

ECONOMIC GROWTH CENTER

YALE UNIVERSITY

Box 1987, Yale Station
New Haven, Conn. 06520

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THE INDEXABILITY OF ORDINAL MEASURES OF INEQUALITY

John C. H. Fei and Gary S. Fields

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0. Introduction

The comparison of alternative patterns of family income distribution, as in most social welfare judgments, is a difficult and sometimes controversial subject.¹ An ideal method for the design of an ordinal measurement of inequality is an axiomatic approach whereby reasonable properties are explicitly postulated for a complete pre-ordering R defined on the income distribution space Ω^+ , the non-negative orthant of the n -dimensional real space S^n .² Commonly used axioms are those of scale irrelevance (A1), symmetry (or anonymity) (A2), and the desirability of rank-preserving equalization (A3).³ While A1 isolates the 'distribution' of income from the overall 'level,' A2 emphasizes a 'democratic' principle in which all families are treated alike. The third axiom A3 states that equality increases when income is transferred from a relatively rich to a relatively poor family. This set of axioms, to be discussed briefly below, will be taken as the starting point of the present paper.

The central issue addressed here is the indexability of a pre-ordering R . In the conventional approach to the measurement of inequality, use is made of arbitrary indices, i.e., a real-valued function $f(X)$, such as a Gini coefficient or coefficient of variation defined

¹Our discussion in this paper is in terms of inequality of income distribution, but all results apply without modification to inequality measures of any quantifiable magnitude.

²The limitation to non-negative incomes is purely for convenience.

³These three axioms are considered in some detail in our earlier paper [Fields and Fei (1974)].

on Ω^+ . It is well-known that $f(X)$ induces a complete pre-ordering R (in precisely the same way that in the ordinary theory of consumer preference a cardinal utility function $U(Y)$ induces a complete pre-ordering (i.e., the indifference curves) on the commodity space.¹ If we instead adopt an ordinal approach and define R axiomatically, the question naturally arises as to whether or not there exists a continuous real-valued function $f(X)$ which induces R , i.e., whether or not R is indexable. The major theorem of this paper (Theorem 7.7) is that the three axioms mentioned above are almost sufficient to insure indexability, and the only new axiom which needs to be added is an axiom of continuity (A4). To introduce this axiom, we exploit the fact that Ω^+ is a convex set. The first three axioms assure us that Ω^+ contains an ideal point A (or most equal point, analogous to the bliss or saturation point of consumer theory) and that a movement toward A from any point Y along a straight line will strictly increase equality. This property, to be referred to as the ideally centered property of R , is intuitively appealing and indeed useful.

The three axioms also assure the existence of a worst point W , i.e., a least equal point. The line AW , which we refer to as the extreme ray, intersects every indifference (iso-inequality) set. The process of indexing R can conveniently begin with the construction

¹ See Hicks (1939) and Debreu (1959). Much of what follows will draw parallels between the theory of consumer choice and the theory of inequality measurement.

of a real-valued function $f(X)$ which induces R on AW . It is a simple matter to extend $f(X)$ to cover the entire Ω^+ . The major result of our paper is the proof that the extension is continuous.

The above ideas--ideally centered property, extreme ray, and indexability--can in fact be developed generally when R is defined on any convex subset C of S^n , the n -dimension real space. Both C and the ideal point A in it can be quite arbitrary. The general case has economic significance in its own right; e.g., incomes may be negative and there may be a social consensus establishing some point other than perfect equality as the ideal. The indexability theorem of this paper will first be proven for the general case (Theorem 6.4) and the result then applied to the special case (Theorem 7.7).

In Sections 1 and 2, we summarize certain elementary notions associated with the three axioms. Then, in Section 3, we explore the convexity of the income distribution space and the ideally centered property. In Section 4, the axiom of continuity will be introduced. Sections 5-7 introduce the notion of indexability and the basic indexability theorem is proved. The economic significance of the indexability theorem, from a theoretical and empirical point of view, will be discussed in Section 8, while possible directions for future research in the development of new axioms are presented in Section 9.

1. Axioms of Scale Irrelevance and Symmetry

Let $.Y = (Y_1 \ Y_2 \ \dots \ Y_n) \geq 0$, a vector of non-negative real numbers, be a pattern of income distribution to n families (or individuals). The totality of all such patterns is the non-negative orthant.

$$(1.1) \ \Omega^+ = \{Y | Y \geq 0\}$$

of the n -dimensional real space. We shall exclude the origin (i.e., when no family receives any income) from Ω^+ . An ordinal approach to the measurement of inequality is defined by a complete pre-ordering R , i.e., a binary relation defined for ordered pairs (X, Y) in Ω^+ satisfying:

(1.2) (a) Comparability. Precisely one of the following holds:

- (i) $X R Y$ and $Y \not R X$ denoted by $X > Y$,
- (ii) $Y R X$ and $X \not R Y$ denoted by $Y > X$,
- (iii) $X R Y$ and $Y R X$ denoted by $X \simeq Y$.

(b) Transitivity. $X R Y$ and $Y R Z$ implies $X R Z$.

In (i), $X > Y$ means X is more equal than Y and in (iii) $X \simeq Y$ means X and Y are indifferent (which from now on we use synonymously with equally equal). Thus, (1.2.a) means we can unambiguously compare any two patterns of income distribution from the point of view of inequality. To simplify terminology, we shall occasionally refer to R as an ordering.

Let us first impose two axioms on R :

(1.3) A1. Axiom of Scale Irrelevance. If $X = aY$, i.e.,

$$(X_1 \ X_2 \ \dots \ X_n) = (aY_1 \ aY_2 \ \dots \ aY_n), \ a > 0, \ \text{then} \\ X > Y.$$

A2. Axiom of Symmetry. If (i_1, i_2, \dots, i_n) is any

$$\text{permutation of } (1, 2, \dots, n), \text{ then } (X_1 \ X_2 \ \dots \ X_n) \\ \simeq (X_{i_1} \ X_{i_2} \ \dots \ X_{i_n}).$$

The first axiom states that two patterns of income distribution are indifferent when one is a positive scalar multiple of the other. Thus, all points on any ray emanating from the origin in Ω^+ are indifferent under A1. We can normalize the set of income distribution patterns in Ω^+ to arrive at the subset:

$$(1.4) \ (a) \ \Omega_c = \{\theta \mid \theta = (X_1/S_X \ X_2/S_X \ \dots \ X_n/S_X), \ X \in \Omega^+, \ S_X = \sum_{i=1}^n X_i\},$$

satisfying

$$(b) \ \sum_{i=1}^n \theta_i = 1 \ \text{and} \ \theta_i \geq 0 \ \text{for} \ i = 1, \dots, n.$$

The values of θ_i in Ω_c indicate the fractions of income accruing to different families.

