}(X)$ or $\text{sup}(X)$, which is read as “the supremum of X .” In an analogous fashion, we may define “bounded from below,” “lower bound,” and “greatest lower bound.” The greatest lower bound is written as $\text{glb}(X)$ or as $\text{inf}(X)$, read as “the infimum of X .” Clearly $\text{glb}(X) = -\text{lub}(-X)$, where $-X = \{-x \mid x \in X\}$, so that a set that is bounded from below has a greatest lower bound if and only if a set that is bounded from above has a least upper bound.

Least Upper Bound Property: Any set of numbers that is bounded from above has a least upper bound.

Although I do not provide a proof of the following theorem, it is not hard to make one up.

Theorem 4.9: The least upper bound property is equivalent to the completeness property.

The next main result, the Bolzano-Weierstrass theorem, is useful for proving the existence of things, such as the solution of equations. It is used when it is possible to find something that has some property approximately. The theorem then allows one to say that by going to the limit there exists something having the property exactly. In order to state the theorem, we need a few new concepts.

Definition: A subset A of \mathbb{R}^N is bounded if there is a positive number b such that $\|x\| \leq b$, for all $x \in A$.

Definition: A subset A of \mathbb{R}^N is compact if it is closed and bounded.

Definition: A subsequence of the sequence x_1, x_2, \dots is a sequence of the form x_{n_1}, x_{n_2}, \dots , where $n_{k+1} > n_k$, for all k .

Theorem 4.10: If x_1, x_2, \dots is a sequence in \mathbb{R}^N that converges to x , then every subsequence of x_1, x_2, \dots converges to x .

Proof: If ϵ is a positive number, let M be a positive integer such that $\|x_n - x\| < \epsilon$, if $n > M$. If x_{n_1}, x_{n_2}, \dots is a subsequence of x_1, x_2, \dots , then $n_k \geq k$, for all k , so that $\|x_{n_k} - x\| < \epsilon$, if $k > M$. Therefore x_{n_1}, x_{n_2}, \dots converges to x . ■

Theorem 4.11: Every convergent sequence in \mathbb{R}^N is bounded.

Proof: Let x_1, x_2, \dots be a sequence in \mathbb{R}^N that converges to x . There is a positive integer M such that $\|x_n - x\| \leq 1$, if $n > M$. Then $\|x_n\| \leq \max(\|x_1\|, \dots, \|x_M\|, \|x\| + 1)$, for all n , so that x_1, x_2, \dots is bounded. ■

Theorem 4.12 (Bolzano-Weierstrass): A subset A of \mathbb{R}^N is compact, if and only if every sequence in A has a subsequence that converges to a point in A .

Proof: I show that A is closed and bounded, if every sequence in A has a subsequence that converges to a point in A . If A is unbounded, then for every positive integer n , there is an $x_n \in A$ such that $\|x_n\| > n$. Let x_{n_1}, x_{n_2}, \dots be a subsequence of x_1, x_2, \dots that converges to a point in A . Then $\|x_{n_k}\| > n_k \geq k$, so that $\|x_{n_k}\|$ diverges to infinity as k goes to infinity. This is impossible, because x_{n_1}, x_{n_2}, \dots converges and by theorem 4.11 every convergent sequence is bounded. This proves that A is bounded. I show that A is closed. If A is not closed, there is a sequence x_1, x_2, \dots in A that converges to a point y not in A . Let x_{n_1}, x_{n_2}, \dots be subsequence of x_1, x_2, \dots that converges to a point $x \in A$. This is impossible, since by theorem 4.10, x_{n_1}, x_{n_2}, \dots must also converge to y , which is not in A . This contradiction proves that A is closed. Since A is closed and bounded, it is compact.

I show that if A is compact, then every sequence in A has a subsequence that converges to a point in A . Let x_1, x_2, \dots be a sequence in A . Because A is bounded, it is contained in a closed cube $C_1 = \{y \in \mathbb{R}^N \mid -b \leq y_n \leq b, \text{ for } n = 1, \dots, N\}$, where b is some positive number. Divide C_1

in half along each dimension, obtained 2^N congruent closed subcubes, each with edges of length $2b/2 = b$. Since there are only finitely many subcubes, one of these contains the point x_n , for infinitely many integers n . Call this cube C_2 . Suppose that the closed cubes C_1, \dots, C_k have been defined, where $C_1 \supset C_2 \supset \dots \supset C_k$ and, for each k , C_k has edges of length $2b/2^{k-1} = b2^{-k+2}$ and C_k contains x_n , for infinitely many n . Divide C_k in half along each dimension, obtaining 2^N congruent closed subcubes, one of which must contain x_n , for infinitely many values of n . Call this cube C_{k+1} . Each edge of C_{k+1} has length $b2^{-k+2}/2 = b2^{-(k+1)+2}$. I have defined by induction on K a sequence of closed cubes C_1, C_2, \dots such that

- 1) for all k , C_k contains x_n , for infinitely many n ,
- 2) $C_1 \supset C_2 \supset \dots$, and
- 3) for each k , each edge of C_k has length $b2^{-k+1}$.

