MATH CAMP: Lecture 2

Definition: If V and W are vector spaces, $T = V \longrightarrow W$ is *linear* or a *linear transformation* if $T(av_1 + bv_2) = aT(v_1) + \; bT(v_2)$, for all numbers a and b and for all vectors v_1 and v_2 in V.

Example: Let $T : R^N \to R^M$ be defined by $T(x) = Ax$, where A is an $M \times N$ matrix. Then $T(ax + by) = A(ax + by) = aAx + bAy = aT(x) + bT(y)$.

Example: If $f \in V = \{f : [0, 2\pi] \to R | f(0) = f(2\pi)\}$ Let

$$(Tf)(s) = \begin{Bmatrix} f(s+\pi), \text{ if } 0 \leq s \leq \pi \\ f(s-\pi), \text{ if } \pi \leq s \leq 2\pi \end{Bmatrix}$$

$T : V \to V$ is linear. If $[0, 2\pi)$ is thought of as a circle, the transformation T corresponds to rotating the circle counterclockwise 180° and then applying the function f.

Matrices can be used to represent any linear transformation from one finite dimensional vector space to another. Let $T : V \to W$ be linear, let $v_1, ..., v_N$ be a basis for V, and let $w_1, ..., w_M$ be a basis for W. If $v \in V$, there are unique numbers $x_1, ..., x_N$ such that $v = x_1v_1 + \cdots + x_Nv_N$. Since $T(v) \in W$, there are unique numbers, $y_1, ..., y_M$ such that $T(v) = y_1w_1 + \cdots + y_Mw_M$. Since $T(v_n) \in W$, for each n, there are unique numbers $a_{1n}, ..., a_{Mn}$, such that $T(v_n) = a_{1n}w_1 + \cdots + a_{Mn}w_M$. Therefore,

$$T(v) = \sum_{n=1}^{N} x_n T(v_n) = \sum_{n=1}^{N} x_n \sum_{m=1}^{M} a_{mn} w_m = \sum_{m=1}^{M} \left(\sum_{n=1}^{N} a_{mn} x_n \right) w_m = \sum_{m=1}^{M} y_m w_m.$$

Let

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MN} \end{pmatrix}.$$

Then, $y = Ax$. The $M \times N$ matrix A represents T in that there is one and only one linear transformation T corresponding to A and one and only one matrix A corresponding to T given the bases $v_1, ..., v_N$ for V and $w_1, ..., w_M$ for w.

Let $S : W \longrightarrow Q$ be a linear transformation and let $q_1, ..., q_J$ be a basis for Q. Let the $J \times M$ matrix $B = (b_{jm})$ representing S, so that

$$S(w_m) = \sum_{j=1}^{J} b_{jm} q_j.$$

$S \circ T : V \longrightarrow Q$ is the linear transformation defined by $S \circ T(v) = S(T(v))$. Then

$$\begin{aligned} S \circ T(v_n) &= S(Tv_n) = S\left(\sum_{m=1}^{M} a_{mn} w_m\right) = \sum_{m=1}^{M} a_{mn} S(w_m) = \sum_{m=1}^{M} a_{mn} \sum_{j=1}^{J} b_{jm} q_j \\ &= \sum_{j=1}^{J} \left(\sum_{m=1}^{M} b_{jm} a_{mn}\right) q_j = \sum_{j=1}^{J} c_{jn} q_j, \end{aligned}$$

where $c_{jn} = \sum_{m=1}^{M} b_{jm} a_{mn}$, so that the $J \times N$ matrix $C = (c_{jn})$ represents $S \circ T$.

Definition: If A is an $M \times N$ matrix and B is a $J \times M$ matrix, the matrix $C = BA$, the product of B and A, is the $J \times N$ matrix defined by $c_{jn} = \sum_{m=1}^{M} b_{jm} a_{mn}$, and

$$\begin{aligned} C &= \begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & & \vdots \\ c_{J1} & \cdots & c_{JN} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & \cdots & b_{1M} \\ \vdots & & \vdots \\ b_{J1} & \cdots & b_{JM} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MN} \end{pmatrix} = BA. \end{aligned}$$

Example:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark: If the $M \times N$ matrix A represents T and the $J \times M$ matrix B represents S, then the $J \times N$ matrix $C = BA$ represents $S \circ T$.

Note: The order in which matrices are multiplied does not affect the product. That is, if A is an $M \times N$ matrix, B is a $J \times M$ matrix and C is a $K \times J$ matrix, then $(CB)A = C(BA)$.