The second axiom states that two patterns of income distribution are indifferent if one distribution is a permutation of the other. We can identify the $n!$ rank preserving subsets of Ω^+ as follows. Let $i^* = (i_1^*, i_2^*, \dots, i_n^*)$ be a particular permutation of $(1, 2, \dots, n)$. A rank preserving subset $C(i^*)$ is defined as

$$(1.5) \ C(i^*) = \{Y \mid Y \in \Omega^+; Y_{i_1^*} \leq Y_{i_2^*} \leq \dots \leq Y_{i_n^*}\}.$$

Thus all points in $C(i^*)$ have the same ranking of families according to income level. When a complete pre-ordering R is defined on $C(i^*)$, A2 allows R to be extended uniquely to Ω^+ . The first two axioms suggest that we can take the intersection of Ω_c and $C(i^*)$ to arrive at

$$(1.6) \quad \Omega(i^*) = \Omega_c \cap C(i^*) .$$

In each $\Omega(i^*)$, the ranking of family incomes is preserved. Thus we have the following theorem:

Theorem 1.1. If R is defined on any $\Omega(i^*)$, then under A1 and A2, R can be extended uniquely to Ω^+ .

The proof is obvious.

It is particularly convenient to work with that $\Omega(i^*)$ corresponding to the natural order. We refer to this set as the monotonic rank-preserving (sub)set and denote it by

$$(1.7) \quad \Omega_0 = \Omega(1, 2, \dots, n) = \{ \theta \mid \theta = (\theta_1 \ \theta_2 \ \dots \ \theta_n), \ \theta_i \geq 0, \\ \sum \theta_i = 1, \ \theta_1 \leq \theta_2 \leq \dots \leq \theta_n \} .$$

Theorem 1.1 implies that once we know how to order points in Ω_0 , we can in fact order all points in Ω^+ under A1 and A2. Thus, in our paper, additional axioms will be stated as properties of Ω_0 . Economically, this has the advantage that, in the search for new axioms, we need not concern ourselves with rank reversals. This procedure is also appealing methodologically because the properties are stated on a subset of the entire income distribution space.

2. Axiom of Desirability of Rank-Preserving Equalization

In constructing additional axioms, it is necessary to specify what would happen to inequality if a relatively rich family were to transfer a positive amount of income to a relatively poor family. Intuitively, inequality should be lessened. We then have:

Definition. Let X and Y belong to the same rank-preserving subset $\Omega(i^*)$. We shall say ' Y is obtained from X by a rank-preserving equalization,' in notation $Y = E(X)$, if Y is obtained from X by the transfer of a positive amount of income h from a relatively rich family (the q 'th) to a relatively poor family (the p 'th). Thus

$$(2.1) \text{ (a) } Y_p = X_p + h ; Y_q = X_q - h ; X_q > X_p ; h > 0 ; X, Y \in \Omega(1^*) ;$$

$$(b) X_k = Y_k \text{ for all } k \neq p, q .$$

A very reasonable property for R is the desirability of a rank-preserving equalization, which may be stated formally as:

Definition. A complete pre-ordering R has the desirability of rank-preserving equalization property when $Y = E(X)$ implies $Y > X$.

That is, when a relatively poor family receives an income transfer from a relatively rich family without disturbing the rank, the resulting pattern is more equal. Notice that this desired property is defined for the entire income distribution space Ω^+ . The third axiom may now be stated as a property of Ω_0 as follows:

A3. Axiom of Desirability of Rank-Preserving Equalization. If X and Y belong to Ω_0 and if $Y = E(X)$, then $Y > X$.

We have the following theorem:

Theorem 2.1. A complete pre-ordering R satisfying A1-A3 has the property of the desirability of rank-preserving equalization in Ω^+ .

The proof is obvious.¹

The three axioms A1-A3 are familiar properties. It can be shown² (i) A1-A3 form an axiomatic system in that they are consistent and independent, and (ii) many familiar indices of inequality satisfy these axioms.

Suppose now that X is obtained from Y by a finite sequence of rank-preserving equalization, in notation $X = T(Y)$:

$$(2.2) \quad X = T(Y) = E_k(\dots E_2(E_1(Y))\dots) .$$

It follows from the transitivity of R that X is preferred to Y :

$$(2.3) \quad X = T(Y) \text{ implies } X > Y .$$

This may be easily related to Lorenz domination according to the following definition:

Definition. For X and Y in Ω_0 , X Lorenz-dominates Y (in notation, $L_X \geq L_Y$) when

¹The necessity of a proof here illustrates the general methodological point that when an axiom is introduced in Ω_0 , the satisfaction of its properties on Ω^+ must be proven.

²See Fields-Fei (1974).

$$(2.4) \text{ (a) } X_1 + X_2 + \dots + X_j \geq Y_1 + Y_2 + \dots + Y_j \text{ for } j = 1, 2, \dots, n-1,$$

$$\text{and (b) } X_1 + X_2 + \dots + X_j > Y_1 + Y_2 + \dots + Y_j \text{ for some } j < n.$$

If the condition in (2.4.a) is replaced by a strict inequality

$$(2.5) \quad X_1 + X_2 + \dots + X_j > Y_1 + Y_2 + \dots + Y_j \text{ for } j = 1, 2, \dots, n-1,$$

then we shall say X strictly Lorenz-dominates Y (in notation, $L_X > L_Y$).

Notice that

$$(2.6) \quad \sum_{i=1}^n X_i = \sum_{i=1}^n Y_i = 1 \text{ in } \Omega_0.$$

The basic theorem of our previous paper is

Theorem 2.2. $X = T(Y)$ if and only if $L_X \geq L_Y$.

The essential part of the theorem for this paper is the sufficient condition which necessitates the construction of a finite sequence of rank-preserving equalizations within Ω_0 whenever the Lorenz curve of X dominates that of Y .¹ It is this sufficient condition which implies

$$(2.7) \quad L_X \geq L_Y \text{ implies } X > Y$$

by the transitivity of R .

¹The fact that the sequence of equalizations is entirely within Ω_0 is essential for our axiomatic approach, especially the proof of the indexability theorem below. In this respect, our result differs from those of Atkinson (1970) and Rothschild and Stiglitz (1973).

It immediately follows that the point $\Phi = (1/n \ 1/n \dots 1/n)$ in which incomes are equally distributed and the point $U = (0 \ 0 \dots 1)$ in which the income is concentrated entirely in the hands of the wealthiest family are respectively the ideal point and worst point of Ω_0 , i.e., under A3, for all X in Ω_0 ,

$$(2.8) \text{ (a) } X < \Phi = (1/n \ 1/n \dots 1/n) \text{ if } X \neq \Phi \text{ and}$$

$$\text{(b) } X > U = (0 \ 0 \dots 1) \text{ if } X \neq U.$$

It is easily seen that U can be transformed into X and X into Φ by appropriate sequences of rank-preserving equalizations, and thus (2.8) is implied by (2.3).

At several points in this paper, we will make use of the example of a three person economy, which we illustrate geometrically.

