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ON INEQUALITY COMPARISONS

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0. Introduction

Is one distribution (of income, consumption, or some other economic variable) among families or individuals more or less equal than another?[1] Despite the seeming straightforwardness of this question, there has been and continues to be considerable debate over how to go about finding the answer. One can adopt either an ordinal or a cardinal approach to inequality comparisons. If A and B are two alternative income distribution patterns, an ordinal approach specifies whether A is more equal than B while a cardinal approach requires the additional specification of how much more equal A is than B.

The traditional approach has been cardinal. Dating back at least to 1905 and the classic work of Lorenz and Gini, economists have proposed to compare distributions by means of summary measures such as a Gini coefficient, variance of logarithms, and the like. Often these indices seem to have been used more because of their existence than because of a careful examination of the properties they have. The additional information (i.e., how much more equal) is not only a source of controversy but also redundant for purposes of ranking the inequality of two distributions.[2]

In recent years, a number of writers have reversed the direction of inquiry.[3] The new approach is to start by specifying as axioms a relatively

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[1]Throughout this paper, we shall talk in terms of income distributions among families. All results apply, however, without modification to comparisons of inequality in the distribution of any quantifiable economic magnitude.

[2]Cardinality of inequality is redundant and controversial for purposes of ranking of distributions in the same sense that cardinal utility is redundant and controversial in the analysis of consumer choices. See Hicks (1939, p. 17).

[3]Much of this literature is summarized in Sen (1973).

small number of properties which a "good" index of inequality should have and then examining which if any of the various measures now in use satisfy those properties. The key issue is the reasonableness of the postulated properties. Work to date has shown the barrenness of the Pareto criterion,[1] but has not yet sought to develop an alternative axiomatic structure. The primary purpose of this paper is to contribute to such a development.

We shall postulate three axioms: scale irrelevance, symmetry, and desirability of rank-preserving equalization. The axiomatic system so constructed is intentionally incomplete. The advantage of an incomplete system is that we can then show that several indices in current use (the Gini coefficient, coefficient of variation, Atkinson index, and Theil index) satisfy our axioms.[2] This lends support to their reasonableness. However, they differ in ways which lie outside the scope of our axioms. Hopefully, future research will uncover additional axioms which will narrow down this incompleteness.

The plan of the paper is as follows. Three axioms which have been utilized in various contexts for inequality comparisons will be introduced in Section 1. We will then use these axioms to investigate and strengthen previous results on the consistency of alternative orderings in terms of Lorenz domination (Section 2). Then we design a general method to show

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[1]For an axiomatic development of the Pareto criterion, see Sen (1973).

[2]Indices of inequality, including those mentioned above, are cardinal measures which naturally introduce a pre-ordering. Thus, rigorously, it is the pre-ordering R induced by the index which satisfies our axioms.

that the orderings induced by many of the popular indices satisfy the first three axioms (Section 3). The concluding section reviews the highlights of this type of approach to inequality comparisons and suggests directions for future research extending these results.

1. Three Axioms for Inequality Comparisons

Suppose there are n families in an economy whose incomes may be represented by the non-negative row vector

$$(1,1)\quad X = (X_1\ X_2\ \ldots\ X_n) \geq 0$$

in the non-negative orthant of the n-dimension income distribution space Ω^+. A point in Ω^+ is a pattern of income distribution. In this paper, we shall exclude the origin (0 0 ... 0) (i.e., when no family receives any income) from Ω^+. The object of inequality comparisons between two such patterns is to be able to say that one is more or less equal than the other. More specifically, we wish to introduce a complete pre-ordering[1] of all points in Ω^+, i.e., a binary relation "G" defined on ordered pairs in Ω^+ satisfying the conditions of comparability and transitivity:

(1.2) (a) Comparability. For any X and Y in Ω^+, exactly one of the following is true:

(i) XGY ... in which case we write $X \succ Y$ and read "X is more equal than Y"

(ii) YGX ... in which case we write $Y \succ X$ and read "Y is more equal than X"

(iii) XGY and YGX ... in which case we write X = Y and read "X and Y are equally unequal."

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[1] Intuitively, a complete pre-ordering has exactly the same meaning as the ranking of commodity bundles by ordinary (ordinal) indifference curves in consumer analysis.

(b) Transitivity. XGY and YGZ implies XGZ.

From now on, we shall denote a complete pre-ordering by R.

We now introduce three properties which we shall propose as axioms for inequality comparisons. Not only do these seem reasonable to us but in addition they have been used by previous writers on inequality.

First, suppose two distributions X and Y are scalar multiples of one another:

(1.3) $X = aY$, i.e., $(X_1\ X_2\ \dots\ X_n) = (aY_1\ aY_2\ \dots\ aY_n)$, $a > 0$.

Because inequality in the distribution of income and the level of income enter as separate arguments into judgments of social well-being, it would seem reasonable and desirable for comparisons of inequality to be independent of the level of income. For this reason, we require that the two distributions X and Y in (1.3) be ranked as equally unequal.[1] Hence, we postulate:

A1. Axiom of Scale Irrelevance. $X = aY$ $(a > 0)$ implies $X \approx Y$.

This axiom allows us to normalize all distributions X in Ω^+ according to the fraction of income received by each family:

(1.4) $[X = (X_1\ \ X_2\ \dots\ X_n)] \approx [\theta = (\theta_1\ \theta_2\ \dots\ \theta_n)]$ where

$\theta_i = X_i\ /\ (X_1 + X_2 + \dots + X_n)$ for $i = 1, 2, \dots, n$.

The totality of all such normalized patterns, Ω^c, is the subset of points $\theta = (\theta_1\ \theta_2\ \dots\ \theta_n)$ of Ω^+ satisfying the conditions

(1.5) $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.

[1]Following Atkinson (1970), we would note that this condition is analogous to constant relative inequality aversion. For further applications of this notion to inequality comparisons, see also the papers by Rothschild and Stiglitz (1973) and Dasgupta, Sen, and Starrett (1973).

Axiom 1 assures us:

Lemma 1.1. If R is first defined on Ω^c, then it can be extended uniquely to Ω^+.

Next, suppose the elements in one vector X are a permutation of the elements of Y, i.e., the frequency distributions of income are the same but different individuals receive the income in the two cases. On the principle of treating all individuals or families as the same with regard to income distributions, these two patterns can be characterized by the same degree of inequality. Hence, we state:

A2. Axiom of Symmetry.[1] If $(i_1, i_2, \ldots, i_n)$ is any permutation of $(1, 2, \ldots, n)$, then $(X_1\ X_2\ \ldots\ X_n) = (X_{i_1}\ X_{i_2}\ \ldots\ X_{i_n})$.

Let $(i_1^*, i_2^*, \ldots, i_n^*)$ be a particular permutation of $(1, 2, \ldots, n)$. Then those $\theta = (\theta_1\ \theta_2\ \ldots\ \theta_n)$ in Ω^c which satisfy the condition

$$(1.6)\quad \theta_{i_1^*} \leq \theta_{i_2^*} \leq \ldots \leq \theta_{i_n^*}$$

comprise a rank-preserving subset of Ω^c. There are altogether n! such rank-preserving subsets. Suppose R is defined for any one of them. Then A2 allows us to extend it uniquely to the entire set Ω^c and, by Lemma 1.1, to the full income distribution space Ω^+. For convenience, we shall work with

[1] A2 is sometimes referred to as the axiom of anonymity in the literature [see Sen (1973)]. Sen also includes an illuminating discussion highlighting the conflicts between A2 and a Benthamite utilitarian approach to social judgments (in which social welfare is taken as the sum of individual utilities).

the permutation with the natural order (1, 2, ..., n). Denote the corresponding rank-preserving subset as Ω_0, which includes all points satisfying the conditions

$$(1.7)\quad \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n;\ \theta_i \geq 0;\ \sum_{i=1}^{n} \theta_i = 1$$

Ω_0 will be referred to as the monotonic rank-preserving set. A1 and A2 allow us to state the following:

Lemma 1.2. Under A1 and A2, if R is first defined on the monotonic rank-preserving set Ω_0, then it can be extended uniquely to Ω^+.

Notice from Lemma 1.2 that after postulating A1 and A2, we can restrict our search for "reasonable" properties to the space Ω_0.