Definition: An $N \times N$ matrix A is *invertible* if there is an $N \times N$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

I is called the $N \times N$ identity matrix and represents the identity function $id_V : V \longrightarrow V$, where V is an $N-$dimensional vector space and $id_V(v) = v$, for all $v \in V$. Clearly, $IA = AI = A$, for any $N \times N$ matrix A.

Lemma: If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \\ (AB)(B^{-1}A^{-1}) &= AIA^{-1} = AA^{-1} = I. \end{aligned} \qquad \blacksquare$$

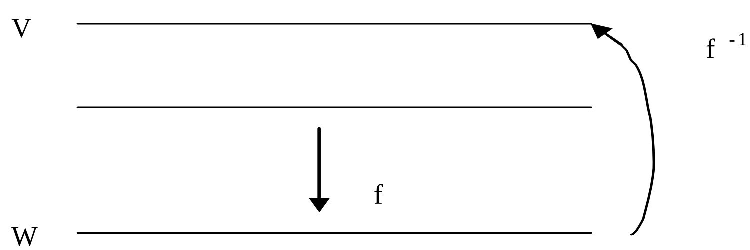
Definition: A function $f : V \to W$ is *invertible,* if there exists $f^{-1} : W \longrightarrow V$ such that $f \circ f^{-1} = id_W$ and $f^{-1} \circ f = id_V$. That is, $f(f^{-1}(w)) = w$, for all $w \in W$ and $f^{-1}(f(v)) = v$, for all $v \in V$.

Definition: $f : V \to W$ is *onto,* if for every $w \in W$, there exists a $v \in V$ such that $f(v) = w$.

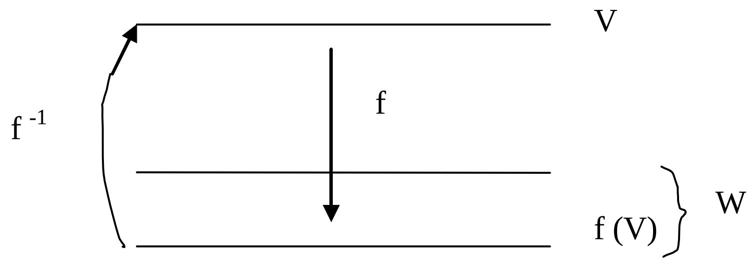
Definition: $f : V \to W$ is *one to one*, if for every $\underline{v} \in V$ and $\bar{v} \in V$ such that $\underline{v} \neq \bar{v}$, $f(\underline{v}) \neq f(\bar{v})$.

Remarks:

1. $f : V \to W$ is onto if and only if there exists $f^{-1} : W \to V$ such that $f(f^{-1}(w)) = w$, for all $w \in W$.



2. f is one to one, if and only if there exists $f^{-1} : f(V) \to V$ such that $f^{-1}(f(v)) = v$, for all $v \in V$, where $f(V) = \{f(v) \mid v \in V\}$.



3. f is one to one and onto, if and only if f is invertible.

Theorem: If $T : V \to W$ is an invertible linear transformation, them T^{-1} is linear.

Proof: Let $w_1, w_2 \in W$ and $c_1, c_2 \in R$. Let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Since T is linear, $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) = c_1w_1 + c_2w_2$. Therefore,

$$c_1T^{-1}(w_1) + c_2T^{-1}(w_2) = c_1v_1 + c_2v_2 = T^{-1} \circ T(c_1v_1 + c_2v_2) = T^{-1}(c_1w_1 + c_2w_2). \quad \blacksquare$$

Proposition: Let $T : V \to V$ be a linear transformation and let $v_1, ..., v_N$ a basis for V. If A is the $N \times N$ matrix representing T with respect to $v_1, ..., v_N$, then T is invertible if and only if A is invertible.

Proof: If T is invertible and A^{-1} represents T^{-1}, then $id_V = T^{-1} \circ T$, $A^{-1}\ A$ represents $T^{-1} \circ T$, and I represents id_V. Hence $A^{-1}A = I$. Similarly $AA^{-1} = I$.

If A is invertible, A^{-1} represents a linear transformation $T^{-1} : V \to V$ and $A^{-1}A = I = AA^{-1}$, which implies that $T^{-1} \circ T = T \circ T^{-1} = id_V$. $\blacksquare$

Theorem: If A is an $N \times N$ matrix, then the following are equivalent:
i) A is invertible,
ii) there is an $N \times N$ matrix B such that $BA = I$, and
iii) the system $Ax = 0$ has no non-zero solution.

