Lecture #15

I now derive some properties of $\int_0^T g(t)\,dW_t$. In order to do so, I need two theorems, which I do not prove, though the proof of the second is easy.

Theorem 15.1: If $(S, \mathbb{S}, P)$ is a probability space and $x : S \to R$ is a random variable such that $E(x^2) < \infty$, then $|x|$ is integrable and $E|x| \leq ||x||_2$.

Theorem 15.2: If $(S, \mathbb{S}, P)$ is a probability space and $x : S \to R$ is an integrable random variable, then $|Ex| \leq E|x|$.

I may now prove the following.

Theorem 15.3: $E\left[\int_0^T g(t)\,dW_t\right] = 0$.

Proof: By definition, $\int_0^T g(t)\,dW_t = \lim_{N \to \infty} V_N$, where V_N is as above. By equation 14.3, $E\left[V_N\right] = 0$. Let $V = \int_0^T g(t)\,dW_t$. Then

$$0 \leq |EV| = \left|EV - E\left[V_N\right]\right| = \left|E\left[V - V_N\right]\right| \leq E\left|V - V_N\right| \leq \left\|V - V_N\right\|_2,$$

where the second inequality follows from theorem 15.2 and the third inequality follows from theorem 15.1. Since $\lim_{N \to \infty} \left\|V - V_N\right\|_2 = 0$, it follows that $EV = 0$. ■

Theorem 15.4: $||\int_0^T g(t)\,dW_t||_2^2 = \int_0^T ||g(t)||_2^2 dt$.

Proof: Let $V_N(s)$ be defined by equation 14.2. By equation 14.5,

$$\lim_{N \to \infty} ||V_N||_2^2 = \int_0^T E[g(t)^2]\,dt = \int_0^T ||g(t)||_2^2 dt.$$

By the definition of $\int_0^T g(t)\,dW_t$,

$$\lim_{N \to \infty} ||V_N - \int_0^T g(t)\,dW_t||_2 = 0.$$

By the triangle inequality for the norm $||.||_2$,

$$\left| ||V_N||_2 - ||\int_0^T g(t)\,dW_t||_2 \right| \le ||V_N - \int_0^T g(t)\,dW_t||_2 .$$

Hence

$$\int_0^T ||g(t)||_2^2 dt = \lim_{N \to \infty} ||V_N||_2^2 = ||\int_0^T g(t)\,dW_t||_2^2 . \qquad \blacksquare$$

The next theorem should be obvious.

<u>Theorem</u> 15.5: If a and b are numbers and $f(t) : S \to R$ and $g(t) : S \to R$ are stochastic processes that have properties (1) - (3), then

$$\int_0^T [af(t) + bg(t)]\,dW_t = a\int_0^T f(t)\,dW_t + b\int_0^T g(t)\,dW_t .$$

If for all t, f(t, s) = 0 with probability 1, then

$$\int_0^T f(t)\,dW_t = 0 .$$

The stochastic integral $\int_0^T g(t)\,dW_t$ inherits properties (1) - (3) of the stochastic process g(t). It should be plausible that the random variable $\int_0^T g(t)\,dW_t$ is measurable with respect to $\mathfrak{S}_T$. We know that it belongs to $L_2(S, \mathfrak{S}, P)$. As a function from $[0, \infty)$ to $L_2(S, \mathfrak{S}, P)$, $\int_0^T g(t)\,dW_t$ is continuous with respect to T, for suppose that T_n is a sequence in $[0, \infty)$ converging to T. Then

$$\lim_{n \to \infty} \left\| \int_0^{T_n} g(t)\,dW_t - \int_0^T g(t)\,dW_t \right\|_2^2 = \lim_{n \to \infty} \left\| \int_T^{T_n} g(t)\,dW_t \right\|_2^2$$

$$= \lim_{n \to \infty} \left| \int_T^{T_n} ||g(t)||_2^2 \right| = 0 ,$$

where the second equation follows from theorem 15.4 and last equation applies because $||g(t)||_2^2$ is a continuous function of t by assumption 3.

