MATH CAMP: Lecture 1

1 Linear Algebra

Simultaneous Linear Equations:

Example:

$$\begin{aligned} 3x_1 + 2x_2 &= 6 \\ 6x_1 + 7x_2 &= -2 \end{aligned}$$

Now solve. Subtract twice first equation from second

$$\begin{array}{rcr} 6x_1 + 7x_2 & = & -2 \\ -6x_1 - 4x_2 & = & -12 \\ \hline 3x_2 & = & -14 \\ x_2 & = & \dfrac{-14}{3} \end{array}$$

Substitute this into equation 1

$$\begin{aligned} &3x_1 + 2\left(\frac{-14}{3}\right) = 6 \\ &3x_1 = 6 + \frac{28}{3} = \frac{18+28}{3} = \frac{46}{3} \\ &x_1 = \frac{46}{9} \end{aligned}$$

The next two equations are equivalent to the first two.

$$\begin{aligned} 3x_1 + 2x_2 &= 6 \\ 3x_2 &= -14 \end{aligned}$$

These are equivalent to the following pairs of equations.

$$\begin{array}{ccc} x_1 + \frac{2}{3}x_2 = 2 & \text{or} & x_1 = 2 - \frac{2}{3}\left(\frac{-14}{3}\right) = \frac{46}{9} \\ x_2 = \frac{-14}{3} & & x_2 = \frac{-14}{3} \end{array}$$

The last pair is said to be *row reduced and in echelon form.* We want to do this more generally

$$\begin{array}{lcl} a_{11}x_1 + a_{12}x_2 & + \cdots + & a_{1N}x_N = y_1 \\ a_{21}x_1 + a_{22}x_2 & + \cdots + & x_{2N}x_N = y_2 \\ \quad\vdots & & \\ a_{M1}x_1 + a_{M2}x_2 & + \cdots + & a_{MN}x_N = y_M \end{array}$$

The a_{mn} and y_n are numbers and $x_1, ..., x_N$ are unknown. In order to be more systematic we write the equation as:

$$\begin{aligned}
Ax &= y, \text{where} \\
A &= \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MN} \end{pmatrix} \quad M \times N \text{ matrix} \\
x &= \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad N\text{-vector of unknowns} \\
y &= \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \quad M\text{-vector of numbers.}
\end{aligned}$$

Ax is the M -vector

$$\begin{matrix} a_{11}x_1 & +\cdots+ & a_{1N}x_N \\ a_{21}x_1 & +\cdots+ & a_{2N}x_N \\ \vdots & & \\ a_{M1}x_1 & +\cdots+ & a_{MN}x_N \end{matrix}$$

Consider the following so called elementary operations on $M \times N$ matrix A:

1. Multiply a row of A by a non-zero number.

2. Replace a row by that row plus c times another row, where c is a non-zero number.

3. Interchange two rows.

If the $M \times N$ matrix B is obtained from A by any one of these operations, then A and B are equivalent in the sense the equations $Bx = 0$ and $Ax = 0$ have the same solutions. (Think this through.)

Similarily, if the $M \times (N+1)$ matrix $(B \vdots z)$ is obtained from the $M \times (N+1)$ matrix $(A \vdots y)$ by an elementary row operation, then the systems $Bx = z$ and $Ax = y$ have the same solutions.

Elementary row operations can transform any system $Ax = y$ into a system $Bx = z$ where

a) the first non-zero entry in any row of B is 1, and

b) each column of B that contains the leading non-zero entry of some row has all its other entries 0.

Definition: Such a matrix B is said to be *row-reduced.*

Example:

$$\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ is row reduced.}$$

Example:

$$
\begin{aligned}
&\begin{pmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 6 & 8 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \\
\rightarrow\ &\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} \\
\rightarrow\ &\begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 1 & 3/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix} \\
\rightarrow\ &\begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/2 \\ 0 \end{pmatrix} \\
\rightarrow\ &\begin{pmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3/2 \\ 0 \end{pmatrix}
\end{aligned}
$$

The matrix $\begin{pmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{pmatrix}$ is an examle of a row reduced echelon matrix.

Definition: A matrix is a *row reduced echelon matrix* if:

a) it is row reduced,

b) any row of zeros lies below all non-zero rows, and

c) if the first r rows are non-zero and the leading non-zero entry of row m is in column n_m for $m = 1, \ldots, r$ then $n_1 < n_2 < \cdots < n_r$.

Example: The matrix $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a row reduced echelon matrix

Theorem: Every matrix is equivalent to a row reduced echelon matrix.