Proof: (i) $\to$ (ii) Let $B = A^{-1}$.

(ii) $\to$ (iii). If $BA = I$ and $Ax = 0$, then $0 = BAx = Ix = x$, so that x is zero.

(iii) $\to$ (i) By a previous theorem, A is equivalent to the $N \times N$ identity matrix I. Equivalence is established via elementary row operations. Each elementary row operation on A corresponds to left multiplication by an invertible matrix P. I check this statement.

a) Multiplication of the rth row of A by $c \neq 0$ corresponds to PA, where

$$P = \begin{pmatrix} 1 & 0 & . & . & . & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & . & . & . & . & 0 \\ . & & . & & & . & & & & . \\ . & & & . & & . & & & & . \\ . & & & & 1 & 0 & & & & . \\ 0 & . & . & . & 0 & c & 0 & . & . & 0 \\ . & & & & & 0 & 1 & & & . \\ . & & & & & . & & . & & . \\ . & & & & & . & & & . & 0 \\ 0 & . & . & . & . & 0 & . & . & 0 & 1 \end{pmatrix} \text{row } r$$

$\uparrow$
column r

and

$$P^{-1} = \begin{pmatrix} 1 & 0 & . & . & . & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & . & . & . & . & 0 \\ . & & . & & & . & & & & . \\ . & & & . & & . & & & & . \\ . & & & & 1 & 0 & & & & . \\ 0 & . & . & . & 0 & c^{-1} & 0 & . & . & 0 \\ . & & & & & 0 & 1 & & & . \\ . & & & & & . & & . & & . \\ . & & & & & . & & & . & 0 \\ 0 & . & . & . & . & 0 & . & . & 0 & 1 \end{pmatrix} \text{row } r$$

$\uparrow$
column r

b) Replacement of the rth row of A by row r plus c times row s, where $c \neq 0$, corresponds to PA, where

$$
P = \begin{array}{c}
\begin{array}{ccccccccccccccc}
 & & & & \text{column } s & & & & \text{column } r & & & & & & \\
 & & & & \downarrow & & & & \downarrow & & & & & &
\end{array} \\
\begin{pmatrix}
1 & 0 & 0 & . & 0 & . & . & . & 0 & . & . & . & . & . & 0 \\
0 & 1 & 0 & . & . & & & & . & & & & & & 0 \\
. & & & & . & & & & . & & & & & & . \\
. & & & & 0 & & . & & . & & & & & & . \\
0 & . & . & 0 & c & 0 & . & 0 & 1 & 0 & . & . & . & . & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & 1 & 0 & . & . & . & 0 \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & . & & & & . \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & & . & & & . \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & & & . & & . \\
. & & & & . & & & & . & & & & & & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & . & . & . & 0 & 1 & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & . & . & . & . & 0 & 1
\end{pmatrix}
\end{array}
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \text{row } r \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
$$

and

$$
P^{-1} = \begin{array}{c}
\begin{array}{ccccccccccccccc}
 & & & & \text{column } s & & & & \text{column } r & & & & & & \\
 & & & & \downarrow & & & & \downarrow & & & & & &
\end{array} \\
\begin{pmatrix}
1 & 0 & 0 & . & 0 & . & . & . & 0 & . & . & . & . & . & 0 \\
0 & & . & & . & & & & . & & & & & & 0 \\
. & & & & . & & & & . & & & & & & . \\
. & & & & 0 & & . & & . & & & & & & . \\
0 & . & . & 0 & -c & 0 & . & 0 & 1 & 0 & . & . & . & . & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & 1 & 0 & . & . & . & 0 \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & . & & & & . \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & & . & & & . \\
. & & & & . & & & & . & & & & & & . \\
. & & & & . & & & & . & & & & . & & . \\
. & & & & . & & & & . & & & & & & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & . & . & . & 0 & 1 & 0 \\
0 & . & . & . & 0 & . & . & . & 0 & . & . & . & . & 0 & 1
\end{pmatrix}
\end{array}
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \text{row } r \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
$$

c) Interchange of rows r and s of A corresponds to multiplication of A on the left by the matrix