Example 1. In Figure 1, let ABC be an equilateral triangle in which the distance of the perpendicular line AD is defined to be one unit. If Z is any point inside ABC , then the sum of the distances of the perpendicular lines $(ZZ_1, ZZ_2, \text{ and } ZZ_3)$ is one. Thus the totality of points in ABC corresponds to the normalized set Ω_c . The three perpendicular lines $AD, BE,$ and CF partition Ω_c into $3! = 6$ rank-preserving subsets (1.6). The monotonic rank-preserving subset Ω_0 (1.7) is represented by the triangle ΦCD where Φ is the ideal point (2.8). (In Ω_0 , $ZZ_1 \leq ZZ_2 \leq ZZ_3$.) Let parallel lines such as $v_1 v_2$ and $v'_1 v'_2$ in Ω_0 represent the indifference (i.e., iso-inequality) sets of R . By A2, R can be extended symmetrically to Ω_c , and the indifference curves are now "rings" (e.g., $v_1 v_2 v_3 v_4 v_5 v_6$). In Ω_0 , let $a_2 b_2$ be parallel to AB . Moving from point X toward Y along $a_2 b_2$ represents a transfer of income from the richest family to the poorest family leaving the income of the middle family unchanged. Thus $Y = E(X)$ and hence $Y > X$. Hence, the slope of the indifference curves in Ω_0 must be less than the slope of AB if A3 is to be satisfied. A ring closer to the ideal point Φ represents a more equal indifference set.

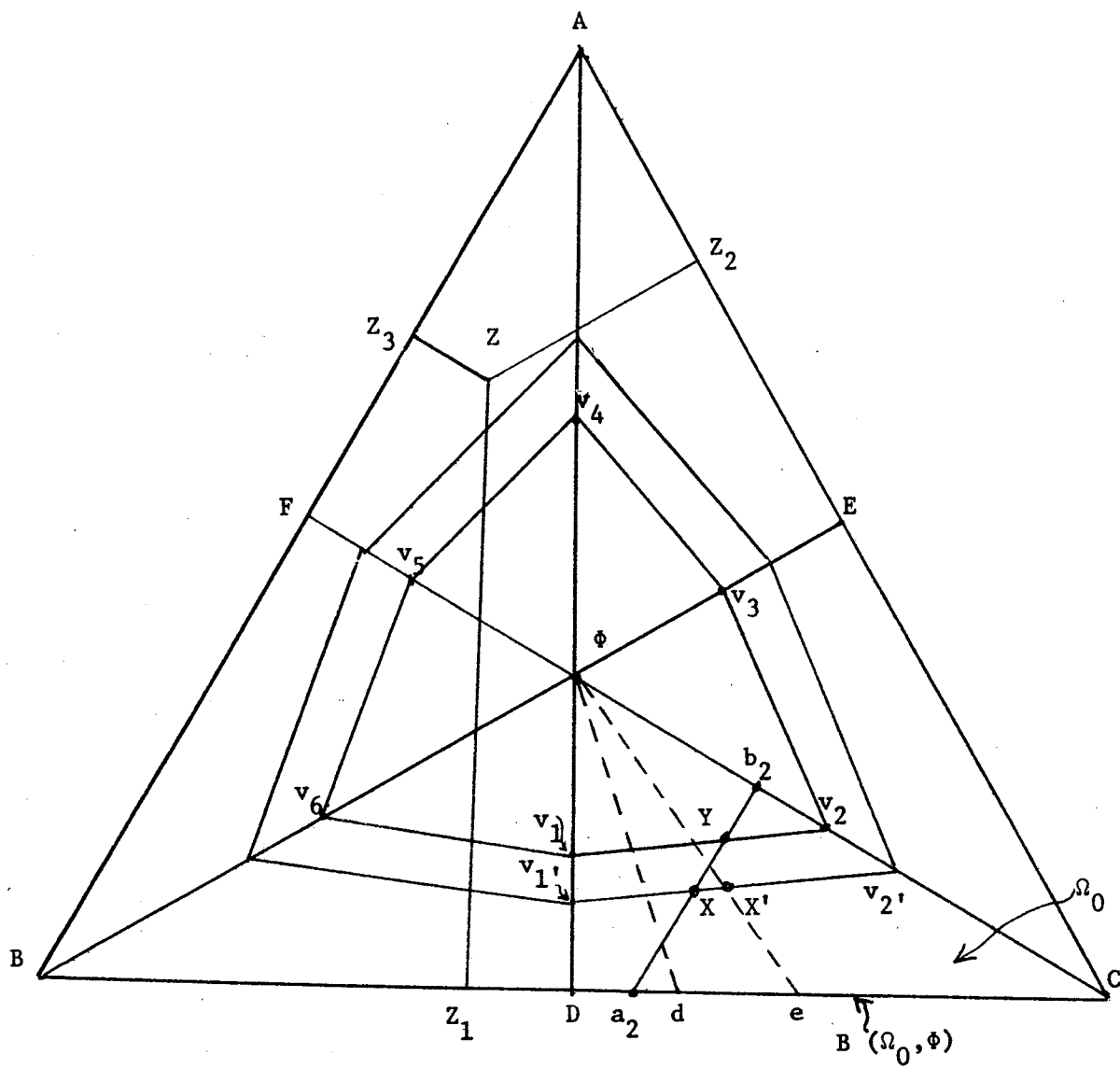


Figure 1

3. Ideally Centered Property

We now make use of two properties of Ω_0 : (i) that it is a compact, convex set and (ii) under A1-A3 there is an ideal point Φ . Let Y be any point in Ω_0 . From convexity, the line segment ΦY is entirely within Ω_0 . As we move from Y toward Φ along this line segment, it can be shown (see Theorem 4.1 below) that the equality strictly increases, i.e., if X lies between Φ and Y on ΦY , then $Y < X < \Phi$. This property will be referred to as the ideally centered property of the inequality pre-ordering.

More generally, we can define the ideally centered notion when a complete pre-ordering R is defined on any convex subset C (not necessarily Ω_0) of the n -dimension real space. We only require that, under R , C have an ideal point A (i.e., $X < A$ for all $X \neq A$ in C) as well as a worst point W (i.e., $X > W$ for all $X \neq W$ in C). By the remark in the last paragraph, the results of this section can be applied to the special case when the three axioms are postulated.

Certain elementary properties of the convex set C may now be stated. Let X , Y , and E be three points of C . If $X \neq Y \neq E$, we shall say Y is closer to E than is X (or Y lies between X and E) when

$$(3.1) \text{ (a) } Y = tE + (1-t)X, \quad 0 < t < 1, \quad \text{or}$$

$$\text{(b) } Y = X + t\epsilon \quad \text{where}$$

$$\text{(c) } \epsilon = (\epsilon_1 \ \epsilon_2 \ \dots \ \epsilon_n) = (E_1 - X_1 \ E_2 - X_2 \ \dots \ E_n - X_n),$$

i.e., where Y is a strictly convex linear combination of X and E .

Notice that the e_i are deviations of X from E . Thus, " Y is closer to E than is X " means that Y can be obtained from X by a proportional adjustment of the deviations where the magnitude of adjustment is a positive fraction t .

If E is any point of C , then C is the union of line segments EX for all X in C . If, in addition, C is compact (closed and bounded), there is a base $B(C,E)$ of C relative to E .¹ The base is the set of all base points in C , satisfying the following conditions:

(3.2) (a) $B(C,E) \subset C$,

(b) $b \in B(C,E)$ implies b does not lie between E and any other point of C .