I define a subsequence x_{n_k} of x_n by induction on k . Let $n_1 = 1$. Suppose that n_1, \dots, n_k have been defined such that $x_{n_k} \in C_k$, for $k = 1, \dots, K$ and $n_1 < n_2 < \dots < n_k$. Since C_{k+1} contains infinitely many members of the sequence x_1, x_2, \dots , there exists an $x_{n_{k+1}} \in C_{k+1}$ such that $n_{k+1} > n_k$. I have defined x_{n_1}, x_{n_2}, \dots such that $x_{n_k} \in C_k$ and $n_k < n_{k+1}$, for all k .

Since the diameter of C_k is $b\sqrt{N}2^{-k+1}$, which converges to 0 as k goes to infinity, it follows that $\lim_{K \rightarrow \infty} \sup_{k \geq K, m \geq K} ||x_{n_k} - x_{n_m}|| = 0$ and so the subsequence x_{n_1}, x_{n_2}, \dots is Cauchy. By the completeness property of the real numbers, there exists an $x \in \mathbb{R}^N$ such that $\lim_{k \rightarrow \infty} x_{n_k} = x$. Since A is closed and $x_{n_k} \in A$, for all k , it follows that $x \in A$. That is, the subsequence x_{n_1}, x_{n_2}, \dots converges to a point in A . ■

The above proof is illustrated in the next figure.

Theorem 4.13 (Heine-Borel): A subset A of \mathbb{R}^N is compact if and only if every open cover of A contains a finite subcover.

Proof: I show that A is closed and bounded if every open cover of A contains a finite subcover. In order to show that A is closed, let $x \in \mathbb{R}^N \setminus A$ and for $m = 1, 2, \dots$, let

$U_m = \{y \in \mathbb{R}^N \mid \|y - x\| > 1/m\}$. U_1, U_2, \dots is an open cover of A , because $\bigcup_{m=1}^{\infty} U_m = \mathbb{R}^N \setminus \{x\}$ and

$A \subset \mathbb{R}^N \setminus \{x\}$. Therefore for some M , $A \subset \bigcup_{m=1}^M U_m = U_M$. It follows that $B_{1/M}(x) \cap A = \emptyset$. Hence

$\mathbb{R}^N \setminus A$ is open and so A is closed. In order to show that A is bounded, let $U_m = \{x \in \mathbb{R}^N \mid \|x\| < m\}$, for $m = 1, 2, \dots$. Then U_1, U_2, \dots is an open cover of A , so that for

some M , $A \subset \bigcup_{m=1}^M U_m = U_M$. Hence $\|x\| < M$, for all $x \in A$ and so A is bounded.

I show that if A is compact then every open cover \mathcal{U} of A contains a finite subcover. Suppose that \mathcal{U} contains no finite subcover of A . Since A is bounded, there is a closed cube C_1 in \mathbb{R}^N that contains A . As in the previous proof, divide C_1 into 2^N congruent closed subcubes that intersect along sides. One of the subcubes, C_2 , is such that $A \cap C_2$ is not covered by a finite subcover of \mathcal{U} , since if there were no such subcube, the intersection of A with each subcube would have a finite subcover, and the union of these finitely many subcovers would be a finite subcover of A , contrary to hypothesis. Suppose that the closed cubes C_1, \dots, C_K have been defined such that $C_1 \supset C_2 \supset \dots \supset C_K$ and, for each $k = 1, \dots, K$, $A \cap C_k$ has no finite subcover. The cube C_K is the union of 2^N congruent cubes that intersect along sides. One of these subcubes, C_{K+1} , is such that $A \cap C_{K+1}$ has no finite subcover. By induction on K , I have defined a sequence of cubes C_1, C_2, \dots such that $C_1 \supset C_2 \supset \dots$ and $\lim_{k \rightarrow \infty} \text{diam}(C_k) = 0$ and for each k , $A \cap C_k$ has no finite subcover.

Since $A \cap C_k$ has no finite subcover, it is not empty and so contains a point x_k . Since $\lim_{k \rightarrow \infty} \text{diam}(C_k) = 0$, the sequence x_1, x_2, \dots is Cauchy and so by the completeness of the real numbers converges to a point x . Since A is compact, it is closed and so $x \in A$. Since \mathcal{U} covers A , $x \in U$, for some $U \in \mathcal{U}$. Since U is open, there is a positive number ε such that $B_\varepsilon(x) \subset U$. Because $x_k \in C_k$, for all k , and $\lim_{k \rightarrow \infty} x_k = x$ and $\lim_{k \rightarrow \infty} \text{diam}(C_k) = 0$, there is a positive integer K such that $C_K \subset B_\varepsilon(x) \subset U$. Therefore U covers C_K and hence $A \cap C_K$, contrary to the hypothesis that $A \cap C_k$ has no finite subcover. This contradiction proves that every open cover of A contains a finite subcover. ■

I now apply the Heine-Borel theorem to prove that every continuous function achieves a maximum and a minimum on a compact set.