$$
P \quad = \quad \begin{pmatrix}
1 & 0 & . & . & . & 0 & . & . & . & . & . & . & . & . & 0 \\
0 & 1 & 0 & . & . & 0 & . & . & . & . & . & . & . & . & 0 \\
. & & . & & & . & & & & & & . & & & . \\
. & & & . & & . & & & & & & . & & & . \\
0 & . & . & 0 & 1 & 0 & . & . & . & . & . & 0 & . & . & 0 \\
0 & . & . & . & . & 0 & . & . & . & . & 0 & 1 & 0 & . & 0 \\
0 & . & . & . & . & 0 & 1 & 0 & . & . & . & 0 & . & . & 0 \\
. & & & & & . & & . & & & & . & & & . \\
. & & & & & . & & & . & & & . & & & . \\
. & & & & & . & & & & . & & . & & & . \\
. & & & & & 0 & & & & & . & . & & & . \\
0 & . & . & . & 0 & 1 & 0 & . & . & . & . & 0 & . & . & . \\
 & & & & & 0 & & & & & & . & 1 & & . \\
 & & & & & . & & & & & & . & & . & 0 \\
0 & . & . & . & . & 0 & . & . & . & . & . & 0 & . & 0 & 1
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ \\ \text{row } r \\ \\ \\ \\ \\ \\ \text{row } s \\ \\ \\ \\ \end{matrix}
$$

$$\uparrow \text{ column } r \qquad\qquad \uparrow \text{ column } s$$

$$P^{-1} = P.$$

So, $I = P_1P_2, ..., P_QA$, where P_q is invertible, for all q. Let $P = P_1P_2, ..., P_Q$. $P^{-1} = P_Q^{-1}, ..., P_1^{-1}$, so that P is invertible. $I = PA$, since A is row reduced to I via left multiplication by the matrix P. $P^{-1} = P^{-1}PA = (P^{-1}P)A = IA = A$. Therfore, $I = P^{-1}P = AP$. Since $PA = I = AP$, A is invertible. ■

Corollary: If A is an $N \times N$ matrix and $BA = I$, for some $N \times N$ matrix B, then $B = A^{-1}$.

Proof: By the theorem, A is invertible. Therefore, $BA = I$ implies that $(BA)A^{-1} = A^{-1}$, so that $B = BI = B(AA^{-1}) = (BA)A^{-1} = A^{-1}$.

Definition: If $T : V \to W$ is a function, the *image* or *range* of T is $\{T(v) \mid v \in V\}$.

Definition: If $T : V \to W$ is a linear transformation, the *null space* or *kernel* of T is $\{v \in V \mid T(v) = 0\}$.

Theorem: If $T : V \to W$ is a linear transformation, then the range of T is a subspace of W and the kernel of T is a subspace of V.

Proof: $c_1T(v_1) + c_2T(v_2) = T(c_1v_1 + c_2v_2)$. $T(v_1) = 0$ and $T(v_2) = 0$ imply that $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) = c_1(0) + c_2(0) = 0$. ■

Definition: If $T : V \to W$ is a linear transformation, the *rank* of T is the dimension of the range of T and the *nullity* of T is the dimension of the null space of T.

Definition: If A is an $M \times N$ matrix, the *column rank* of $A =$ the dimension of the linear span of the columns of A, and the *row rank* of $A =$ the dimension of the linear span of the rows of A.

I will later show that the row rank of A equals its column rank.

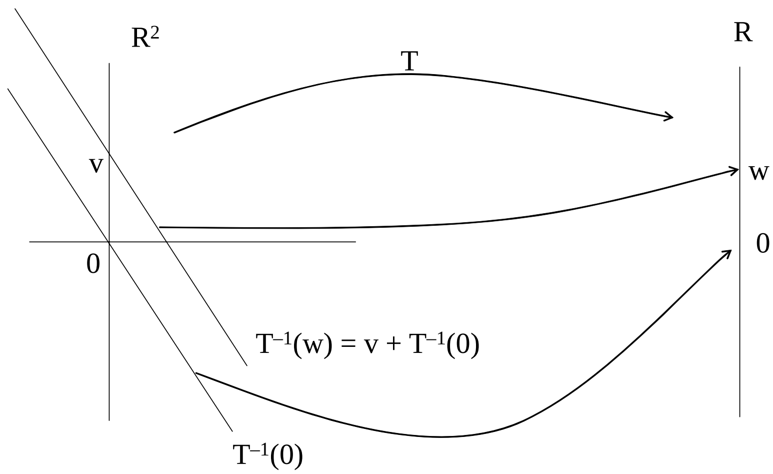
Theorem: Let $T : V \to W$ be a linear transformation. Then, rank T + nullity $T = \dim V$.