Thus, C can be spanned by the base rays (i.e., C is the union of all lines radiating from E to the base points). Two distinct base rays intersect only at E , which implies that the base $B(C,E)$ lies in the boundary of C . Thus, when E is specified, $B(C,E)$ is uniquely determined.

Suppose an ordering R is defined on C with an ideal point A . We shall say that R is ideally centered at A according to the following definition:

Definition. A complete pre-ordering R defined on a convex set C is ideally centered at A if A is an ideal point and $Y > X$ whenever Y is closer to A than is X .

¹If C is closed but not bounded, it may still have a base. See example 3 below.

In other words, a movement toward A along any line in C leads to a strictly greater ranking. It is apparent that in an ideally centered ordering the ideal point A is unique.

Example 2. Consider the convex set C shown in Figure 2. Relative to point A , the base $B(C,A)$ is the curved portion of the boundary $GabcD$. C is the union of all base rays such as aA , bA , and cA . If an ordering is ideally centered at A , the iso-inequality curves, shown by the dotted lines, are such that a movement along any base ray (e.g., from X to Y) toward A will intersect an iso-inequality curve with a more equal index.

Suppose now that $I(Z)$ is the set of all points indifferent to Z . Every base ray intersects $I(Z)$ at most once. However, it is possible that some base rays (e.g., Ac) do not intersect some indifference sets (e.g., $I(Z)$) even once. It is reasonable to regard the base point c as superior to Z (i.e., $c > Z$) for otherwise (i.e., if $c < Z$), movement along Ac toward A would, if the ordering is continuous, run into a point with the same inequality as Z .

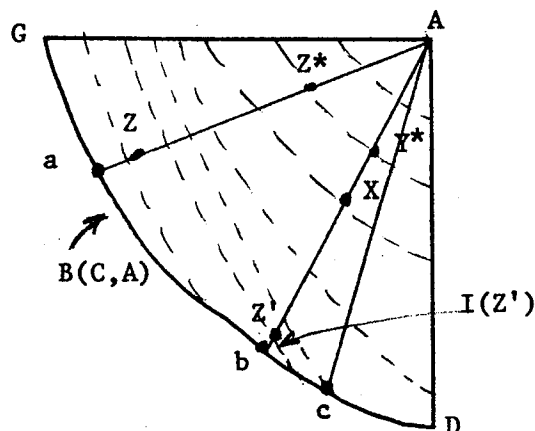


Figure 2

This example suggests the following definition:

Definition. An ideally centered complete pre-ordering R on C is continuously ideally centered at A if $X < Z^*$ implies the existence of a Y^* between X and A such that $Y^* \sim Z^*$.¹

¹When $B(C,A)$ is not empty, an alternative way to state the definition is to replace ' X ' by 'a base point b '.

When $B(C,A)$ is not empty, if Z is superior to a base point b (i.e., $Z > b$) we can always find a point Z' on the base ray bA which is indifferent to Z .

Let us now consider a continuously ideally centered ordering R with an ideal point A and a worst point W . It is clear that W must be a base point. The end points A and W of the base ray AW are extremes in that every other pattern of income distribution X in C satisfies $A > X > W$. We shall refer to AW as the extreme base ray. We have:

Theorem 3.1. If R is continuously ideally centered at A and has a worst point W , there exists a unique choice function $h(X)$ such that:

$$(3.3) \quad (a) \quad C \xrightarrow{h(X)} AW, \\ (b) \quad h(X) \simeq X,$$

i.e., $h(X)$ maps C onto the base ray AW (3.3.a) such that $h(X)$ and X are indifferent (3.3.b).

This theorem, which we shall make use of later, can be illustrated by the following example.

Example 3. In the XY plane of Figure 3.a, let C be the infinite horizontal strip bounded by the vertical axis and the two horizontal half lines through points $W = (X = 0, Y = 1)$ and $M = (X = 0, Y = -1)$. For the point $A = (X = 1, Y = 0)$, the base $B(C,A)$ coincides with the boundary of C . From point A , draw an auxiliary line Aa_Xa' which approaches the upper boundary WV asymptotically. Draw the lines AW and AM . For every point $X > 1$ on the horizontal axis, construct a rectangle $a_Xb_Xc_Xd_X$ with the aid of these auxiliary lines. As X increases, these rectangles approach

the boundary $B(C,A)$ as a limit. Defining A to be the ideal point and treating these rectangles and $B(C,A)$ as indifference sets, an ideally centered complete pre-ordering R can be constructed whereby the boundary $B(C,A)$ is the worst indifference set. Notice that R is continuously ideally centered, i.e., if $Z > b$, a base point, there exists a point Z^* on the base ray bA such that Z and Z^* are indifferent. The ordering we have constructed is an example of an R which is continuously ideally centered and has no worst point.

Now construct another complete pre-ordering which is the same as the above except that now a point V on the boundary is defined as the worst point. This new ordering is an example of a complete pre-ordering which is ideally centered and has a worst point, but it is no longer continuously ideally centered, because $E > V$ and yet there is no point of the base ray AV which is indifferent to E .

Finally, let us construct a complete pre-ordering which is continuously ideally centered and has an extreme base ray. The convex set C and the auxiliary lines of Figure 3.a are reproduced in Figure 3.b. Through a point $X > 1$ on the horizontal axis, construct the pentagon $a u v c d$ which approaches the "open pentagon" $VEOMN$ in the limit. These are the new indifference sets. In addition, the parallel lines in the triangle WOE are also indifference sets such that W is the worst point. We then have a continuously ideally centered complete pre-ordering with extreme base ray AW (i.e., the ray connecting the ideal point A with the worst point W). Notice that the extreme base ray intersects every indifference set. It is this fact that guarantees the existence of a choice function $h(Y)$ mapping C onto WA , e.g., $h(X) = Y$ and $h(X') = Y'$.

The ideally centered property presented in this section is really quite general in the sense that no restrictions are placed on the convex set or on the location of the ideal point (A) and worst point (W). For example, families may receive negative incomes, hence C may not be a subset of the positive orthant. Or if every family is guaranteed a minimum income, C may be a proper subset of Ω^+ . Whatever space C is, the location of the ideal and worst points is a matter of judgment. In nearly all discussions of income distribution inequality, the ideal point A is specified as the point of perfect equality Φ and the worst point W as the point of perfect inequality U (see (2.8)). However, some respectable philosophical schools of thought