Next, let X and Y be two alternative distribution in Ω_0 such that X is obtained from Y by the transfer of a positive amount of income h from a relatively rich family j to a poorer family i, i < j. We shall write X = E(Y) and say that X is obtained from Y by a rank-preserving equalization. For a particular pair i, j (i <j), there is a maximum amount which can be transferred if the rank is to be preserved. Formally,

Definition. Rank-Preserving Equalization. X = E(Y) if for some i,j (i<j) and h>0,

(1.8) (a) $X_k = Y_k$ for $k \neq i,j$,

$X_i = Y_i + h$,

$X_j = Y_j - h$, where:

(b) If $j = i + 1$, $h \leq 1/2\ (Y_j - Y_i)$;

If $j > i+1$, $h \leq \min\ [(Y_{i+1} - Y_i), (Y_j - Y_{j-1})]$.

Example 1. Rank-Preserving Equalization.

Let Y = (.01 .02 .04 .06 .07 .10 .70) and suppose the sixth family is to transfer income to the second family. Then the maximum rank-preserving amount of transfer is min $(Y_3 - Y_2, Y_6 - Y_5)$ = .02. If this amount is transfered, the new distribution $X = E(Y) = (.01\ .04\ .04\ .06\ .07\ .08\ .70)$. For any $h > .02$, the second and third families will switch rank and the transfer would not be rank-preserving. Suppose instead that the sixth family were to transfer a positive amount of income to the fifth family. The maximum amount that could be transferred without reversing the rank is $(Y_6 - Y_5)/2 = .015$ and for such a transfer the new distribution is $X = E(Y) = (.01\ .02\ .04\ .06\ .085\ .085\ .70)$. As before, the transfer of any larger amount would not be rank-preserving. Although this example illustrates rank-preserving equalizations in Ω_0, a similar definition will be stated on Ω^+ later.

The next axiom which we shall introduce is:

A3. <u>Axiom of Rank-Preserving Equalization.</u> In Ω_0, if $X = E(Y)$, then $X \succ Y$.[1]

The intuitive justification for this axiom is simply that it is reasonable to regard as more equal a distribution which can be derived from another by a richer person giving a part of his income to a poorer person.[2] Defining the <u>perfect equality point</u> as $\Phi = (1/n\ 1/n\ \dots\ 1/n)$, any income distribution point X in Ω_0 can be transformed into Φ by a finite sequence of rank-preserving equalizations.[3] Thus A3 and the transitivity of the ordering imply:

<u>Lemma 1.3.</u> $\Phi = (1/n\ 1/n\ \dots\ 1/n) \succ X$ for all $X \neq \Phi\ \epsilon\ \Omega_0$.

The proof is immediate.

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[1]Precedent for this axiom dates back at least to Dalton (1920), who called this the "principle of transfers."

[2]Note the importance of rank-preservation in this axiom. With reference to the above example, suppose contrary to our construction that the sixth individual transferred $Y_j - Y_i = .08$ to the second individual. Then each would wind up with the other's original income and, by the Axiom of Symmetry (A2), the new situation would be characterized by the same degree of inequality as the old.

[3]This assertion is easily proven by constructing a sequence of transfers from families above the mean to those below.

Notice that A3 has been introduced only on Ω_0. Suppose now we introduce an R on Ω_n satisfying all three axioms. Bv Lemma 1.2, R can be extended uniquely to the entire income distribution space Ω^+. It is clear that the property of A3 is automatically extended. Formally:

<u>Definition</u>. Let X and Y be two patterns of income distribution in Ω^+. We shall say that X is obtained from Y <u>by a rank preserving equalization</u>, in notation $X = E(Y)$, if

a) X and Y belong to the same rank preserving subset[1]

b) X is obtained from Y by the transfer of a positive amount of income h from a relatively rich family (e.g. $Y_q = X_q - h$) to a relatively poor family (e.g. $Y_p = X_p + h$) for $Y_q > Y_p$.

Notice that $X = E(Y)$ is now defined for the entire income distribution space Ω^+. However, this definition coincides with the previous definition (1.8 a,b) where both X and Y belong to Ω_0. Thus

<u>Lemma 1.4</u>. If R is first defined on the monotonic rank-preserving set Ω_0 satisfying A1-A3, the unique extension of R to Ω^+ also possesses the property of desirability of rank preserving equalization, i.e., if $X=E(Y)$ then $X \succ Y$.

A1-A3 constitute an axiomatic system. To show this is so, we require that the axioms be consistent (i.e., there exists an ordering satisfying all three axioms) and independent (i.e., there exist orderings satisfying each pair of axioms but not the third). We illustrate these ideas in example 2 below.

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[1] For some permutation $i_1\ i_2 \ldots i_n$, if $Y_{i_1} \le Y_{i_2} \le \ldots \le Y_{i_n}$ then $X_{i_1} \le X_{i_2} \le X_{i} \ldots \le X_{i_n}$.

Example 2: A Two Person Economy

Suppose there are two individuals with non-negative incomes X_1 and X_2. The income distribution space Ω^+ is the positive quadrant of Figure 1 excluding the origin. The Axiom of Scale Irrelevance (A1) corresponds to linear iso-inequality rays emanating from the origin. The Axiom of Symmetry (A2) requires that the iso-inequality rays be symmetric about the 45° line. Lemma 1.1 allows us to confine our attention to the line AB rather than the entire positive quadrant. There are $2! = 2$ rank preserving subsets, $\Omega(1,2) = \Omega_o$ (represented by the line segment $B\Phi$) and $\Omega(2,1)$ (represented by $A\Phi$), separated by the perfect equality point Φ. Lemma 1.2 lets us limit our attention further to line segment $B\Phi$ satisfying $X_1 \leq X_2$. The Axiom of Rank-Perserving Equalization (A3) requires that starting from an initial point Y on $B\Phi$, when the richer family's income is reduced and the poorer family's raised, the new point X be preferred to Y ($X \succ Y$). Graphically, this occurs when X is closer to Φ than Y. For example, we can define $X \succ Y$ when the distance $|\Phi-X|<|\Phi-Y|$. This determines a pre-ordering R on $B\Phi$ which can be extended symmetrically to $A\Phi$ and projectively to the entire non-negative quadrant. This example shows that A1-A3 are consistent.[1]

To show independence, we must give examples of orderings which satisfy two of the axioms but not the third. If the iso-inequality rays of Figure 1 were replaced by a set of non-linear symmetric (and symmetrically indexed) curves such as those depicted in Figure 2, A1 would be violated although A2 and A3 would be satisfied. Next, with the linear iso-inequality rays of Figure 1, define the ordering on the line segment $B\Phi$ as before, but now for a given $k > 1$, for any point X' on line segment $A\Phi$, define $X' = X$ for X on $B\Phi$ if the distance $|\Phi-X'| = k\,|\Phi-X|$. Then A1 and A3 are satisfied and A2 is violated. Finally, for Figure 1, on $B\Phi$ (and symmetrically on $A\Phi$), define $Y \succ X$ when the distance $|\Phi-Y|<|\Phi-X|$. Then A1 and A2 are satisfied and A3 is violated. Thus, the three axioms are independent.

2. Inequality Comparisons: Zones of Ambiguity and Lorenz Domination

In the last section, we showed that if we postulate a set of "reasonable" axioms for R on Ω_0, then R can be extended from Ω_0 to the entire income distribution space Ω^+. We have not as yet considered whether the three axioms

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[1]Notice that in this example, A1-A3 completely determine R.