Proof: Let $v_1, ..., v_K$ be a basis for the null space of T. Extend $v_1, ..., v_K$ to a basis $v_1, ..., v_K, v_{K+1}, ..., v_N$ of V. I show that $T(v_{K+1}), ..., T(v_N)$ is a basis for the range of T. The vectors $T(v_1), ..., T(v_N)$ span the range of T. Since $T(v_n) = 0$, if $n \leqq K$, $T(v_{K+1}), ..., T(v_N)$ span the range of T. I show that $T(v_{K+1}), ..., T(v_N)$ are independent, so that $T(v_{K+1}), ..., T(v_N)$ is a basis for the range of T and hence rank $T = N - K$.

$$\begin{aligned}\sum_{n=K+1}^{N} c_n T(v_n) &= 0 = T\left(\sum_{n=K+1}^{N} c_n v_n\right) \Longrightarrow \sum_{n=K+1}^{N} c_n v_n \\ &= \sum_{n=1}^{K} b_n v_n \Longrightarrow \sum_{n=1}^{K} b_n v_n - \sum_{n=K+1}^{N} c_n v_n = 0\end{aligned}$$

$\Longrightarrow b_n = 0$ and $c_n = 0$, for all n, since $v_1, ..., v_N$ are independent. Therefore, rank of T + nullity of $T = N - K + K = N = \dim V$. ■

If $T : V \to W$ is a linear transformation, then for any $w \in W$, $T^{-1}(w) = v + T^{-1}(0)$, where v is any vector in V such that $T(v) = w$ and where $v + T^{-1}(0) = \{v + z | z \in T^{-1}(0)\}$. It is possible to visualize the meaning of these assertions by considering a linear transformation $T : R^2 \to R$.



The function T portrayed in this diagram may be thought of as a projection of R^2 followed by a linear function from the vertical axis onto R.

Definition: If $T : V \to W$ is a linear transformation, T is *non-singular* if the kernel of T is $\{0\}$.

Remark: T is non-singular if and only if T is one to one, since $T(v_1) = T(v_2)$ if and only if $T(v_1 - v_2) = 0$.

Lemma: If $T : V \to W$ is a linear transformation, then T is non-singular if and only if $T(v_1), ..., T(v_N)$ are linearly independent whenever $v_1, ..., v_N$ are linearly independent.

Proof: Suppose that T is non-singular. If $v_1, ..., v_N$ are linearly independent, then

$$\begin{aligned} c_1T(v_1) + \cdots + c_NT(v_N) &= 0 \Longrightarrow T(c_1v_1 + \cdots + c_Nv_N) = 0 \\ &\Longrightarrow c_1v_1 + \cdots + c_Nv_N = 0 \Longrightarrow c_1 = c_2 = \cdots = c_N = 0. \end{aligned}$$

Suppose T carries independent vectors to independent vectors. Let $v \neq 0$, where $v \in V$. The vector v by itself is independent. Therefore, $T(v)$ is independent. Therefore, $T(v) \neq 0$, since 0 is dependent. Therefore, the kernel of T is $\{0\}$. ■

Theorem: Let $T : V \to W$ be linear and suppose that $\dim V = \dim W$. Then, the following are equivalent.

1) T is invertible.
2) T is non-singular.
3) T is onto.
4) If $v_1, ..., v_N$ is a basis of V, then $T(v_1), ..., T(v_N)$ is a basis of W.
5) There is a basis $v_1, ..., v_N$ of V such that $T(v_1), ..., T(v_N)$ is a basis of W.

Proof: $1 \Longrightarrow 2$. Obvious.

$2 \Longrightarrow 3$. Suppose that T is non-singular. Let $v_1, ..., v_N$ be a basis of V. By the previous theorem, $T(v_1), ..., T(v_N)$ are independent. Since $\dim W = N$, $T(v_1), ..., T(v_N)$ is a basis of W. If $w \in W$, $w = c_1 T(v_1) + \cdots + c_N T(v_N) = T(c_1 v_1 + \cdots + c_N v_N)$. Therefore, T is onto.

$3 \Longrightarrow 4$. Let $v_1, ..., v_N$ be a basis of V. Since these vectors span V and T is onto, $T(v_1), ..., T(v_N)$ span W. Since $\dim W = N, T(v_1), ..., T(v_N)$ are independent. Therefore $T(v_1), ..., T(v_N)$ is a basis of W.

$4 \Longrightarrow 5$. Obvious.

$5 \Longrightarrow 1$. Suppose that there is a basis $v_1, ..., v_N$ of V such that $T(v_1), ..., T(v_N)$ is a basis of W. Then, rank $T = \dim W = \dim V$. Therefore, by a previous theorem, nullity of $T = 0$. Therefore, T is one to one. Since rank $T = \dim W$, T is onto. Therefore, T is invertible. ■