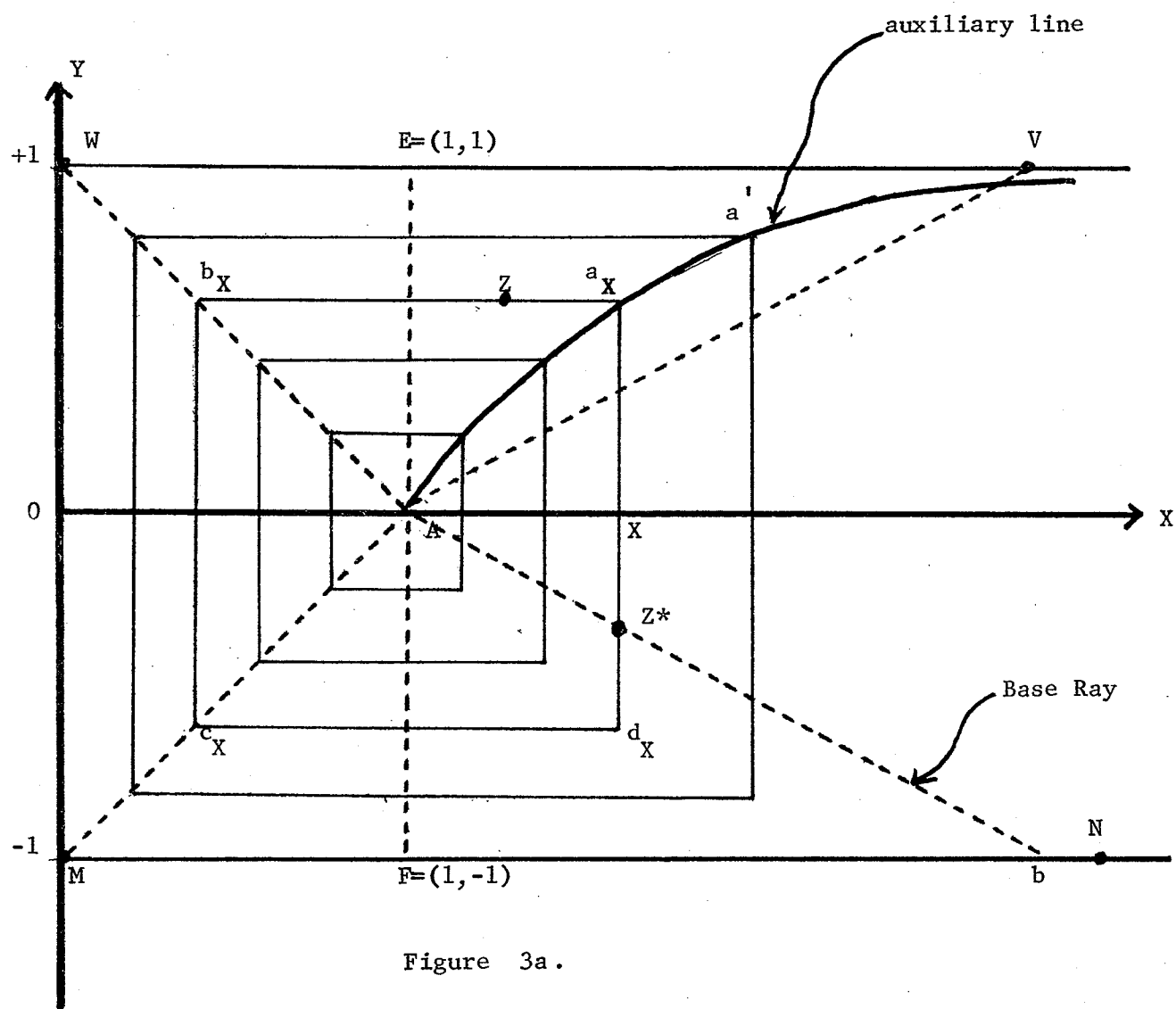


Figure 3a.

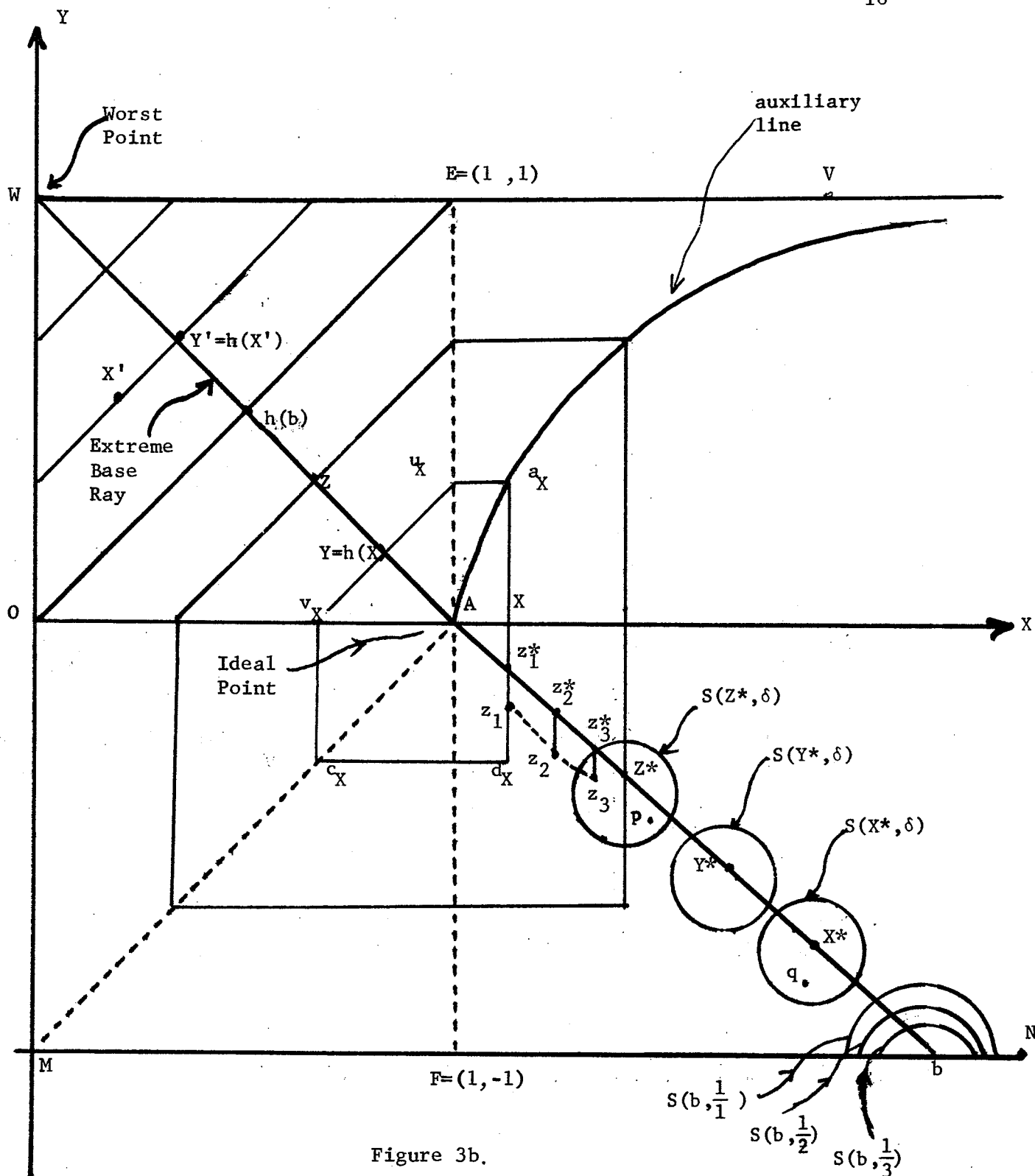


Figure 3b.

may not share these judgments.¹ Notice that the analysis of this section and of Section 6 below remains intact for any choice of C, A, and W.