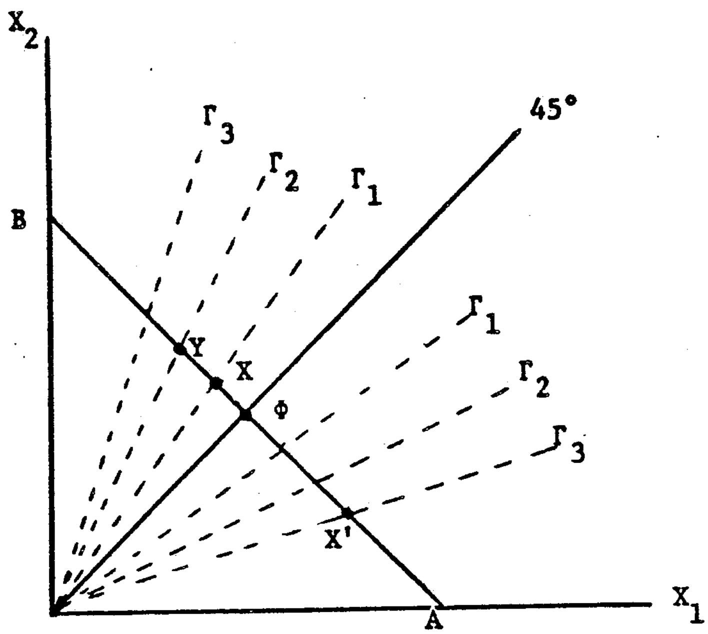

Figure 1

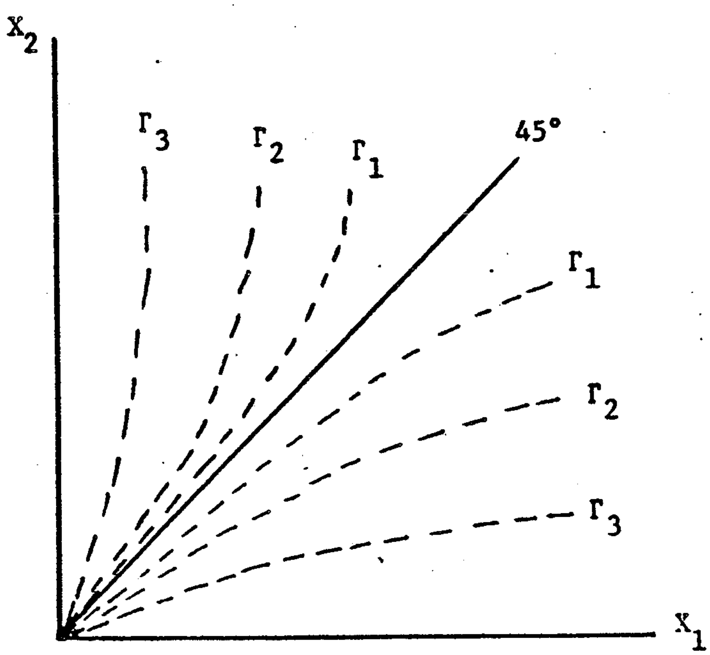

Figure 2

are sufficient to allow us to compare any two points X,Y in Ω^+ according to the comparability condition (1.2.a). A noteworthy feature of the two person case introduced above is that our three axioms do have this power. However, this is not true for $n \geq 3$. In this section, therefore, we examine when inequality comparisons can or cannot be made using A1-A3.

A. Zones of Ambiguity

We shall now show that there are well-defined ranges in which inequality comparisons can be made using A1-A3 alone and other well-defined zones of ambiguity where comparisons cannot be made without further specification of the rules of ordering. We begin by illustrating these relationships for the case n = 3.

Example 3: A Three Person Economy (n = 3)

We illustrate zones of comparability and ambiguity in the three person case by relying on a property of an equilateral triangle (ABC in Figure 3), namely, that the sum of the perpendicular distances to the sides from any point Z ($ZZ_1 + ZZ_2 + ZZ_3$) is the same for all points in the triangle and equal to AD.[1] If AD = 1 (i.e., has unit distance), the point set bounded by ABC represents Ω^c for n = 3. The perpendicular bisectors AD, BE, and CF partition Ω^c into 3! = 6 rank-preserving subsets ($\Omega(1,2,3)$, $\Omega(2,1,3)$, ...) according to (1.6). The monotonic rank-preserving subset $\Omega_0 = \Omega(1,2,3)$ is depicted as the lower right region DCΦ, where Φ is the perfect equality point.

For any arbitrary point Y in Ω_0, the rest of the region can be partitioned into six zones I-VI according to the direction of income transfers needed to go from Y. A single rank-preserving transfer, holding one person's income constant, is depicted as a movement along one of the three auxiliary lines a_1b_1, a_2b_2, a_3b_3 parallel to the sides of the triangle passing through Y. For instance, points along a_1b_1 correspond to a single rank-preserving equalization or disequalization involving the second and third families holding family 1's income constant. All other points in Ω_0 are obtained from Y by a combination of rank-preserving equalizations or disequalizations.

The set of points in Zone I and II are denoted by Y*. Any point in Y* can be obtained from Y by a finite sequence of rank-preserving equalizations. For example, we can transform Y into the point U in three steps through the points a_1 and T, where the line a_1T parallels AB and the line TU parallels AC.

Thus, by A3 and the transitivity of R, $X \succ Y$ for every X in Y^*. Similarly, denote the set of points in Zones IV and V by Y_*; then $Y \succ X$ for every X in Y_*. Finally, the set of points in Zones III and VI are denoted by M^* and M_* respectively. A point such as Q in M^* can be obtained from Y in two steps: (i) a rank-preserving equalization from Y to W, and (ii) a rank-preserving disequalization from W to Q. A3 tells us that W is more equal than Y (as income is transfered from the middle income to the poor family) and Q is less equal than W (as income is transfered from the middle income to the high income family). These zones (M^* and M_*) might then be thought of as "zones of ambiguity" (relative to Y), for without additional specification of the relative weights we wish to give to the respective income transfers, our axioms A1-A3 are insufficient to tell us which distribution (Y or Q) is the more equal. The transfers in these six cases are summarized in Table 1.

B. Lorenz Domination

The ideas in example 3 will now be generalized to Ω_0 for the general n person case. We shall also establish that there is a direct one-to-one correspondence between the zones of ambiguity and the more familiar concept of Lorenz domination, which we examine below.

The first concept we need to introduce is a sequence of equalizations from a given point $Y \in \Omega_0$ according to the following definition:

Definition. X is obtained from Y by a finite sequence of equalizations, $X = T(Y)$, when

(2.1) $X = E_k(\ldots E_2(E_1(Y))\ldots)$.

Starting from a given point Y, we can define three sets Y^*, Y_*, and M as follows:

(2.2) (a) $Y^* = \{X | X = T(Y)\}$,

(b) $Y_* = \{X | T(X) = Y\}$,

(c) $M = \Omega_0 - Y^* \cup Y^*$.

Y^* is the set of all points in Ω_0 obtained from Y by a sequence of equalizing transfers, while Y_* includes those points in Ω_0 from which a sequence of equalizing transfers will lead to Y. We can also talk about disequalizing

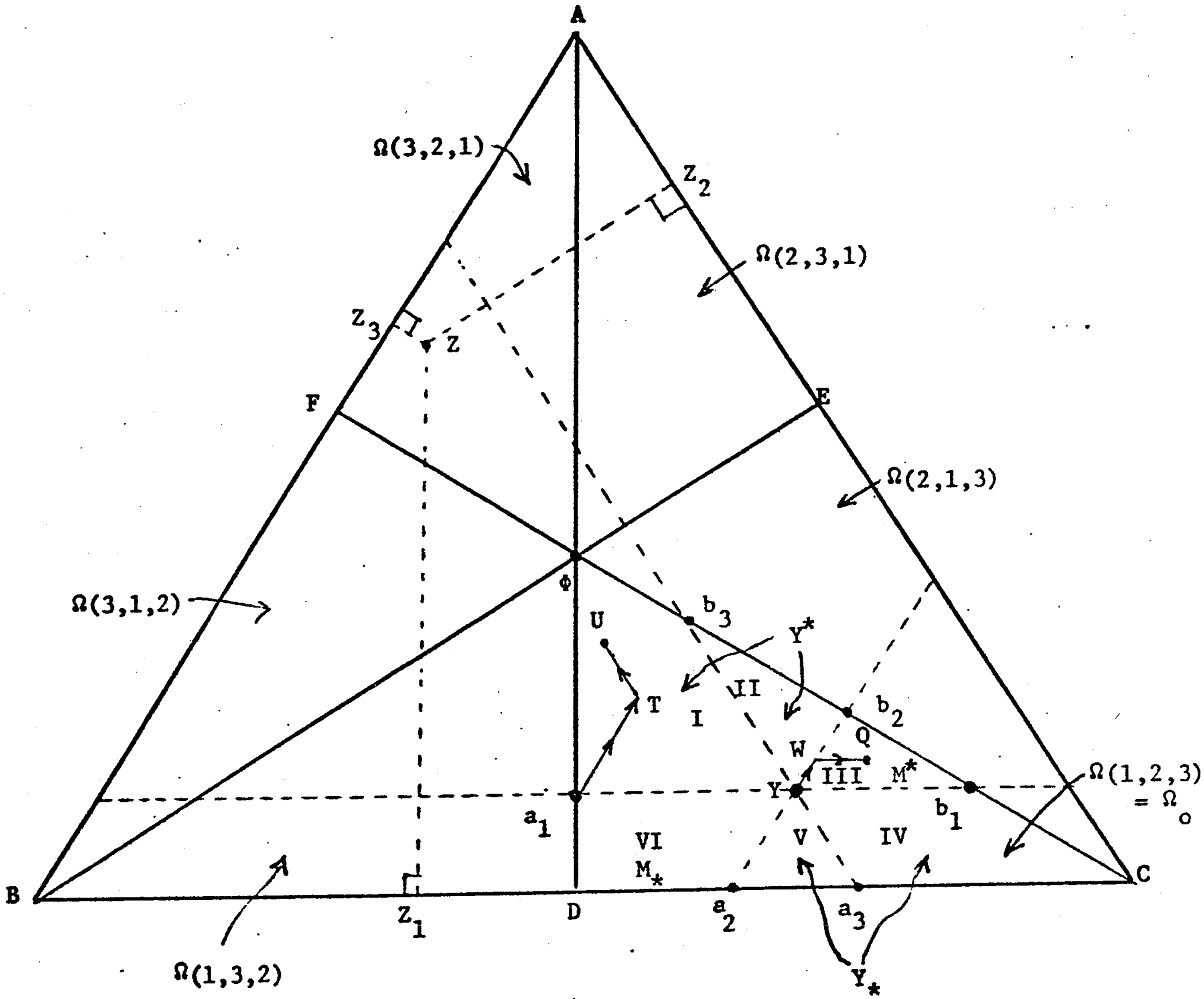
A
Ω(3,2,1)
Z_2
Ω(2,3,1)
Z_3
Z
F
E
Ω(2,1,3)
Φ
Ω(3,1,2)
b_3
U
Y^*
T
II
I
b_2
Q
W
III
M^*
Y
b_1
Ω(1,2,3)
= Ω_o
a_1
VI
M_*
V
IV
B
Z_1
D
a_2
a_3
C
Ω(1,3,2)
Y_*