Stochastic Differential Equations

We now turn to the study of stochastic processes x(T, s) that satisfy the equation

$$x(T,\ s) = x(0,\ s) + \int_0^T \mu(t,\ x(t,\ s))\,dt + \int_0^T \sigma(t,\ x(t,\ s))\,dW_t(s)\,, \qquad (15.1)$$

for all T and s, where T varies over $[0, \infty)$, $\mu : [0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$, and $\sigma : [0, \infty) \times (-\infty, \infty) \to [0, \infty)$. Such a process is called an Ito process. Equation 15.1 is called a stochastic differential equation or an Ito differential equation. Notice that if $\sigma = 0$, then equation 15.1 becomes

$$x(T,\ s) = x(0,\ s) + \int_0^T \mu(t,\ x(t,\ s))\,dt,$$

which, for each s, is the same as the ordinary differential equation

$$\frac{dx(t,\ s)}{dt} = \mu(t,\ x(t,\ s))$$

with the initial condition x(0, s). For this equation to have a unique solution, we need to assume that μ is continuous and satisfies what is known as a Lipschitz condition with respect to x. The <u>Lipschitz condition</u> is that there is a positive number K such that

$$|\mu(t,\ x) - \mu(t,\ y)| \le K|x - y|,$$

for all x and y. The Lipschitz condition is stronger than continuity and applies if μ is differentiable with respect to x and the derivative is uniformly bounded, which means that there is some $b > 0$ such that $\left|\frac{\partial \mu(t,\ x)}{\partial x}\right| < b$, for all t and x. I make a similar assumption about σ. More formally,

<u>Assumption</u> 15.6: Both μ and σ are continuous with respect to t and x and both satisfy a Lipschitz condition with respect to x.

<u>Theorem</u> 15.7: Under assumption 15.6, the stochastic differential equation 15.1 has a unique solution x(t, s) with initial condition x(0, s), provided the function of s, x(0, s), belongs to $L_2(S, \mathcal{S}, P)$ and is measurable with respect to $\mathcal{S}_0$. Furthermore $x(t,\ s) \in L_2(S, \mathcal{S}, P)$, for every t, and x(t, s) is a continuous function of t with probability 1.

I do not attempt a proof of this important theorem.

We can imagine finding an approximate solution to a stochastic differential equation by dividing time into tiny intervals of length Δ and if the non-negative integer $n = n(t, \Delta)$ is such that $n\Delta < t \leq (n+1)\Delta$, we could form the function

$$x_\Delta(t, s) = x(0, s) + \sum_{k=0}^{n-1} \mu(k\Delta, x_\Delta(k\Delta, s))\,\Delta + \mu(n\Delta, x_\Delta(n\Delta, s))\,(t - n\Delta)$$

$$+ \sum_{k=0}^{n-1} \sigma(k\Delta, x_\Delta(k\Delta, s))\left(W_{(k+1)\Delta}(s) - W_{k\Delta}(s)\right) + \sigma(n\Delta, x_\Delta(n\Delta, s))\left(W_t(s) - W_{n\Delta}(s)\right),$$

where $W_0(s) = 0$ and where the functions $x_\Delta(k\Delta, s)$ are defined by induction on k beginning with $x_\Delta(0\Delta, s) = x(0, s)$, for $k = 0$, and by the formula

$$x_\Delta((k+1)\Delta, s) = x_\Delta(k\Delta, s) + \mu(k\Delta, x_\Delta(k\Delta, s))\,\Delta$$

$$+ \sigma(k\Delta, x_\Delta(k\Delta, s))\left(W_{(k+1)\Delta}(s) - W_{k\Delta}(s)\right)$$

The functions $x_\Delta(t, s)$ are approximate solutions to the stochastic differential equation that converge to the actual solution.

The point I want to make here is that from the equations defining $x_\Delta(t, s)$ and the central limit theorem, we can guess that the differences $W_{(k+1)\Delta}(s) - W_{k\Delta}(s)$ need not be normally distributed. All that matters is that they be independent of $\mathcal{S}_{k\Delta}$, have mean zero, and that

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \mathrm{Var}\left[W_{t+\Delta} - W_t\right] = 1 .$$

The difference $W_{(k+1)\Delta} - W_{k\Delta}$ are to be thought of as the dW_t encountered in lecture 14. We can ignore terms with a variance of order dt^2 or any order higher than dt. In order to see that this is so, recall that the stochastic integral is the limit of sums of independent random variables with variance of order dt, where dt is the time elapsed between the moments when the random variables occur. If the integral is from 0 to T there are T/dt such random variables. The variance of sum of all these random variables is the sum of their variances, which is of order $(T/dt)dt = T$. Suppose that the variances of the individual random variables was of order dt^a, where $a > 1$. Then the variance of the sum of all the random variables would of order $(T/dt)dt^a = Tdt^{a-1}$, which goes to zero as dt goes to zero. Therefore integral of such terms would be 0.