¹ Confucianism and Platoism, for example, believe in the existence of a class structure in an ideal society, in which case the ideal point cannot be the perfect equality point Φ .

4. Axiom of Continuity

Returning to the special case of Sections 1 and 2, the monotonic rank preserving income distribution space Ω_0 , which is both compact and convex, we recall that the perfect equality point Φ is indeed an ideal point (2.8). Furthermore, the property of being ideally centered at Φ is in fact ensured by A1-A3. Specifically,

Theorem 4.1. Under A1-A3, any complete pre-ordering R satisfies two conditions:

(a) R is ideally centered at the perfect equality point

$$\Phi = (1/n \ 1/n \ \dots \ 1/n) ;$$

(b) Let Y lie between X and Φ (i.e., $Y = t\Phi + (1-t)X$ for $0 < t < 1$). Then Y strictly Lorenz dominates X , i.e., $L_Y > L_X$.

Proof: Let $(d_1 \ d_2 \ \dots \ d_n) = t(\Phi - X) = t(\frac{1}{n} - X_1 \ \frac{1}{n} - X_2 \ \dots \ \frac{1}{n} - X_n)$.

Then we have:

(4.1) (a) $d_1 \geq d_2 \geq \dots \geq d_n$ (because the X_i are monotonically non-decreasing)

(b) $d_1 + d_2 + \dots + d_n = 0$ (because $\sum_{i=1}^n X_i = \sum_{i=1}^n Y_i = 0$)

(c) $d_1 > 0$ and $d_n < 0$ (because $X \neq \Phi$).

Define $V_k = d_1 + d_2 + \dots + d_k$ for $k = 1, 2, \dots, n$. By (a), V_k monotonically increases from a strictly positive value ($V_1 = d_1$) for X_i below the mean and then monotonically decreases to $V_n = 0$ (by (b)).

By (c), $V_{n-1} > 0$. Thus $V_k > 0$ for $k = 1, 2, \dots, n-1$. Thus

$L_Y > L_X$ and hence $Y > X$ by Theorem 2.1.

Q.E.D.

Since $1/n$ is the mean income, A1-A3 insure that R is sensitive to proportional adjustments of deviations from the mean. Although Theorem 4.1 establishes that an ordering which satisfies A1-A3 is ideally centered, there is no assurance that the ordering will be continuously ideally centered. In order to guarantee that the ordering possesses this property, we need a new axiom:

A4. Weak Axiom of Continuity. A complete ordering R is continuously ideally centered at the perfect equality point \bar{x} .¹

The following example shows that A4 is independent of A1-A3 and the four axioms are consistent.

Example 4. Return to the pre-ordering R represented by the indifference curves in Figure 1. The base $B(\Omega_0, \bar{x})$ is the line CD and the dotted lines such as $\bar{x}d$ and $\bar{x}e$ are base rays. A1-A3 assure that a movement upward along such rays toward \bar{x} increases equality. One can easily see that the continuously ideally centered property (at \bar{x}) is satisfied for this example; e.g., if $X > e$, then there exists a point X' on the base ray $\bar{x}e$ such that $X \simeq X'$. Thus the four axioms are consistent.

To show A4 is independent of A1-A3, we shall now construct a new ordering which satisfies A1-A3 but not A4. Take the pair of points (X, X') which belong to the same indifference set and define a new ordering which coincides with the above ordering everywhere except for the pair (X, X') which is now defined so that $X > X'$. The new ordering is seen to satisfy A1-A3 because it is not true that $X = T(X')$ or $X' = T(X)$ and hence A3 is not violated anywhere. However, now A4 is violated because $X > e$ and yet there is no point on the base ray $\bar{x}e$ with the same inequality as X . Thus A4 is independent of A1-A3.

¹Two points of a methodological nature may be noted. The Weak Axiom of Continuity is meaningful only when the ordering possesses the ideally centered property to begin with. Theorem 4.1 tells us that this property is indeed ensured by A1-A3. Also notice that the Lorenz domination condition is essential for the proof of Theorem 4.1. Hence the basic results of Section 2 (Theorem 2.1 and eq. (2.7)) involving rank-preserving equalizations are prerequisite for the very introduction of the fourth axiom.

A4 was called the weak axiom of continuity, because it only guaranteed the continuous ideally centered property toward the ideal point Φ . The following example illustrates a certain type of irregularity which might arise even when all four axioms are satisfied.

Example 5. Inadequacy of Weak Continuity. The set Ω_0 from Fig. 1 is reproduced in Figure 4. Suppose the indifference curves comprise three systems of straight lines: those parallel to ab , those parallel to cb , and those (dotted) connecting points on ac with points infinitely close to but not containing b . Both sets of solid curves are constructed to be flat enough to satisfy A3 (see example 1). Although all four axioms are satisfied, this pre-ordering nevertheless exhibits an irregularity, namely, point d (lying between a and c) is seen to be superior to U and yet we cannot find a point on the horizontal line $U\Phi_1$ with the same inequality as d . This occurs despite the fact that as we move from U toward Φ_1 horizontally the ordering increases strictly monotonically. Notice that the line $U\Phi_1$ is the base of the ideally centered ordering and hence the irregularity occurs on the base.

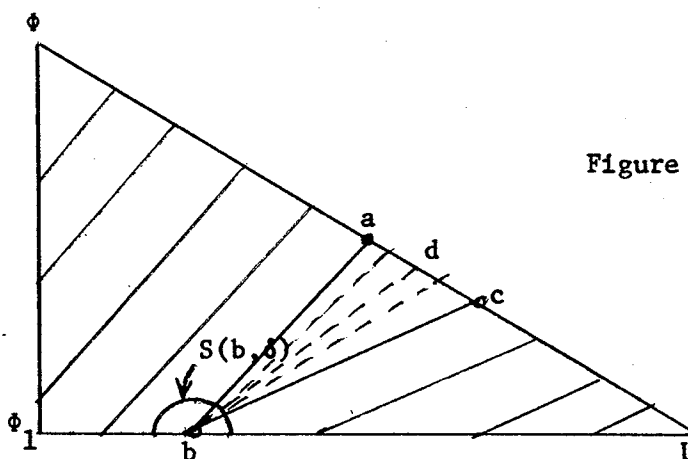


Figure 4

The above example shows that while the pre-ordering may be continuous toward the ideal point Φ , under the weak continuity axiom, it may not be continuous toward some other points (Φ_1 in the above example) for which we have reason to expect that the ordering should

also be continuous. The above example also shows that the irregularity occurs on the base, which may now be denoted by $B(\Omega_0, \Phi_0)$, i.e., the base of Ω_0 relative to the perfect equality point Φ_0 in it. The following lemma identifies the base as those patterns of income distribution in which the poorest family receives nothing.