Figure 3

TABLE 1

Zone	All Points in Zone Contained in:	As Compared with Y_i: Y_1	Y_2	Y_3	Crossing Index (r)	Comparisons Possible Using A1-A3:
I	Y^*	+	+	-	2	Yes
II	Y^*	+	-	-	2	Yes
III	M^*	+	-	+	3	No
IV	Y_*	-	-	+	2	Yes
V	Y_*	-	+	+	2	Yes
VI	M_*	-	+	-	3	No

transfers as the transfer of income from a relatively poor to a relatively rich family, in which case Y_* is the set of all X which can be obtained from Y by a sequence of disequalizing transfers. The set M contains all other points of Ω_0.

It follows directly from (2.2.c) that the set M contains all points which are not unambiguously comparable with Y under A1-A3. A point Z in M can always be transformed into Y by a finite sequence of rank-preserving transfers. However, any such sequence necessarily involves at least one equalization and at least one disequalization---which is why Z cannot be compared with Y. The theorem we prove below, Theorem 2.1, implies that the Lorenz curves of Z and Y necessarily cross each other.

Another concept we need is Lorenz-domination. For two points X and Y in Ω_0, the Lorenz Curve of X is said to dominate that of Y according to the following definition:

Definition. X Lorenz-dominates Y (in notation, $L_X \geq L_Y$) when

(2.3) (a) $X_1 + X_2 + \ldots + X_j \geq Y_1 + Y_2 + \ldots + Y_j$ for $j = 1, 2, \ldots, n-1$

(b) $X_1 + X_2 + \ldots + X_j > Y_1 + Y_2 + \ldots + Y_j$ for some $j<n$.

Notice that

(2.4) $$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i = 1 \text{ in } \Omega_0.$$

In other words, one distribution Lorenz-dominates another if the Lorenz Curve of the first distribution never lies below that of the second and lies above it at least one point.

The basic theorem of this section is:

<u>Theorem 2.1.</u> $X \in Y^*$ if and only if $L_X \geq L_Y$.

Thus, the Lorenz Curve of Y is dominated by the Lorenz Curves of all $X \in Y^*$, dominates those of $X \in Y_*$, and crosses those of $X \in M$, i.e., neither dominates the other.

The necessary condition of the theorem (i.e., $X \in Y^*$ implies that the Lorenz curve of X dominates that of Y) is a well-known result.[1] The sufficient condition of the theorem states that whenever the Lorenz curve of X dominates that of Y (i.e., $L_X \geq L_Y$), X can be obtained from Y by a sequence of rank-preserving equalizations within Ω_0. This sufficient condition, when proved, along with A3 and the transitivity of R, will allow us to conclude that for all X in Y^*, $X \succ Y$. This may be summarized as:

<u>Corollary 2.2.</u> Under A3, for X, Y in Ω_0, $L_X \geq L_Y$ implies $X \succ Y$.

The proof of the sufficient condition, which requires the construction of a sequence of rank-preserving equalizations within Ω_0 when (2.3) is satisfied,[2] will be given after the theorem is illustrated for the case n=3.

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[1]See Atkinson (1970), Rothschild and Stiglitz (1973), and Dasgupta, Sen, and Starrett (1973).

[2]Rothschild and Stiglitz have proven that when the Lorenz Curve of X dominates that of Y, it is possible to construct a sequence of transfers which may or may not be rank-preserving, i.e., they may move out of and back into Ω_0. The sufficient condition which we shall prove in the text is a stronger version requiring that such a sequence be rank-preserving and stay within Ω_0.

Example 4. Illustration of Theorem 2.1 in the Three Person Case.

The equilateral triangle ABC of Example 3 is reproduced in the left-hand side of Figure 4. For a given point Y in Ω_0, the sets M* and M_* are now shaded. AD has unit distance, so we construct the unit square CGEF beside the triangle. The vertical lines pq and st mark off the thirds of the unit distance on the horizontal axis. For the income distribution pattern Y, we can now construct its Lorenz curve in the unit square by the following procedure. The point d_1 (on pq) has the same vertical distance as Y, and thus qd_1 is the income of the poorest family. Next from the point Y, locate point Z (on AC) by extending the line a_2b_2 (parallel to AB). The sum of the incomes of the first two families equals the vertical distance ZW. We can then locate the point d_2 (on st) which has the same vertical distance as Z. The Lorenz curve of Y, L_Y, then passes through d_1 and d_2.

If X is a point in Y*, its Lorenz curve, L_X, constructed by the same rule, will lie above and therefore dominate L_Y. (This is because e_1 is higher than d_1 and e_2 higher than d_2 as long as X is in Y*.) Similarly, the Lorenz curve of any point in Y_* will be dominated by L_Y. Finally, the Lorenz curve of any point in the shaded regions M* and M_* will cross L_Y from above and below respectively.

C. Proof of the Sufficient Condition of Theorem 2.1

The sufficient condition of Theorem 2.1 holds that whenever X Lorenz-dominates Y, there exists a sequence of rank-preserving equalizations leading from Y to X. In order to prove the validity of this part of the theorem, we must produce a rule for finding the necessary sequence. Let us first illustrate the procedure for a simple numerical example.

Example 5. Illustration of Sequence of Rank Preserving Equalizations

Suppose we have two distributions X and Y in Ω_0:

	1	2	3	4	5	6	7	8	9	10
X = (	.010	.020	.030	.040	.050	.060	.070	.080	.090	.550),
Y = (	.005	.016	.030	.045	.052	.055	.068	.070	.080	.570)

Let their difference be denoted by the vector

$$d = X - Y = (\underbrace{+.005 \quad +.004 \quad 0}_{D_1} \quad \underbrace{-.005 \quad -.002}_{D_2} \quad \underbrace{+.005 \quad +.002 \quad +.010 \quad +.010}_{D_3} \quad \underbrace{-.029}_{D_4})$$

$$S_1 = +.009 \qquad S_2 = -.007 \qquad S_3 = +.027 \qquad S_4 = -.029$$

That X Lorenz dominates Y may be seen by observing that the cumulative value of X must be no less than the cumulative value of Y, or equivalently, the cumulative value of d must be non-negative. This can be more readily verified when the positive d_i and the negative d_i are grouped up separately as shown. The Lorenz domination can be seen from the fact that

$$S_1 > 0, \quad S_1 + S_2 > 0, \quad S_1 + S_2 + S_3 > 0, \quad S_1 + S_2 + S_3 + S_4 = 0.$$

The following sequence of rank-preserving equalizations would convert Y into X:

(i)	Take .010	from person 10,	give to person 9;
(ii)	.010	10,	8;
(iii)	.002	10,	7;
(iv)	.005	10,	6;
(v)	.002	10,	2;
(vi)	.002	4,	2;
(vii)	.003	4,	1;
(viii)	.002	5,	1.

The reader may easily verify that each equalization is in fact rank-preserving, so that Y is transformed into X entirely within Ω_0.