As an example of the application of the idea just explained, suppose we have an equation of the form

$$dx(t, s) = \mu(t, x(t, s))dt + \sigma(t, x(t, s))dZ_t(s) ,$$

where $Z_t(s)$ is a stochastic process such that $Z_{t+\Delta}(s) - Z_t(s)$ is independent of $\mathfrak{S}_t$, $E\left[Z_{t+\Delta} - Z_t\right] = 0$, and $\mathrm{Var}\left[Z_{t+\Delta} - Z_t\right] = \Delta^2$, for small Δ. Then because Δ^2 is another order of small than Δ we can ignore $Z_t($ and the stochastic differential equation has the same solutions as does the ordinary differential equation

$$\frac{dx(t)}{dt} = \mu(t, x(t, s)) .$$

To take another application of the same idea, consider the stochastic process defined by the equation

$$dx(t, s) = \mu(t, x(t, s))dt + \sigma(t, x(t, s))dW_t(s) + \gamma(t, x(t, s))\left(dW_t(s)\right)^2,$$

where $W_t(s)$ is the Weiner process. Since the mean of $\left(dW_t\right)^2$ is dt, we can rewrite this equation as

$$dx(t, s) = [\mu(t, x(t, s)) + \gamma(t, x(t, s))]dt + \sigma(t, x(t, s))dW_t(s)$$

$$+ \gamma(t, x(t, s))\left[\left(dW_t(s)\right)^2 - dt\right].$$

The term $\left[\left(dW_t(s)\right)^2 - dt\right]$ has mean zero and variance $3(dt)^2$, since $dW_t(s)$ is normally distributed with mean 0 and variance dt. Since $\lim_{dt \to 0} \frac{1}{dt}(dt)^2 = 0$, we may ignore the term $\gamma(t, x(t, s))\left[\left(dW_t(s)\right)^2 - dt\right]$ and the write the equation as

$$dx(t, s) = [\mu(t, x(t, s)) + \gamma(t, x(t, s))]dt + \sigma(t, x(t, s))dW_t(s) .$$

The next topic is Ito's lemma. Suppose that x(t, s) is an Ito process satisfying the equation

$$dx(t, s) = \mu(t, x(t, s))dt + \sigma(t, x(t, s))dW_t(s) .$$

Let f(t, x) be a twice differentiable function of t and x and let y(t, s) = f(t, x(t, s)). Ito's lemma shows that y(t, s) is also an Ito process. In order to see how the argument works, we apply Taylor's theorem to the function f to obtain

$$dy(t, s) = \frac{\partial f(t, x(t, s))}{\partial t}dt + \frac{\partial f(t, x(t, s))}{\partial x}dx(t, s)$$

$$+ \frac{1}{2}\frac{\partial^2 f(t,\ x(t,\ s))}{\partial t^2}dt^2 + \frac{\partial^2 f(t,\ x(t,\ s))}{\partial x \partial t}dx(t,\ s)dt + \frac{1}{2}\frac{\partial^2 f(t,\ x(t,\ s))}{\partial x^2}(dx(t,\ s))^2$$

$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\Big[\mu(t,\ x(t,\ s))\,dt + \sigma(t,\ x(t,\ s))\,dW_t(s)\Big]$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial x \partial t}dt\Big[\mu(t,\ x(t,\ s))\,dt + \sigma(t,\ x(t,\ s))\,dW_t(s)\Big]$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big[\mu(t,\ x(t,\ s))^2 dt^2 + 2\mu(t,\ x(t,\ s))\,\sigma(t,\ x(t,\ s))\,dtdW_t(s)$$

$$+ \sigma(t,\ x(t,\ s))^2\big(dW_t(s)\big)^2\Big]$$

$$= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right]dt$$

$$+ \left[\frac{1}{2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial t}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mu^2\right]dt^2 + \frac{\partial f}{\partial x}\sigma dW_t$$

$$+ \left[\frac{\partial^2 f}{\partial x \partial t}\sigma + \frac{\partial^2 f}{\partial x^2}\mu\sigma\right]dtdW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\Big[\big(dW_t\big)^2 - dt\Big].$$