Lemma 4.2. $B(\Omega_0, \Phi_0)$ contains all $\theta = (\theta_1 \ \theta_2 \ \dots \ \theta_n)$ of Ω_0 for which $\theta_1 = 0$.

Proof: Let $X = (X_1 \ X_2 \ \dots \ X_n) \neq \Phi$ be any point of Ω_0 . It is sufficient to show that $X = t\Phi + (1-t)\theta$ for some $0 < t < 1$ and some point $\theta = (\theta_1 \ \theta_2 \ \dots \ \theta_n) \in \Omega_0$ with $\theta_1 = 0$. The coordinates of θ may easily be found by solving for that t which gives $\theta_1 = 0$. We have:

$$(i) \quad \theta_1 = \frac{1}{1-t} X_1 - \frac{t}{1-t} \frac{1}{n} = 0$$

which implies a value of t satisfying

$$(ii) \quad 0 < t = nX_1 < 1 \quad (\text{because } X_1 < \frac{1}{n} \text{ if } X \neq \Phi).$$

For $i > 1$, we construct

$$(iii) \quad \theta_i = \left(\frac{1}{1-t} \right) X_i - \frac{t}{1-t} \frac{1}{n}.$$

Since $X_1 \leq X_2 \leq \dots \leq X_n$, we see $\theta_1 (= 0) \leq \theta_2 \leq \dots \leq \theta_n$. Furthermore, from (iii),

$$(iv) \quad \sum_{i=1}^n \theta_i = \frac{1}{1-t} \sum_{i=1}^n X_i - \frac{t}{1-t} = 1.$$

Thus $(\theta_1 = 0 \ \theta_2 \ \dots \ \theta_n) \in \Omega_0$.

Q.E.D.

Notice that the base $B(\Omega_0, \Phi_0)$ has a natural economic interpretation as a rank preserving subspace of Ω_0 corresponding to an economy with $n-1$ persons. The ideal point of this subspace is $\Phi_1 = (0 \ 1/(n-1) \ 1/(n-1) \dots 1/(n-1))$ when the income is evenly distributed to $n-1$ persons. Under the first three axioms, the pre-ordering is also ideally centered at Φ_1 , as will be illustrated by example 6. It would seem reasonable to expect that the ordering should also be continuously ideally centered at Φ_1 , a condition which is not guaranteed by the weak continuity axiom.

More generally, we can define the rank-preserving sub-spaces Ω_i and the ideal point Φ_i in each Ω_i as:

(4.2) (a) Ω_i contains all $(\theta_1 \ \theta_2 \ \dots \ \theta_n)$ of Ω_0 for which

$$\theta_1 = \theta_2 = \dots = \theta_i = 0 ,$$

(b) $\Phi_i = (e_1 \ e_2 \ \dots \ e_n) \in \Omega_i$ where $e_1 = e_2 = \dots = e_i = 0$,

$$e_{i+1} = \dots = e_n = \frac{n-i}{n} .$$

We see that Ω_i contains all distribution patterns in which the income of the first i families is zero, and Φ_i is the perfect equality point in it. Applying Lemma 4.2 inductively, we have

(4.3) (a) $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_{n-1} = (0 \ \dots \ 0 \ 1) = U$,

(b) $B(\Omega_i, \Phi_i) = \Omega_{i+1}$,

(c) Ω_{i+1} lies in the boundary of Ω_i .

In words, Ω_{i+1} is the base (and lies in the boundary) of Ω_i relative to the ideal point Φ_i . Ω_{n-1} is the worst point U . Theorem 4.1 and Lemma 4.2 immediately imply:

Corollary 4.3. A complete pre-ordering R satisfying A1-A3 has the ideally centered property in each rank-preserving subspace Ω_i .

This corollary may be illustrated by the following example.

Example 6. Referring to Figure 4, for $n = 3$, we see that $\Omega_0 = \Phi_0 U \Phi_1$ and $\Omega_1 = U \Phi_1$. We see $B(\Omega_0, \Phi_0) = \Omega_1$ and $B(\Omega_1, \Phi_1) = \Omega_2 = U$, the worst point, i.e., Ω_1 spans Ω_0 and U spans Ω_1 . Since the pre-ordering indicated satisfies all four axioms, we see that within each rank-preserving subspace, R is ideally centered. However, this does not prevent the occurrence of the irregularity observed in example 5.

Example 6 shows that although an ordering is ideally centered in a subspace Ω_i (e.g., Ω_1 in the above example), there may be a base ray $U \Phi_1$ of Ω_i with an end point U inferior to some point d not necessarily in Ω_i and yet there is no point on this base ray which is inferior to d . It is to rule this irregularity that we need the following axiom:

A4'. Strong Axiom of Continuity. If b is a base point of Ω_0 and X is superior to b and inferior to the ideal point Φ_i in the subspace Ω_i (i.e., $\Phi_i > X > b$), then there exists a point Z on the base ray $b \Phi_i$ with the same inequality as X (i.e., $Z \simeq X$).

The property implied by A4' may be referred to as the strong continuously ideally centered property. It implies the (weak) continuously ideally centered property (A4) as a special case. Both axioms are intuitively reasonable for the same reasons.

The importance of A4' is that, along with our earlier axioms, it allows us to prove an important indexability theorem. We shall take up the notion of indexability in Section 6, but first we shall show that many of the familiar cardinal inequality measures satisfy A1-A4'.

5. Indices of Inequality

In empirical research on inequality, an index of inequality

$$(5.1) \quad I = f(X) = f(X_1 \ X_2 \ \dots \ X_n)$$

is defined on the income distribution space Ω^+ , where conventionally $f(Y) < f(X)$ is interpreted as "Y is more equal (less unequal) than X." Thus (5.1) naturally induces a complete pre-ordering R under the rule

$$(5.2) \quad Y R X \text{ when } f(Y) < f(X),$$

which obviously satisfies the conditions of comparability and transitivity of a complete pre-ordering (1.2). In this way, the induced R preserves the ordinal ranking of (5.1) while discarding its cardinality.¹ The reasonableness of an index (5.1) as a measurement of inequality is then seen to depend on the reasonableness of the R which it induces.

Relative to the purpose of the present paper, we note the following result:

Theorem 5.1. A real-valued function $f(X)$ induces a complete pre-ordering R satisfying A1-A4' when $f(X)$ satisfies the following:

(5.3) (a) Homogeneous of Degree Zero. $f(X) = f(aX)$, $a > 0$;

(b) Symmetry. $f(X_{i_1} \ X_{i_2} \ \dots \ X_{i_n}) = f(X_1 \ X_2 \ \dots \ X_n)$,

where (i_1, i_2, \dots, i_n) is a permutation of $(1, 2, \dots, n)$;

(c) Monotonicity of Partial Derivative.

$$\frac{\partial f}{\partial X_i} = f_i(X) < \frac{\partial f}{\partial X_j} = f_j(X) \text{ for } i < j \text{ and } X \in \Omega_0.$$

¹The analogy of this procedure to consumer theory is again noted.