To infer a general rule for rank-preserving equalizations from Example 5, given any two distributions X and Y in Ω_0, let their difference be denoted by

(2.5) (a) $d = (d_1 \; d_2 \; \ldots \; d_n) = (X_1 - Y_1 \;\; X_2 - Y_2 \; \ldots \; X_n - Y_n)$

(b) $\Sigma \, d_i = 0.$

Given (2.5.b), the n elements of d can be partitioned consecutively into r subsets $(D_1 \; D_2 \; \ldots \; D_r)$ according to the following rules:

(2.6) (a) Every d_i belongs to one D_j. (Jointly exhaustive)

(b) If $d_i \in D_j$ and $d_p \in D_q$, then $j < q$ implies $i < p$.

(Consecutive partition)

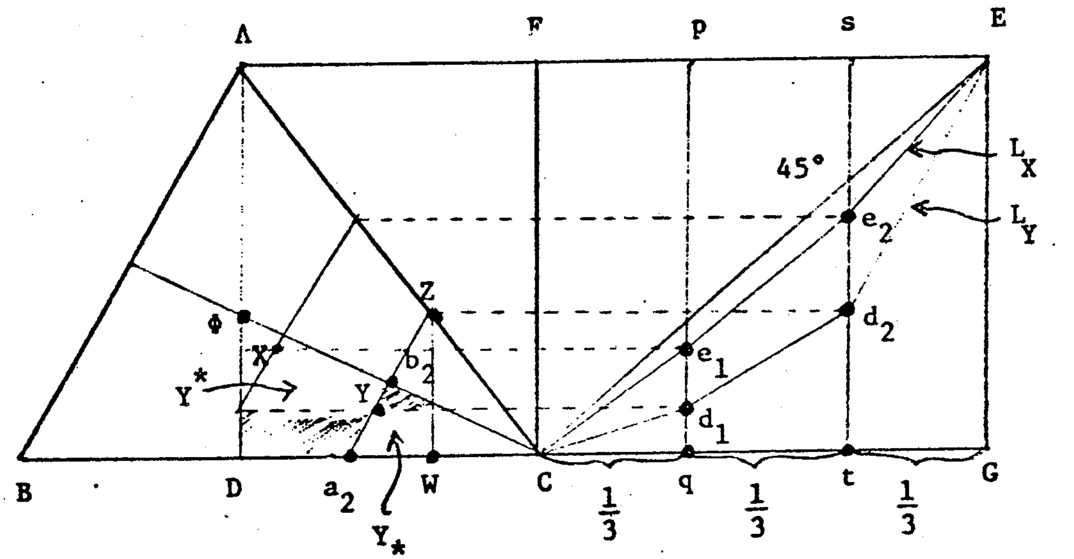
A
F
p
s
E
45°
L_X
L_Y
Z
e_2
d_2
Φ
X^*
b_2
e_1
Y^*
Y
d_1
B
D
a_2
W
C
q
t
G
Y_*
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$

Figure 4

(c) All $d_i \in D_j$ are non-positive or non-negative [D_j is called <u>positive</u> (or negative) according to the signs of the d_i in D_j.]

(Sign preserving)

(d) The first element of $D_2, D_3, \ldots, D_r$ is non-zero.

(e) The D_j alternate in sign.

It can be easily shown, as in Example 5, that the partition is unique. Furthermore, if $X \neq Y$, then there is at least one strictly positive d_i and one strictly negative d_j. Thus,

(2.7) If $X \neq Y$, $r \geq 2$.[1]

We can also define

(2.8) $$S_j = \sum_{d_i \in D_j} d_i \text{ for } j = 1, \ldots, r$$

with the properties

(2.9) (a) $S_j \neq 0$, $j = 1, 2, \ldots, r$,

(b) $S_1, S_2, \ldots, S_r$ alternate in sign

(c) $$\sum_{j=1}^{r} S_j = \sum_{i=1}^{n} d_i = 0.$$

Note how these conditions determine the groupings in Example 5.

[1]The number (r-1) may be thought of as a <u>crossing index</u>, since if we were to plot the two distributions X and Y with two curves they would cross (r-1) times. [Cf. Table 1.]

We may now state a <u>general rule for rank preserving equalizations</u> from X to Y for which Example 5 is an illustration:

(a) Identify the groups according to (2.6).

(b) With each transfer, eliminate the gap between X and Y of one family's income by

(i) Taking from the poorest family (the p'th) with non-zero d in the richest group (S_r),

(ii) Giving to the richest family (the q'th) with non-zero d in the next lower group (S_{r-1}),

(iii) Compute the amount of transfer as the smaller of d_p and $-d_q$.

(c) Repeat these steps (a,b) again, each time eliminating the gap for another family's income.

To prove the validity of this rule, we need to draw on the Lorenz domination condition of Theorem 2.1 by the following lemma.

<u>Lemma 2.3.</u> When $X \neq Y$, $L_X \geq L_Y$ is equivalent to (2.10.a) and (2.10.b):

$$(2.10)\ (a) \quad \sum_{j=1}^{i} d_j \geq 0 \text{ for } i = 1, \ldots, n$$

$$(b) \quad \sum_{j=1}^{p} S_j \geq 0 \text{ for } p = 1, \ldots, r$$

Proof: (2.3. a, b) imply (2.10.a). Conversely (2.10.a) implies (2.3.a) and, since $X \neq Y$, it also implies (2.3.b). Thus, (2.3.a,b) and (2.10.a) are equivalent when $X \neq Y$. It follows directly from (2.8) that (2.10.a) implies (2.10.b). Thus we need only prove the reverse implication. Suppose $d_i \in D_q = (d_{a+1}\ d_{a+2} \ldots d_{a+m_q})$ Then define

$$V_i = \sum_{j=1}^{i} d_j = S_1 + S_2 + \ldots + S_{q-1} + d_{a+1} + \ldots + d_i.$$

We want to prove $V_i \geq 0$. In this expression,

(2.11) (a) $S = S_1 + S_2 + \ldots + S_{q-1} \geq 0$, (by (2.10.b))

(b) $S_q = d_{a+1} + \ldots + d_{a+m_q}$, where all d's have same sign,

(c) $V_{a+m_q} = S + S_q \geq 0$ (by 2.10.b).

Thus, d_i is one member of a sequence $(V_{a+1}, V_{a+2}, \ldots, V_{a+m_q})$ which either (i) is monotonically increasing from $S \geq 0$ if D_q is a positive set, or (ii) is monotonically decreasing to $S+S_q \geq 0$ if D_q is a negative set. In either case, $V_i \geq 0$. Q.E.D.

Notice that (2.9.a,c) and (2.10.b) imply $S_1 > 0$ and $S_r < 0$. Thus (2.9.b) implies r is even. Hence,

Lemma 2.4. $L_X \geq L_Y$ implies r is even and the S_i can be grouped into r/2 pairs with the indicated signs:

(2.12) $(S_1^+ \; S_2^-) \; (S_3^+ \; S_4^-) \ldots (\overset{+}{S}_{r-1} \; S_r^-)$.

This is illustrated in Example 5 where r=4. Then when $L_X \geq L_Y$, families in the last group $\bar{S}_r$ of X must be poorer than those in Y. The opposite is true for the group $\overset{+}{S}_{r-1}$.

Before we can prove the validity of this rule we need an additional lemma. In this lemma suppose $Y' = (Y_1' \; Y_2' \ldots Y_n')$ is obtained from Y by a single rank-preserving equalization. Let

(2.13) $d' = (d_1' \ldots d_n') = (X_1 - Y_1' \;\; X_2 - Y_2' \ldots X_n - Y_n')$.

Then,

Lemma 2.5. If $L_X \geq L_Y$, there exists $Y' \in \Omega_0$ such that:

(2.14) (a) $Y' = E(Y)$,

(b) $L_X \geq L_Y'$,

(c) $d_i = 0$ implies $d_i' = 0$,

(d) there is at least one integer j such that $d_j \neq 0$ and $d_j' = 0$.

Proof: Suppose the last non-zero d_i in S_{r-1}^+ in (2.12) is d_p and the first non-zero d_i in S_r^- is d_q $(q > p)$. Thus, by this choice, we have

$$(2.15) \quad d_{p+1} = d_{p+2} = \ldots = d_{q-1} = 0.$$

Let

$$(2.16) \quad h = \min(d_p, -d_q) = \min(X_p - Y_p, Y_q - X_q) > 0.$$

When h is transferred from the q'th family to the p'th family of Y, let the result be denoted by Y'. Then obviously (2.14.a,c,d) are satisfied. To prove (b), we have

$$(2.17) \quad d_1' + d_2' + \ldots + d_i' = \begin{cases} d_1 + d_2 + \ldots + d_i \text{ for } i < p \text{ or } i \geq q \\ d_1 + d_2 + \ldots + d_i - h = d_1 + d_2 + \ldots + d_{p-1} + \\ \qquad \text{for } p \leq i < q. \end{cases}$$

The first sum $d_1 + d_2 + \ldots + d_i \geq 0$ by (2.10.a). In the second sum, $d_1 + d_2 + \ldots + d_{p-1}$ is non-negative by Lorenz-domination (2.10.a) and $(d_p - h)$ is non-negative because $h \leq d_p$. Thus $d_1' + d_2' + \ldots + d_i' \geq 0$ and $L_X \geq L_Y'$ by (2.10.a). Q.E.D.