The stochastic terms are multiples of dW_t, $dtdW_t$, and $\Big[(dW_t)^2 - dt\Big]$. The variance of dW_t is dt. That of $dtdW_t$ is dt^3, and that of $(dW_t)^2 - dt$ is $3dt^2$. As argued earlier, the stochastic terms with variance of order dt^2 and dt^3 can be ignored. We can also ignore terms with a deterministic differential of order dt^2. After dropping the terms of order dt^2 and dt^3, we are left with the stochastic differential equation

$$dy(t,\ s) = \left[\frac{\partial f(t,\ x(t,\ s))}{\partial t} + \frac{\partial f(t,\ x(t,\ s))}{\partial x}\mu(t,\ x(t,\ s))\right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 f(t,\ x(t,\ s))}{\partial x^2}\sigma^2(t,\ x(t,\ s))\right]dt + \frac{\partial f(t,\ x(t,\ s))}{\partial x}\sigma(t,\ x(t,\ s))\,dW_t(s)\,.$$

That y(t, s) satisfies this stochastic differential equation is known as Ito's lemma.

Example: What is known as geometric Brownian motion satisfies the stochastic differential equation

$$dx(t,\ s) = \mu x(t,\ s)dt + \sigma x(t,\ s)dW_t(s)\ ,$$

where μ is a constant and σ is a positive constant. The non-stochastic version of this equation is $dx(t)/dt = \mu x(t)$, which has the solution $x(t) = e^a e^{\mu t}$, where a is an arbitrary constant. This solution may be found by considering the corresponding differential equation for $y(t) = \ln(x(t))$, which is $dy(t)/dt = \mu$. The solution of this differential equation is $y(t) = a + \mu t$. Taking the exponential of this equation, we obtain the equation $x(t) = e^a e^{\mu t}$. Let us do the same thing in the stochastic case and let $y(t, s) = \ln x(t, s)$ and apply Ito's lemma with $f(t, x) = \ln x$. Notice that the function $\mu(t, x)$ is μx and the function $\sigma(t, x)$ is σx. Then

$$dy(t, s) = \left[\frac{1}{x(t, s)}\mu x(t, s) + \frac{1}{2}\left(\frac{-1}{x(t, s)^2}\right)\sigma^2 x(t, s)^2\right]dt + \frac{1}{x(t, s)}\sigma x(t, s)dW_t(s)$$

$$= \left[\mu - \frac{1}{2}\sigma^2\right]dt + \sigma dW_t(s)\,.$$

The solution of this equation is

$$y(t, s) = a + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t(s)\,.$$

The corresponding solution for x(t, s) is

$$x(t, s) = e^a e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} e^{\sigma W_t(s)}\,.$$

Stochastic Optimal Control Theory

I now discuss briefly optimal control theory, where the differential equation to be controlled is stochastic rather than deterministic. Consider the problem

$$\begin{aligned}
&\max_u E\left[\int_0^T f(x(t, s), u(t, s))dt\right] \\
&\text{s.t. } dx(t, s) = \mu(x(t, s), u(t, s))\,dt \\
&\quad + \sigma(x(t, s), u(t, s))\,dW_t(s), \\
&\qquad \text{for all s and t, and} \\
&\qquad x(0, s) \text{ is given.}
\end{aligned} \tag{15.2}$$

The maximization is over stochastic processes u(t, s) such that u(t, s) is measurable with respect to $\mathfrak{S}_t$, for all t, $u(t, s) \in L_2(S, \mathfrak{S}, P)$, for all t, and the mapping from t to u(t, s) is a continuous function from [0, T] to $L_2(S, \mathfrak{S}, P)$. That is, the control at time t depends only on information available at time t. It follows that the functions x(t, s) are measurable with respect to $\mathfrak{S}_t$, for all t. Note that the expectations in the objective function is over all states. It is not conditional on the information available at time 0. One can add a terminal condition at time T, but I ignore this possibility.

Assume the following. (These assumptions are stronger than what is needed.)

Assumptions 15.8:

1) $f(x, u)$ is a continuous function.

2) Both $\mu(x, u)$ and $\sigma(x, u)$ satisfy a Lipschitz condition with respect to x and u and hence are continuous with respect to x and u.