Corollary 4.14: If A is a compact subset of \mathbb{R}^N and $f: A \rightarrow \mathbb{R}^M$ is continuous, then $f(A) = \{f(x) \mid x \in A\}$ is compact.

Proof: Let \mathcal{U} be an open cover of $f(A)$. Since f is continuous, $f^{-1}(U)$ is open in A , for every $U \in \mathcal{U}$. By theorem 4.3, for each $U \in \mathcal{U}$, there is an open subset V_U of \mathbb{R}^N such that $A \cap V_U = f^{-1}(U)$. Then $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$ is an open cover of A . Since A is compact, \mathcal{V} has a finite subcover V_{U_1}, \dots, V_{U_k} of A . Then U_1, \dots, U_k is an open cover of $f(A)$. This proves that every open cover of $f(A)$ has a finite subcover and hence that $f(A)$ is compact. ■

Theorem 4.15: If B is a compact and non-empty subset of \mathbb{R} , then $\text{glb}(B) \in B$ and $\text{lub}(B) \in B$.

Proof: Since B is compact, it is bounded and hence $\text{glb}(B)$ and $\text{lub}(B)$ exist. Since B is non-empty, it follows from the definition of $\text{lub}(B)$ that there is a sequence x_1, x_2, \dots in B such that $\lim_{n \rightarrow \infty} x_n = \text{lub}(B)$. Since B is closed, $\lim_{n \rightarrow \infty} x_n \in B$, and so $\text{lub}(B) \in B$. A similar argument proves that $\text{glb}(B) \in B$. ■

Theorem 4.16: If A is a compact and non-empty subset of \mathbb{R}^N and $f: A \rightarrow \mathbb{R}$ is continuous, then there exist \underline{x} and \overline{x} in A such that $f(\underline{x}) \leq f(x) \leq f(\overline{x})$, for all $x \in A$.

Proof: Since A is compact and f is continuous, $f(A)$ is compact by corollary 4.14. Therefore theorem 4.15 implies that $\text{glb}(f(A)) \in f(A)$ and $\text{lub}(f(A)) \in f(A)$. Let \underline{x} and \overline{x} be members of A such that $f(\underline{x}) = \text{glb}(f(A))$ and $f(\overline{x}) = \text{lub}(f(A))$. Then $f(\underline{x}) \leq f(x) \leq f(\overline{x})$, for all $x \in A$. ■

This theorem says that a continuous function defined on a compact set achieves its minimum and maximum on the set.

The next two theorems are useful applications of the completeness of the real numbers.

Definition: A sequence of numbers x_n is non-decreasing if $x_{n+1} \geq x_n$, for all n . A sequence of numbers x_n is non-increasing if $x_{n+1} \leq x_n$, for all n .

Theorem (Monotone Convergence Theorem) 4.17: Every non-decreasing sequence of numbers that is bounded from above converges, and every non-increasing sequence of numbers that is bounded from below converges.

Proof: Let x_n be a non-decreasing sequence of numbers that is bounded from above. The completeness of the real numbers implies that $a = \sup_{n=1,2,\dots} x_n < \infty$. Let ε be a positive number. By the definition of the supremum, there exists an N such that $x_N \geq a - \varepsilon$. Since x_n is non-decreasing, $a - \varepsilon \leq x_n \leq a$, for all $n \geq N$. Therefore $\lim_{n \rightarrow \infty} x_n = a$.

A similar argument applies to any non-increasing sequence of numbers that is bounded from below. ■

Theorem (Intermediate Value Theorem) 4.18: If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function, where $a < b$, and if r is a number such that $f(a) \leq r \leq f(b)$ or $f(b) \leq r \leq f(a)$, then there is a number $c \in [a, b]$ such that $f(c) = r$.

Proof: Without loss of generality, assume that $f(a) \leq f(b)$. Let $S = \{x \in [a, b] \mid f(x) \leq r\}$. S is not empty, because $a \in S$. Since S is bounded above by b , the completeness of the real numbers implies that $c = \sup S$ exists and $c \leq b$. I show that $f(c) = r$. By the definition of $\sup S$, there is a sequence x_n in S such that $\lim_{n \rightarrow \infty} x_n = c$. Because $x_n \in S$, $f(x_n) \leq r$, for all n . Since f is continuous and x_n converges to c , $f(c) \leq r$. Suppose that $f(c) < r$. Then $c < b$, since $f(b) \geq r$. Since f is continuous, there exists a number ε such that $0 < \varepsilon < b - c$ and $f(x) < r$, if $|x - c| < \varepsilon$. But then $\sup S \geq c + \varepsilon$, which contradicts $c = \sup S$. ■

Corollary 4.19: If $g: [a, b] \rightarrow \mathbb{R}$ and $h: [a, b] \rightarrow \mathbb{R}$ are continuous functions, where $a < b$, and if $g(a) > h(a)$ and $g(b) < h(b)$, then there is a number $c \in (a, b)$ such that $g(c) = h(c)$.

Proof: Apply the intermediate value theorem to the function $f(x) = g(x) - h(x)$, with $r = 0$. ■