Lemma 2.5 assures us that we can repeat the same operation on Y' by reducing one additional non-zero entry of d'. Since there are only a finite number of non-zero d_i we have:

Lemma 2.6. If $L_X \geq L_Y$, then there exists a sequence of transfers T such that X = T(Y) and T involves at most M steps, where M is the number of non-zero d_i in d (as given by (2.5)).

The proof of the sufficient condition of Theorem 2.1 follows directly from Lemma 2.6, as does the validity of the rule presented above.

D. Application of Theorem 2.1 to Zones of Ambiguity

When seeking to compare two distributions X and Y, we can use (2.12) to devise a simple rule for determining when L_X crosses L_Y (i.e., when $X \varepsilon M$). Following (2.12), the rule is simply to examine the sign of the first and last non-zero d_i and, if they have the same sign, the Lorenz curves must cross.[1]

If there are n individuals, the total number of possible ways in which the d's can vary is

$$P = 2^n - 2.$$ [2]

On the other hand, the number of cases when the rule applies is

$$N = 2^{n-1} - 2.$$ [3]

Thus, the ratio

$$\frac{N}{P} = \frac{2^{n-1}-2}{2^n - 2} \rightarrow \frac{1}{2}$$

from below as $n \rightarrow \infty$. When n is large, therefore, we can tell by inspection in roughly half the cases that the Lorenz curves cross. Small sample percentages are summarized in Table 2.[4]

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[1] However, if they have opposite sign, they may or may not cross and it is necessary to compare the full distributions.

[2] The two cases which are not possible are those in which all d's are either positive or negative. These are ruled out by the fact that $\Sigma\, d_i = 0$.

[3] This is because in half of the total cases (2^{n-1}) the first and last d's have opposite sign.

[4] In the case n=3, the P=6 cases correspond to the six regions of Figure 3 (see Example 4). The N=2 cases correspond to M^* and M_*. The crossing indices r of the frequency distributions for the six cases are shown in Table 1. When r is odd, the Lorenz curves cross once; when even, not at all.

TABLE 2

n	P	N	N/P
2	2	0	0
3	6	2	.33
4	14	6	.43
5	30	14	.47
6	62	30	.48
7	126	62	.49
8	254	126	.50
.	.	.	.
.	.	.	.
.	.	.	.

3. Traditional Approach to Inequality Comparisons

A. Inequality Indices

The traditional approach for comparing the inequality of two distributions is to compute an index of inequality I (i.e., a real-valued function with domain Ω^+):

$$(3.1)\quad I = f(X) = f(X_1\ X_2 \dots X_n),\ X_i \geq 0.$$

Examples of such indices are the Gini coefficient, coefficient of variation, range, and others which we shall consider below. Whenever such an index is given, it naturally induces a complete pre-ordering R according to the following definition:

Definition. Pre-Ordering Induced by an Index. A real-valued index of inequality $I = f(X)$ induces a pre-ordering R as follows: for all $X, Y \in \Omega^+$, X G Y when $f(X) \leq f(Y)$.[1]

[1]Notice that (3.1) measures inequality and therefore a more equal distribution has a lower index.

Notice that the cardinality of the index (3.1) is unnecessary for the question of determining which is the more equal of two distributions, since the essential information for this purpose is all contained in the pre-ordering R which f(X) induces.

It is the purpose of this section to show that the R's induced by many familiar inequality indices indeed satisfy the three axioms introduced in Section 1. We begin with two elementary ideas. The first is the equivalence of two cardinal indices:

Definition. Equivalence. Two indices $I_1 = f_1(X)$ and $I_2 = f_2(X)$ are equivalent (in notation, $I_1 \overset{e}{=} I_2$) when $f_1(X) \geq f_1(Y)$ if and only if $f_2(X) \geq f_2(Y)$.

Two equivalent indices obviously induce the same ordering. The second elementary idea is that two indices are equivalent when one is a strictly monotonic transformation of the other. Formally,

Lemma 3.1. $I_1 \overset{e}{=} I_2$ if and only if there exists a real-valued monotonic function g defined on the domain of real numbers such that $I_2(X) = g(I_1(X))$ for all $X \in \Omega^+$.

When a particular index $I = f(X)$ in (3.1) satisfies the restrictions specified below, the following theorem insures that R satisfied A1-A3:

Theorem 3.2 The pre-ordering R induced by an index $I = f(X)$ satisfies A1-A3 when:

(3.2) (a). Homogeneous of Degree Zero. $f(X) = f(aX)$, $a > 0$;

(b). Symmetry. $f(X_{i_1}\ X_{i_2} \ldots X_{i_n}) = f(X_1\ X_2 \ldots X_n)$, where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1,2, \ldots, n)$;

(c). Monotonicity of Partial Derivative.

$$\frac{\partial f}{\partial X_i} = f_i(X) < \frac{\partial f}{\partial X_j} = f_j(X) \quad \text{for } i < j \text{ and } X \in \Omega_0.$$

Proof: (3.2.a) insures that the induced ordering satisfies A1. Similarly, (3.2.b) insures that it satisfies A2. To show A3 holds on Ω_0, suppose X is obtained from $Y \in \Omega_0$ by a rank-preserving equalization (i.e., $X = E(Y)$) brought about by the transfer of a positive amount of income h from a relatively rich family j to a relatively poorer family i. Then the difference $I(X) - I(Y)$ is $D(h) = f(X) - f(Y) = f(Y_1 \ldots Y_i + h \ldots Y_j - h \ldots Y$ $- f(Y_1 \ldots Y_i \ldots Y_j \ldots Y_n)$, which is a function of h. Partially differentiating D(h) with respect to h, we have $\frac{\partial D(h)}{\partial h} = f_i(X) - f_j(X)$ which, when evaluated at $X = Y$, is negative by (3.2.c). Thus, A3 is satisfied in Ω_0. Q.E.D.

B. Relationship between Inequality Indices and A1-A3.

We now want to show that four of the most well-known indices of inequality --- the coefficient of variation, the Gini coefficient, the Atkinson index, and the Theil index --- satisfy conditions (3.2) and hence A1-A3.

Consider first the Coefficient of Variation:

(3.3) $C = C(X) = \sigma/\bar{X}$ where $\sigma = \sqrt{\sum_i (X_i - \bar{X})^2/n}$ and $\bar{X} = \sum_i X_i/n$.

Since both σ and $\bar{X}$ are homogeneous of degree one, C is homogeneous of degree zero and (3.2.a) is satisfied. Obviously, C also satisfies (3.2.b). To verify that C satisfies (3.2.c), we state first

Lemma 3.3. In Ω_0, the Coefficient of Variation C(X) is equivalent to $C* = \theta_1^2 + \theta_2^2 + \ldots + \theta_n^2$ where $\theta_i = X_i/\Sigma X_i$.

Proof: In Ω_0, $\bar{\theta} = 1/n$. Therefore, $C(\theta) = \sigma n = \sqrt{\Sigma(\theta_i - 1/n)^2}$

$$\stackrel{e}{=} \Sigma(\theta_i - 1/n)^2 - \Sigma\theta_i^{\,2} - 1/n \stackrel{e}{=} \Sigma\theta_i^{\,2}. \quad \text{Q.E.D.}$$

...ously satisfies (3.2.c). Since, by Lemma 3.3, C* and C are equivalent, ...lows that C also satisfies A1-A3 in Ω_0.

Next, consider the Gini Coefficient (G) defined for points ... $\theta_2 \dots \theta_n) \in \Omega_0$. Such a point is represented by the curve $f(\theta)$ ... in panel (b) of figure 5. The cumulative value of $f(\theta)$ is the Lorenz ... L_θ shown in the unit square in panel (a). Formally, a Lorenz Curve is ...l-valued function defined on a finite domain (1/n 2/n ... 1). Thus, ...renz Curve for θ is

(3.4) $L_\theta = \theta_1 + \theta_2 + \dots + \theta_r$ for r=1/n 2/n, ..., r/n,

... the Gini coefficient is defined as

(3.5) $G = \frac{A}{A+B}$,

... A and B are the areas indicated in Figure 5. It is clear that G ...sfies (3.2.a,b). To verify that G satisfies (3.2.c), we state first

Lemma 3.4. In Ω_0, the Gini coefficient G(X) is equivalent to ... $\theta_1^2 + 2\theta_2^2 + 3\theta_3^2 + \dots + n\theta_n^2$.