3) The function of s, $x(0, s)$ belongs to $L_2(S, \mathcal{S}, P)$ and is measurable with respect to $\mathcal{S}_0$.

Notice that if $\sigma(x, u)$ is Lipschitz and $x(t, s)$ and $u(t, s)$ are continuous functions from t to $L_2(S, \mathcal{S}, P)$, then $\sigma(x(t, s), u(t, s))$ is a continuous function from t to $L_2(S, \mathcal{S}, P)$, so that the stochastic integral $\int_0^T \sigma(s(t, s), u(t, s))\, dW_t(s)$ exists and the stochastic differential equation $dx(t, s) = \mu(s(t, s), u(t, s))dt + \sigma(x(t, s), u(t, s))dW_t(s)$ should have a solution.

The value function V is defined as

$$V(t, x(t, s), s) = \max_u E\left[\int_t^T f(x(\tau, s), u(\tau, s))\, d\tau \,\middle|\, \mathcal{S}_t \right]$$

$$\text{s.t. } dx(\tau, s) = \mu(x(\tau, s), u(\tau, s))\, d\tau + \sigma(x(\tau, s), u(\tau, s))\, dW_\tau(s),$$

for all τ and s, and

$x(t, s)$ is given,

where the expectation $E\left[. \mid \mathcal{S}_t \right]$ is the expectation conditional on information available at time t, that is, conditional on $\mathcal{S}_t$.

I have not defined expectation conditional on a σ-field, but I will try to convey what it means. Let $\mathcal{A}$ be a σ-field that is a subfield of $\mathcal{S}$. If $y : S \rightarrow R$ is a random variable measurable with respect to $\mathcal{S}$ and if $A \in \mathcal{A}$ is such that $\text{Prob}(A) > 0$, then $E[y \mid A]$ is the expected value of y given knowledge that the true state belongs to A. That is, $P(A)E[y \mid A] = \int_A y(s)\, P(ds)$. The conditional expectation $E[. \mid \mathcal{A}]$ is a random variable $E[. \mid \mathcal{A}] : S \rightarrow R$ that is measurable with respect to $\mathcal{A}$ and is such that $P(A)E[y \mid A] = \int_A E[. \mid \mathcal{A}](s)\, \text{Prob}(ds)$, for all $A \in \mathcal{A}$ such that $\text{Prob}(A) > 0$. That is, $P(A)E[y \mid A]$ is the integral over A of the random variable $E[. \mid \mathcal{A}](s)$. For this reason, as a function of s, $V(t, x(t, s), s)$ is measurable with respect to $\mathcal{S}_t$. Important facts about conditional expectations is that if $y : S \to R$ is measurable with respect to $\mathcal{A}$, then $E[\,y \mid \mathcal{A}] = y$. Also if $y : S \to R$ and $z : S \to R$ are stochastically independent random variables that are measurable with respect to $\mathcal{S}$, then $E[\,yz \mid \mathcal{A}] = E[\,y \mid \mathcal{A}]E[\,z \mid \mathcal{A}]$.

It would seem that the value function $V(t, x(t, s), s)$ should depend on s directly as well as on t and $x(t, s)$. This dependence does not exist, however, and I now try to explain why. Recall that the states s in S should be thought of as functions $s(t)$ from $[0, \infty)$ to R. To say that a function $g(s)$ is measurable with respect to $\mathcal{S}_t$ means that $g(s)$ depends only on the values of $s(\tau)$, for $\tau \le t$. Now imagine that you control the variable u, that you are in state s and know $s(\tau)$, for $\tau \le t$, and that you wish to solve the optimal control problem. You must determine the value of u over a small interval of time from t to t + dt. The problem you must solve is

$$\max_u \left\{ f(x(t, s), u)dt + E\left[V(t+dt, x(t+dt, s), s) \mid \mathcal{S}_t \right](s) \right\} = \max_u \{ f(x(t, s), u)dt$$

$$+ E\left[V\left(t+dt, x(t, s) + \mu(x(t, s), u)dt + \sigma(x(t, s), u)\left(W_{t+dt}(s) - W_t(s)\right), s\right) \middle| \mathcal{S}_t \right](s) \right\}.$$

Let us look at this problem carefully. The term $f(x(t, s), u)dt$ does not depend on s except through $x(t, s)$. Within the term

$$E\left[V\left(t+dt, x(t, s) + \mu(x(t, s), u)dt + \sigma(x(t, s), u)\left(W_{t+dt}(s) - W_t(s)\right), s\right) \middle| \mathcal{S}_t \right](s),$$