Proof: This may be achieved by elementary row operations. ∎

Theorem: If A is an $M \times N$ matrix and $M < N$, then the system $Ax = 0$ has a non-zero solution.

Proof: Let B be a row reduced echelon matrix equivalent to A. $Ax = 0 \iff Bx = 0$. Let r be the number of non-zero rows of B. Then $r \leq M < N$ and rows $1, ..., r$ are the non-zero rows of B. For $1 \leq m \leq r$, let the leading non-zero entry of row m be in column n_m, where $n_1 < n_2 < \cdots < n_r$. Since $r < N$, there is an n such that $1 \leq n \leq N$ and $n \neq n_m$ for any m. For such an n, let $x_n = 1$. For k such that $1 \leq k \leq N$, let $x_k = 0$, if $k \neq n_m$, for all m. It is now possible to solve for x_n, for $n = n_1, ..., n_r$, so that $Bx = 0$. Then $Ax = 0$ and $x \neq 0$. ■

Theorem: If A is an $N \times N$ matrix and if $Ax = 0$ has no non-zero solution, then A is row equivalent to the $N \times N$ identity matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$.

Proof: Let B be a row reduced echelon matrix that is row equivalent to A. $Bx = 0$ has no non-zero solution. Hence the number of non-zero rows of B is N. Hence, $B = I$. ■

2 Vector Spaces

Consider $R^N = \{v = (v_1, ..., v_N) \,|v_n \in R$, for all $n\}$, where R is the set of real numbers. We can define the following operations on R^N. If v and w belong to R^N, $v + w = (v_1, ..., v_N) + (w_1, ..., w_N) = (v_1 + w_1, ..., v_N + w_N)$. If $c \in R$ and $v \in R^N$, $cv = c(v_1, ..., v_N) = (cv_1, ..., cv_N)$. Let $0 = (0, 0, ..., 0) \in R^N$. Observe that

a) $x + y \in R^N$, if $x \in R^N$ and $y \in R^N$
b) $x + y = y + x$,
c) there is a $0 \in R^N$ and $0 + x = x$, for all $x \in R^N$,
d) for all $x \in R^N$, there is a unique $-x \in R^N$ such that $x + (-x) = 0$,
e) $1x = x$, for $x \in R^N$
f) $(c_1c_2)x = c_1(c_2x)$, for all numbers $c_1$and c_2 and for all $x \in R^N$,
g) $c(x + y) \equiv cx + cy$, for numbers c and for x and y in R^N,
h) $(c_1 + c_2)x = c_1x + c_2x$, for all numbers c_1 and c_2 and for all $x \in R^N$.

Definition: A *vector space* consists of a set, V, together with operations of addition and multiplication by numbers, denoted $x + y$ and rx, where $x \in V, y \in V$ and $r \in R$ and these operations satisfy (a)–(h) above with R^N everywhere replaced by V.

Definition: W is a *subspace* of V, if $W \subset V$ and W is a vector space under the operations on V.

Note: $W \subset V$ is a subspace of V if $v + w \in W$, for all $v, w \in W$, and if $cw \in W$ whenever $c \in R$ and $w \in W$.

Example: $V = \{f : [0, 2\pi] \to R | f(0) = f(2\pi)\}$ is a vector space under the operations $(f + g)(s) = f(s) + g(s)$, for all $s \in [0, 2\pi]$ and $(cf)(s) = cf(s)$, for all s.

$W = \{a \sin + \, b \cos \mid a \in R$ and $b \in R\}$ is a subspace of V. W is the linear span of sin and cos.

Definition: If V is vector space, $v \in V$ is said to be a *linear combination* of $w_1, ..., w_N \in V$, if there are numbers $c_1, ...c_N$ such that $v = c_1 w_1 + \cdots + c_N w_N$.

Definitions: If $w_1, ..., w_N \in V$, their *linear span* is the set of all linear combinations of $w_1, ..., w_N$. The linear span of $w_1, ..., w_N$ is a subspace of V and is the smallest subspace containing $w_1, ..., w_N$. The vectors, $w_1, ..., w_N$ are said to span V, if V is the linear span of $w_1, ..., w_N$.

Now I try to get at the idea of the *dimension* of a vector space.

Definition: Vectors $v_1, ..., v_N$ in V are *linearly dependent* if there exist numbers $c_1, ...c_N$, not all zero, such that $c_1 v_1 + \cdots + c_N v_N = 0$.

Definition: Vectors $v_1, ..., v_N$ in V are *linearly independent* if there are not dependent.