Proof: $G = \frac{A}{A+B} = 1-2B \stackrel{e}{=} (-B)$ where

$$B = \frac{1}{2}\frac{1}{n}(\theta_1 + \theta_2 + \dots + \theta_n) + \frac{1}{n}[(n-1)\,\theta_1 + (n-2)\theta_2 + \dots + \theta_{n-1})]$$

$$= \frac{1}{2n} - \frac{1}{n}(1\theta_1 + 2\theta_2 + \dots + n\theta_n) + \frac{1}{n}(n\theta_1 + n\theta_2 + \dots + n\theta_n)$$

$$= \frac{1}{2n} + 1 - \frac{1}{n}(\theta_1 + 2\theta_2 + \dots + n\theta_n).$$

... $G \stackrel{e}{=} \theta_1 + 2\theta_2 + \dots + n\theta_n$. Q.E.D.

... is clearly seen to satisfy (3.2.c). Thus, following the same lines of ...ment as we used for the Coefficient of Variation, G also satisfies A1-A3.

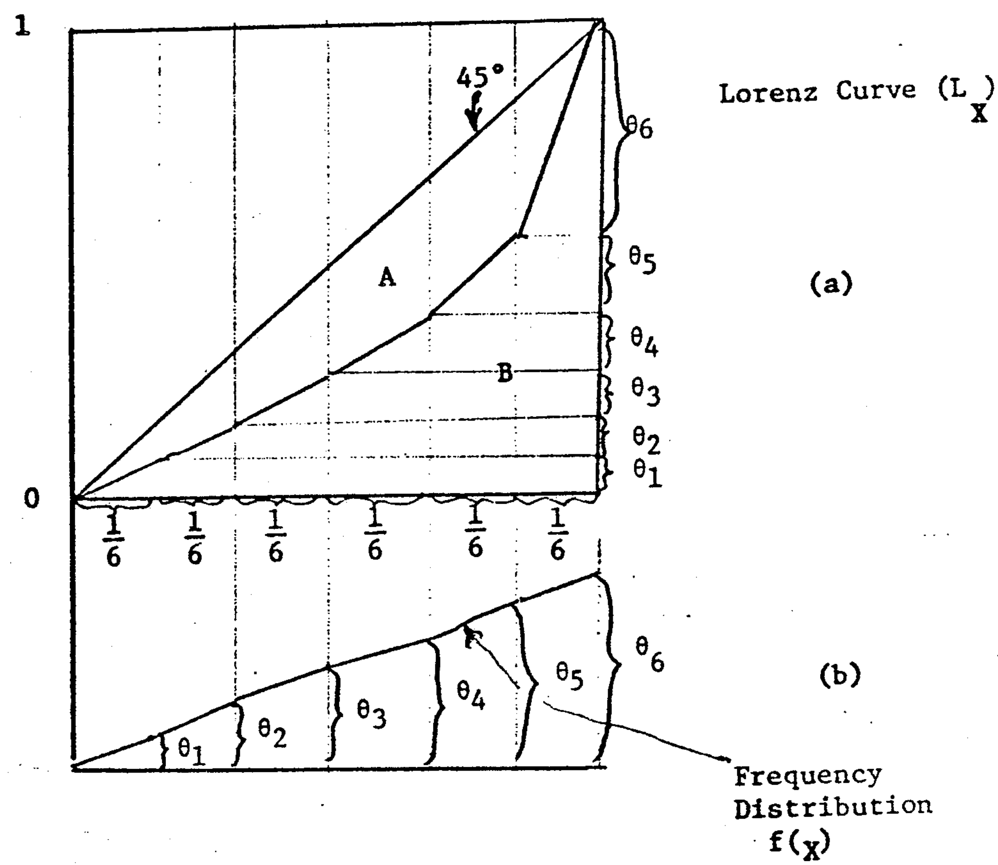
1
45°
Lorenz Curve (L_X)
θ_6
θ_5
A
(a)
θ_4
B
θ_3
θ_2
θ_1
0
$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$
θ_6
θ_5
(b)
θ_4
θ_3
θ_2
θ_1
Frequency
Distribution
$f(X)$

Figure 5

Another index which has recently been proposed is the Atkinson Index,[1] defined as

$$(3.6)\quad A = 1 - \left[\left[\left(\frac{X_1}{\bar{X}}\right)^{1-\varepsilon} + \left(\frac{X_2}{\bar{X}}\right)^{1-\varepsilon} + \ldots + \left(\frac{X_n}{\bar{X}}\right)^{1-\varepsilon}\right]\frac{1}{n}\right]^{\frac{1}{1-\varepsilon}} \quad \varepsilon > 1.$$

It is evident that (3.2.a,b) are satisfied by A. To verify that A satisfies (3.2.c), we first state:

Lemma 3.5. In Ω_0, the Atkinson Index A is equivalent to

$$A^* = \theta_1^{1-\varepsilon} + \theta_2^{1-\varepsilon} + \ldots + \theta_n^{1-\varepsilon}, \ \varepsilon > 1.$$

Proof: In Ω_0, $\bar{X} = 1/n$. Therefore,

$$A = 1 - \left[\left[(n\theta_1)^{1-\varepsilon} + (n\theta_2)^{1-\varepsilon} + \ldots + (n\theta_n)^{1-\varepsilon}\right]\frac{1}{n}\right]^{\frac{1}{1-\varepsilon}}$$

$$= 1 - \left[n^{-\varepsilon}\left[\theta_1^{1-\varepsilon} + \theta_2^{1-\varepsilon} + \ldots + \theta_n^{1-\varepsilon}\right]\right]^{\frac{1}{1-\varepsilon}}$$

$$\stackrel{e}{=} \left[\theta_1^{1-\varepsilon} + \theta_2^{1-\varepsilon} + \ldots + \theta_n^{1-\varepsilon}\right]. \quad \text{Q.E.D.}$$

To verify that A* satisfies (3.2.c), we see

$$\frac{\partial f}{\partial \theta_i} < \frac{\partial f}{\partial \theta_j} \text{ iff } (1-\varepsilon)\,\theta_i^{-\varepsilon} < (1-\varepsilon)\,\theta_j^{-\varepsilon} \text{ iff } \theta_i < \theta_j.$$

Thus, A satisfies A1-A3.

Another recently-proposed measure of inequality is the Theil Index[2]

$$(3.7)\quad T = \ln n - \left[\theta_1 \ln\frac{1}{\theta_1} + \theta_2 \ln\frac{1}{\theta_2} + \ldots + \theta_n \ln\frac{1}{\theta_n}\right].$$

The Theil Index clearly satisfies (3.2.a,b). We now show:

---

[1]See Atkinson (1970).

[2]See Theil (1967).

Lemma 3.6. In Ω_0, the Theil Index T is equivalent to $T^* = \theta_1^{\theta_1}\theta_2^{\theta_2}\ldots\theta_n^{\theta_n}$.

Proof: $T = \ell n n - [\theta_1 \ell n \frac{1}{\theta_1} + \theta_2 \ n \frac{1}{\theta_2} + \ldots + \theta_n \ell n \frac{1}{\theta_n}]$

$$= \ell n n - \ell n [\frac{1}{\theta_1^{\theta_1}\theta_2^{\theta_2}\ldots\theta_n^{\theta_n}}]$$

$$\stackrel{e}{=} \theta_1^{\theta_1}\theta_2^{\theta_2}\ldots\theta_n^{\theta_n}. \quad \text{Q.E.D.}$$

It is obvious that T* satisfies (3.2.c) and therefore T satisfies A1-A3.