$\mu(x(t, s), u)$ and $\sigma(x(t, s), u)$ do not depend on s except through $x(t, s)$ and the random variation $W_{t+dt}(s) - W_t(s)$ is stochastically independent of $\mathcal{S}_t$, so that knowledge of $s(\tau)$, for $\tau \le t$, does not help predict it. If $V(t+dt, x(t+dt, s), s)$ does not depend directly on s, then given knowledge of $x(t, s)$, knowledge of $s(\tau)$, for $\tau \le t$, will not improve your choice of u. So if we assume that $V(t+dt, x(t+dt, s), s)$ does not depend directly on s, then $V(t, x(t, s), s)$ does not depend directly on s either and furthermore the optimal choice of u may be made to depend only on $x(t, s)$. (If there was more than one optimal choice of u, you could make the choice among these optimal choices depend artificially on s.) Now imagine that we divide the time interval [t, T] into a great many little intervals of length dt and make the argument just made by backward induction on the intervals, [t, t +dt], [t + dt, t + 2dt], ... , [T − dt, T]. Since $V(T, x(T, s), s) = 0$, it does not depend on s. By the above argument,

$$V(T - dt, x(T - dt, s), s) = \max_u f(x(T - dt, s), u)\,dt$$

does not depend directly on s and the optimal value of u may be made to depend only on $x(T - dt, s)$. By continuing by backward induction on the intervals, we see that $V(t, x(t, s), s)$ does not depend directly on s. I hope that this informal argument convinces you that we may replace $V(t, x(t, s), s)$ with $V(t, x(t, s))$ or $V(t, x)$ and we may assume that the optimal choice of u at time t is a function of $x(t, s)$ alone. We should not, however, dispense with the conditional expected value in the definition of the value function V.

I now derive in a loose way a version of the Hamilton Jacobi Bellman equation that applies to problem 15.2. Assume that the function V(t, x) is twice differentiable. Applying the Bellman principle and then applying Ito's lemma to the function V(t, x) and the stochastic process x(t, s), we have that

$$V(t, x(t, s)) = \max_u \left\{ f(x(t, s), u)dt + E\left[V(t+dt, x(t+dt, s)) \,\middle|\, \mathfrak{S}_t \right] \right\}$$

$$V(t, x(t, s)) = \max_u \left\{ f(x(t, s), u)dt + E\left[V(t, x(t, s)) + dV(t, x(t, s)) \,\middle|\, \mathfrak{S}_t \right] \right\}$$

$$\begin{aligned} &= \max_u \left\{ f(x(t, s), u)dt + E\left[V(t, x(t, s)) + \left(\frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) \right.\right.\right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \right) dt + \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) dW_t(s) \\ &\quad \left.\left. + \frac{\partial V(t, x(t, s))}{\partial t} dt \,\middle|\, \mathfrak{S}_t \right] \right\} \end{aligned} \tag{15.3}$$

$$\begin{aligned} &= V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} dt \\ &\quad + \max_u \left\{ f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \right\} dt , \end{aligned}$$

where I have used facts (1) - (6) listed below.

1) The function of s, x(t, s), is measurable with respect to $\mathfrak{S}_t$, so that functions of s such as V(t, x(t, s)) that depend on x(t, s) are measurable with respect to $\mathfrak{S}_t$ as well.

2) $$E\left[V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} \,\middle|\, \mathfrak{S}_t \right] = V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t},$$

because $V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t}$ is measurable with respect to $\mathfrak{S}_t$.

3) $$E\left[f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \,\middle|\, \mathfrak{S}_t \right]$$

$$= f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x}\mu(x(t, s), u) + \frac{1}{2}\frac{\partial^2 V(t, x(t, s))}{\partial x^2}\sigma^2(x(t, s), u),$$

because $f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x}\mu(x(t, s), u) + \frac{1}{2}\frac{\partial^2 V(t, x(t, s))}{\partial x^2}\sigma^2(x(t, s), u)$ is measurable with respect to $\mathfrak{S}_t$.

4) $$E\left[\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u)dW_t(s) \middle| \mathfrak{S}_t\right]$$

$$= E\left[\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u) \middle| \mathfrak{S}_t\right] E\left[dW_t(s) \middle| \mathfrak{S}_t\right],$$

since $\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u)$ and $dW_t(s)$ are stochastically independent.