Example: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ are dependent in R^3, since

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are independent, since

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

Example: sin and cos are independent, for suppose that $a \sin + \, b \cos = 0$. Then $0 = a \sin(\pi/2) + b \cos(\pi/2) = a$ and $0 = a \sin(0) + \, b \cos(0) = b$, so that $a = b = 0$, $\sin + \, 2\cos$ and $-2\sin - 4\cos$ are dependent, since $2(\sin + \, 2\cos) + (-2\sin - 4\cos) = 0$

Definition: A *basis* for a vector space V is a set of linearly independent vectors in V that spans V.

Example: Let

$$e_n = (0, ..., 0, \underset{\substack{\uparrow \\ n\text{th slot}}}{1}, 0, ..., 0) \in R^N$$

$e_1, ..., e_N$ is the *standard basis* for R^N.

Theorem: If $v_1, ..., v_M$ span a vector space V, then any independent set of vectors in V has no more than M elements.

Proof: I must show that if $N > M$ and $w_1, ..., w_N$ are in V, then $w_1, ..., w_N$ are linearly dependent. Since $v_1, ..., v_M$ span V, $w_n = \Sigma_{m=1}^{M} a_{mn} v_m$, for all n and for some numbers $a_{1n}, ..., a_{Mn}$. If $x_1, ..., x_N$ are numbers, then

$$\begin{aligned} x_1 w_1 + \cdots + x_N w_N &= \sum_{n=1}^{N} x_n w_n = \sum_{n=1}^{N} x_n \sum_{m=1}^{M} a_{mn} v_m \\ &= \sum_{n=1}^{N} \sum_{m=1}^{M} a_{mn} x_n v_m = \sum_{m=1}^{M} \left(\sum_{n=1}^{N} a_{mn} x_n \right) v_m \end{aligned}$$

Since $N > M$, a previous theorem implies that there exist numbers $x_1, ..., x_N$ not all zero such that $\sum_n a_{mn} x_n = 0$ for $m = 1, ..., M$. Hence, $x_1 w_1 + \cdots + x_N w_N = 0$ and $w_1, ..., w_N$ are linearly dependent. ■

Definition: A vector space is *finite dimensional*, if it has a finite basis.

Corollary: If V is a finite dimensional vector space, then any two bases have the same number of elements.

Proof: If $v_1, ..., v_M$ and $w_1, ..., w_N$ are bases, then $N \leq M$ and $M \leq N$. ■

Definition: The dimension of V is $\dim V =$ the number of vectors in a bases for V.

Lemma: Let $v_1, ..., v_M$ in V be linearly independent and let w in V not belong to the span of $v_1, ..., v_M$. Then $v_1, ..., v_M$, w are linearly independent.

Proof: Suppose that $c_1 v_1 + \cdots + c_M v_M + bw = 0$. If $b \neq 0$, then $w = \frac{-c}{b} v_1 - \cdots - \frac{c_M}{b} v_M \in \text{span}(v_1, ..., v_M)$. This contradiction implies that $b = 0$. Therefore, $c_1 v_1 + \cdots + c_M v_M = 0$. Since, $v_1, ..., v_M$ are linearly independent, $c_m = 0$, for all m.■

Theorem: If W is a subspace of a vector space V of finite and positive dimension and $W \neq V$, then $\dim W < \dim V$.

Proof: Let $M = \dim W$ and $N = \dim V$. I must show that $M < N$. If $W = \{0\}$ then $\dim W = 0 < \dim V$. Suppose that $W \neq \{0\}$. If $w_1, ..., w_M$ are linearly independent vectors in W, they are linearly independent in V and so $M \leq N$. Therefore, there is a linearly independent set of vectors in W with a largest number of elements, say $w_1, ..., w_r$. By the previous lemma $w_1, ..., w_r$ is a basis for W and $r = \dim W$. Since $W \neq V$, there is a v in V such that $v \notin W$. By the previous lemma, $w_1, ..., w_r$, v are independent and hence $N \geq M + 1 > M$. ■

Theorem: If $v_1, ..., v_N$ is a basis for V and $v \in V$, then the numbers $c_1, .., c_N$, such that $v = \sum_{n=1}^{N} c_n v_n$ are unique.

Proof: $\displaystyle\sum_{n=1}^{N} c_n v_n = v = \sum_{n=1}^{N} a_n v_n \Longrightarrow \sum_{n=1}^{N} (c_n - a_n) v_n = 0 \Longrightarrow c_n - a_n = 0$, for all n, since $v_1, ..., v_N$ are independent. ■