C. Generalization

Having observed several indices which satisfy the conditions of Theorem 3.2, we note that there are many other possible measures which also fulfill those conditions. Special cases of the general function of Theorem 3.2 take on the following additive form:

$$(3.9) \quad f(X) = f_1(X_1) + f_2(X_2) + \ldots + f_n(X_n) \text{ with } \frac{\partial f_i}{\partial X_i} < \frac{\partial f_j}{\partial X_j} \ (i<j),$$

a special case of which is

$$(3.10) \quad f(X) = f(X_1) + f(X_2) + \ldots + f(X_n) \text{ with } \frac{\partial f}{\partial X_i} < \frac{\partial f}{\partial X_j}.$$

G* is seen (in Lemma 3.4) to be a special case of (3.9), while C* (Lemma 3.3), A* (Lemma 3.5), and T* (Lemma 3.6) are special cases of (3.10). With the aid of (3.9) and (3.10), we can immediately generate a large number of other indices satisfying A1-A3, e.g.,

$$(3.11) \quad \lambda = \lambda_1\theta_1 + \lambda_2\theta_2 + \ldots + \lambda_n\theta_n, \ \lambda_i < \lambda_j \text{ for } i < j$$

$$(3.12) \quad \varepsilon = \theta_1^{\varepsilon_1} + \theta_2^{\varepsilon_2} + \ldots + \theta_n^{\varepsilon_n} \text{ for } 1<\varepsilon_i < \varepsilon_j$$

which may conceivably help in constructing new measures with more desirable properties for empirical work.

Despite the large number of indices which satisfy our three axioms, there are other indices in common use which violate them, particularly A1 and A3. The difficulty with those indices which violate A1 (e.g., variance) is sometimes stated as "not independent of the level of income," i.e., having larger values for greater total incomes. Those indices which do not satisfy A3 are in some circumstances insensitive to rank-preserving equalizations. Examples are the family of fractile ranges such as the interquartile range; any rank-preserving equalizations within a segment (e.g., within a quartile) leave the index unchanged, in violation of A3. Another example is the Kuznets Ratio:[1]

(3.13) $K = \Sigma|\theta_i - 1/n|$,

which is unchanged by any rank-preserving equalization or disequalization on the same side of the mean. To the extent that A1-A3 are reasonable, all indices which violate them are less than satisfactory; their popular use in empirical work cannot be defended by these axioms and must be justified on other grounds.

We now give an example showing explicitly the iso-inequality set and showing also where A3 is violated by the Kuznets ratio.

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[1]See Kuznets (1957).

Example 6. Kuznets Ratio for the Case n=3

The six regions of the equilateral triangle for the three person case are reproduced in Figure 6. The dotted lines a'a'', b'b'', and c'c'' are drawn parallel to the three sides. In Ω_0 (the triangle DCΦ), to the right (left) of c'c'', the middle family's income is less (greater) than the mean income (= 1/3). Let x be a typical point in c'CΦ through which a line vy_1 is drawn parallel to AB. When x moves along vy_1, the Kuznets ratio is invariant, because income is transferred between two families on the same side of the mean. That A3 is violated can be seen from the fact that the set x^* (i.e., the point set bounded by xwΦv) now contains those points on xv which are not strictly more equal than x, and thus A3 is violated on xv. Similarly, for a typical point z in the triangle c'DΦ, the Kuznets ratio is invariant on the horizontal line segment uy_1 and A3 is violated on uz. The complete pre-ordering induced by the Kuznets ratio is shown by the iso-inequality hexagon $y_1y_2y_3y_4y_5y_6$. We note that if A3 is not to be violated, the contours must be strictly less steep than AC but steeper than BC.

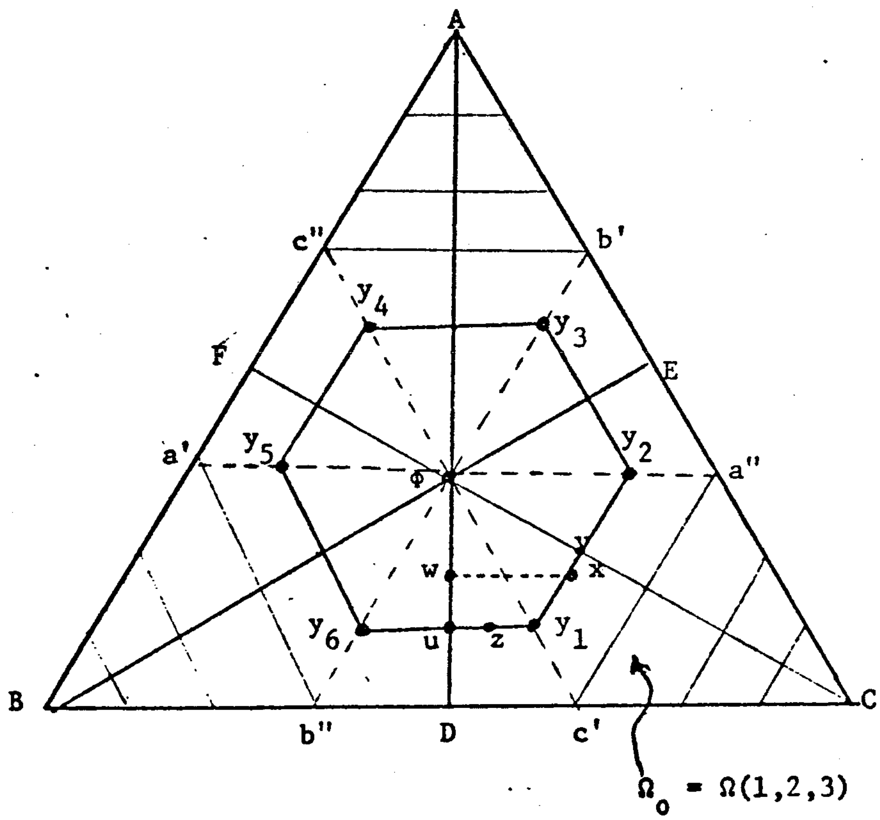

Figure 6

4. Conclusion

In this paper, we have developed an approach to inequality comparisons which differs from the conventional cardinal one. Beginning by postulating three axioms, we showed that many but by no means all of the commonly-used cardinal inequality measures satisfy these axioms. The ones which do satisfy the axioms agree on the ranking of distributions whose Lorenz curves do not intersect. However, when Lorenz curves do intersect, the various measures partition the income distribution space differently. Since the three axioms are insufficient to determine the specific partition to use, the use of any of the conventional measures implicitly accepts the additional welfare judgments associated with that measure.

The key issue for inequality comparisons is the reasonableness of the ordering criterion, which in the case of cardinal measures is the index itself. An axiomatic approach is probably the ideal method for confronting this issue, because the reasonable properties (i.e., the axioms) are postulated explicitly. At minimum, this approach facilitates communication by enabling (and indeed requiring) one to set forth clearly his own viewpoints and value judgments for scrutiny by others. But in addition, to the extent that one person's judgments (such as those in our three axioms) are acceptable to others, controversies over inequality comparisons may be resolved. We have seen that our three axioms are incomplete insofar as they cannot determine the ordinal ranking uniquely. A feasible and desirable direction for future research is to investigate what further axioms could be introduced to complete the axiomatic system or at least to reduce further the zones of ambiguity.

Two of the properties we have postulated, scale irrelevance and symmetry (A1 and A2), permit us to concentrate on the rank-preserving subset Ω_0 in our search for new axioms. When this is done, the ordinal ranking can immediately be extended to the entire income distribution space Ω^+ (Lemma 1.2). As an illustration of this procedure, the third axiom of our paper was first introduced on Ω_0 and then extended to Ω^+ as a matter of logical deduction (Lemma 1.4). This same procedure can be followed in future research with two important advantages. First, as we showed in Section 3, Ω_0 is computationally more convenient than Ω^+. Second, economists have long been aware of the fact that the interdependence of personal (or family) utilities is a vexing problem for social welfare judgments in general and inequality judgments in particular. It may be some consolation to know that in Ω_0, rankings are not disturbed, so one does not have to face the sensitive issues associated with reversals of existing positions in the income hierarchy.

It is conceivable that beyond some point the search for new axioms may turn out to be unrewarding, even on Ω_0. In that case, inequality comparisons will always be subject to arbitrary specifications of welfare weights. In this paper, we have presented new families of such arbitrary indices consistent with our three axioms (see eq. (3.11) and (3.12)). The selection of the proper weights (λ_i and ε_i respectively) by whatever reasonable criterion one cares to exercise is a less desirable but possibly more practical alternative than a strictly axiomatic approach.

Our research has hopefully made clear that inequality comparisons cannot be made without adopting value judgments, explicit or otherwise, about the desirability of incomes accruing to persons at different positions in the income distribution. Even the Lorenz criterion, which permits us to rank the relative inequality of different distributions in only a fraction of the cases, embodies such judgments. The traditional inequality indices such as those considered in Section 3, to the extent they complete the ordering, embody <u>some</u> value judgments beyond our three axioms. These judgments are at present vague, and it would be helpful if future researchers could state these implicit value judgments in axiomatic terms so that when a particular inequality index is used we will know exactly what judgments are being made.

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