5) $$E\left[dW_t(s) \middle| \mathfrak{S}_t\right] = 0.$$

Facts (4) and (5) imply that

6) $$E\left[\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u)dW_t(s) \middle| \mathfrak{S}_t\right]$$

$$= E\left[\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u) \middle| \mathfrak{S}_t\right] E\left[dW_t(s) \middle| \mathfrak{S}_t\right]$$

$$= E\left[\frac{\partial V(t, x(t, s))}{\partial x}\sigma(x(t, s), u) \middle| \mathfrak{S}_t\right](0) = 0.$$

Canceling V(t, x(t, s)) from both sides of equation 15.3 and dividing both sides by dt, we see that

$$\frac{\partial V(t, x(t, s))}{\partial t} = -\max_u\left[f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x}\mu(x(t, s), u) + \frac{1}{2}\frac{\partial^2 V(t, x(t, s))}{\partial x^2}\sigma^2(x(t, s), u)\right]. \tag{15.4}$$

This equation is another Hamilton Jacobi Bellman equation. In deriving it, we have informally verified a maximum principle for the stochastic optimal control problem. Notice that the maximization problem in equation 15.4 is deterministic for each value of s.

A maximum principle and a Hamilton Jacobi Bellman equation apply also to stochastic optimal control problems where the gain is discounted and integrated over an infinite horizon. Consider the problem

$$\begin{aligned}
&\max_u E\left[\int_0^\infty e^{-rt} f(x(t,\ s),\ u(t,\ s))\,dt\right] \\
&\text{s.t. } dx(t,\ s) = \mu(x(t,\ s),\ u(t,\ s))\,dt \\
&\quad + \sigma(x(t,\ s),\ u(t,\ s))\,dW_t(s), \\
&\qquad \text{for all s and t, and} \\
&\qquad x(0,\ s) \text{ is given.}
\end{aligned} \tag{15.5}$$

Assume that assumptions 15.3 apply. Define the value function V by the equation

$$\begin{aligned}
V(t,\ x(t,\ s),\ s) = \max_u E&\left[\int_t^\infty e^{-r(\tau - t)} f(x(\tau,\ s),\ u(\tau,\ s)) \,\middle|\, \mathcal{S}_t\right] d\tau \\
&\text{s.t. } dx(\tau,\ s) = \mu(x(\tau,\ s),\ u(\tau,\ s))\,d\tau \\
&\quad + \sigma(x(\tau,\ s),\ u(\tau,\ s))\,dW_\tau(s), \\
&\qquad \text{for all s and } \tau\text{, and} \\
&\qquad x(0,\ s) \text{ is given.}
\end{aligned}$$

Just as in the case of the undiscounted problem, the value function does not depend directly on s, so that we may write V(t, x(t, s), s) = V(t, x(t, s)). Also we may assume that the optimal value of the control at time t, u(t, s), depends only on x(t, s). The value function does not depend on t either, for suppose that you control u and move with the system through time. The only thing about the optimal control problem that changes when you move from time t to time T > t is that you know more about the state s. In the time interval (t, T], you will have learned $s(\tau)$, for $t < \tau \leq T$. But we know that the value function does not depend directly on the state, but only on the variable x(t, s) and perhaps on t. Similarly the optimal control need not depend directly on the state and may be made to depend only on x(t, s). Hence the optimization problem is the same at time T as at time t, and so we can write the value function as V(x(t, s)) or V(x).

Assume that the function V(x) is twice differentiable. Again applying the Bellman principle and then applying Ito's lemma to the stochastic process y(t, s) = V(x(t, s)), we have that

$$V(x(t,\ s)) = \max_u \left\{ f(x(t,\ s),\ u)\,dt + e^{-rdt} E\left[V(x(t + dt,\ s)) \,|\mathcal{S}_t \right] \right\}$$

$$V(x(t,\ s)) = \max_u \left\{ f(x(t,\ s),\ u)\,dt + e^{-rdt} E\left[V(x(t,\ s)) + dV(x(t,\ s)) \,|\mathcal{S}_t \right] \right\}$$

$$= \max_u \left\{ f(x(t,\ s),\ u)\,dt + \frac{1}{1 + rdt} E[V(x(t,\ s)) \right.$$

$$+\left(\frac{\partial V(t,\ x(t,\ s))}{\partial x}\mu(x(t,\ s),\ u) + \frac{1}{2}\frac{\partial^2 V(t,\ x(t,\ s))\,\sigma^2(x(t,\ s),\ u)}{\partial x^2}\right)dt$$

$$+\ \frac{\partial V(t,\ x(t,\ s))}{\partial x}\sigma(x(t,\ s),\ u)dW_t(s)\,\Big|\,\Im_t\Big]\Big\},$$

so that

$$V(x(t,\ s)) + rV(x(t,\ s))dt = \max_u\Big\{f(x(t,\ s),\ u)dt + rf(x(t,\ s),\ u)dt^2 + V(x(t,\ s))$$

$$+\left(\frac{\partial V(x(t,\ s)}{\partial x}\mu(x(t,\ s),\ u) + \frac{1}{2}\frac{\partial^2 V(x(t,\ s))}{\partial x^2}\sigma^2(x(t,\ s),\ u)\right)dt$$

$$+\,E\left[\frac{\partial V(x(t,\ s))}{\partial x}\sigma(x(t,\ s),\ u)dW_t(s)\,\Big|\,\Im_t\right]\Big\}$$

$$= V(x(t,\ s)) + \max_u\Big\{f(x(t,\ s),\ u)dt + rf(x(t,\ s),\ u)dt^2$$

$$+\left(\frac{\partial V(x(t,\ s)}{\partial x}\mu(x(t,\ s),\ u) + \frac{1}{2}\frac{\partial^2 V(x(t,\ s))}{\partial x^2}\sigma^2(x(t,\ s),\ u)\right)dt\Big\}$$

The steps in each of these equations follow for the same reasons as the corresponding steps did in the undiscounted case. If we cancel V(x(t, s)) from both sides of this last equation, divide the result by dt, and then let dt converge to 0, we see that

$$rV(x(t,\ s))$$

$$= \max_u\left[f(x(t,\ s),\ u) + \frac{\partial V(x(t,\ s))}{\partial x}\mu(x(t,\ s),\ u) + \frac{1}{2}\frac{\partial^2 V(x(t,\ s))}{\partial x^2}\sigma^2(x(t,\ s),\ u)\right]$$

This is still another Hamilton Jacobi Bellman equation. In the stochastic case, its solution is not sufficient for optimality. We have also verified the maximum principle for problem 15.5.

Example: This is a stochastic version of example 13.1. Consider the problem

$$\max_u E\left[\int_0^\infty e^{-rt}(-x^2(t,\ s) - u^2(t,\ s))\,dt\right]$$

$$\text{s.t. } dx(t,\ s) = u(t,\ s)dt + \sigma dW_t(s)$$

and $x(0,\ s)$ is given,

where $\sigma > 0$.

In applying the Hamilton Jacobi Bellman equation, let $f(x, u) = -x^2 - u^2$, $\mu(x, u) = u$, and $\sigma(x, u) = \sigma$. Guess that $V(x) = ax^2 + b$, where $a < 0$ and $b < 0$. Then

$$\frac{dV(x)}{dx} = 2ax \text{ and } \frac{d^2V(x)}{dx^2} = 2a.$$

By the Hamilton Jacobi Bellman equation,

$$r(ax^2 + b) = \max_u[-x^2 - u^2 + 2axu + a\sigma^2]. \tag{15.6}$$

The maximizing value of u is $u = ax$ and

$$\max_u[-x^2 - u^2 + 2axu + a\sigma^2] = -x^2 + a^2x^2 + a\sigma^2,$$

so that equation 15.6 becomes

$$rax^2 + rb = -x^2 + a^2x^2 + a\sigma^2.$$

Hence

$$ra = -1 + a^2 \text{ and } b = \frac{a\sigma^2}{r}.$$

Since $a < 0$, the solution of the quadratic equation in a is

$$a = \frac{r - \sqrt{r^2 + 4}}{2}.$$

The corresponding control is $u(t, s) = ax(t, s)$. If this control is used, $x(t, s)$ evolves according to the stochastic differential equation

$$dx(t, s) = ax(t, s)dt + \sigma dW_t(s).$$

This is only a candidate optimal control, since the Hamilton Jacobi Bellman equation does not provide a sufficient condition for optimality.