It is an easy matter to show that R satisfies A1-A3. By virtue of being differentiable and hence continuous, A4' is satisfied. With the aid of this theorem, it can be shown¹ that the R 's induced by four of the most popular indices of inequality (the Gini coefficient, coefficient of variation, Theil index, and Atkinson index) satisfy all four axioms. Viewed in this light, the four axioms are seen to be very reasonable properties.

It follows from the above theorem that the Strong Axiom of Continuity (A4'), defined in the last section for ordinal rankings, has its origin in the continuity of conventional inequality indices. In the present paper, however, we reserve this approach by defining an ordinal ranking R axiomatically. It is natural to ask if R can be induced by a continuous real-valued function. If this can be done, we can in some sense rehabilitate cardinality in the measurement of inequality.² This rehabilitation is accomplished by appealing not to extra-model value judgments, but rather by appealing to the logic of our axioms. This brings us to the central issue of this paper, the indexability of R .

¹See Fields-Fei (1974).

²As is well-known in consumer preference theory, two cardinal utility functions $f(X)$ and $g(X)$ induce the same indifference ranking if and only if one is a monotonic transformation of the other. In the same sense, the rehabilitation of the cardinality of inequality measurement is unique only up to a monotonic transformation.

6. Indexability

In this section, we work at the same general level of abstraction as in Section 3 by letting a complete pre-ordering R be defined on any convex set C . At issue here is the indexability of R according to the following definition:

Definition. A complete pre-ordering R defined on a convex set C is indexable if there exists a real-valued function $f(X)$ on C which induces R . Furthermore, R is continuously indexable if $f(X)$ is continuous.

The reader is referred to Figure 3.b as a diagrammatic aid to the discussion of this section, but it should be understood that the results are completely general and do not pertain only to that example.

Suppose we have a continuously ideally-centered pre-ordering R with an extreme base ray AW connecting the ideal point A and the worst point W . We can first index the extreme base ray by a real-valued function $g(Y)$ such that $g(W) > g(A)$. For example, the distance $d(A, Y)$ between A and any point Y on AW can be such a function. Since the extreme base ray intersects every indifference set in C , the extension of $g(Y)$ to C is defined by $f(X) = g(h(X))$ where $h(X)$ is the choice function (3.3). Clearly $f(X)$ induces R on C .¹ Thus we have:

¹ Suppose $X > X'$. Then $h(X) > h(X')$ and, because R is ideally centered, $h(X)$ lies between $h(X')$ and A . Thus $f(X') - f(X) = g(h(X')) - g(h(X)) > 0$. Thus $f(X)$ induces R . Q.E.D.

Lemma 6.1. A complete pre-ordering R of a convex set C which is continuously ideally centered at A with a worst point W is indexable by a real-valued function $g(X)$ on the extreme base ray AW . With the aid of the choice function $h(X)$, $g(X)$ can be extended to C by $f(X) = g(h(X))$ which induces R on C .

Lemma 6.1 provides a set of sufficient conditions for the indexability of R . In fact:

Lemma 6.2. A continuously ideally centered complete pre-ordering R with an extreme base ray AW can be indexed by a real-valued function $f(X)$ which is continuous along any base ray.

Proof: Let $g(X)$ in Lemma 6.1 be chosen as a continuous function such as the distance function $d(A, X)$. We now want to prove that $f(X) = g(h(X))$ is continuous along any base ray bA . Notice that, by the ideally-centered property, $f(X)$ decreases monotonically from b to A . To show $f(X)$ is continuous along bA , it is sufficient to show that if r is any real number satisfying $f(A) < r < f(b)$, there exists a point Z^* on bA such that $f(Z^*) = r$. Notice that $h(b)$ is a point on the extreme base ray. Since $f(X)$ is continuous on the line segment from $h(b)$ to A , there is a point Z on this line segment such that $f(Z) = r$.¹ Thus $Z > h(b) \simeq b$ and hence by the continuously ideally-centered property, there exists a Z^* on the base ray bA such that $Z \simeq Z^*$. Thus $f(Z) = f(Z^*) = r$. Q.E.D.

¹This is because the line segment from $h(b)$ to A is a compact set and r is a real number between $f(A) = f(h(A))$ and $f(b)$.

Lemma 6.2 assures that $f(X)$ is continuous when we move toward the ideal point A along a straight line from any direction. Let $S(Z^*, \delta)$ and $S(X^*, \delta)$ stand for the δ -neighborhood ($\delta > 0$) about the points Z^* and X^* respectively. Intuitively, if $f(X)$ is to be continuous, the ordering R must be well-behaved within such neighborhoods. Hence we define:

Definition. A complete pre-ordering R which is ideally-centered at A has the local domination property if, for Z^* lying between X^* and A , there exists a $\delta > 0$ such that if $p \in S(Z^*, \delta)$ and $q \in S(X^*, \delta)$, then $q < p$.

In Figure 3.b, the local domination property states that on a typical base ray such as bA , there exists such neighborhoods so that all points in $S(Z^*, \delta)$ are more equal than all points in $S(X^*, \delta)$. With the aid of this property, we can state the following theorem:

Theorem 6.3. A complete pre-ordering R on a convex set C which

- (a) is continuously ideally centered at A ,
- (b) has a worst point W ,
- (c) has the local domination property

is indexable by a real-valued function $f(X)$ which is continuous at all non-base points.

Proof: Let $f(X)$ be the real-valued function of Lemma 6.2. Let Z^* be any non-base point which lies on a base ray Ab . We want to prove that $f(X)$ is continuous at Z^* . Let z_i be any infinite

sequence in C which converges to Z^* , i.e., $z_i \rightarrow Z^*$. We must show $f(z_i) \rightarrow f(Z^*)$. Since $Z^* \neq b$, there is a point X^* which lies between Z^* and b and hence Z^* lies between X^* and A . Produce the two neighborhoods $S(Z^*, \delta)$ and $S(X^*, \delta)$ according to the local domination property. All but a finite number of z_i lie within $S(Z^*, \delta)$ and hence almost all z_i are unambiguously superior to b . We can assume all $z_i > b$. By the continuously ideally centered property, we can find z_i^* on Ab satisfying $z_i^* \simeq z_i$ for all i . Since the ray Ab is compact, it contains a point of accumulation Y^* and there exists a subsequence $z_{i_j}^* \rightarrow Y^*$. Since $f(X)$ is continuous on Ab , $f(z_{i_j}^*) \rightarrow f(Y^*)$. We need to show $Y^* = Z^*$. If $Y^* \neq Z^*$, either $Y^* > Z^*$ or $Y^* < Z^*$. In either case, apply the local domination property again to produce the two neighborhoods $S(Y^*, \delta)$ and $S(Z^*, \delta)$. Then all except finitely many $z_{i_j}^*$ will be strictly superior (or inferior) to z_{i_j} . But this is impossible because $z_{i_j}^* \simeq z_{i_j}$. Thus $Y^* = Z^*$ and Z^* is the only point of accumulation of z_i^* on Ab . Thus $f(z_i) = f(z_i^*) \rightarrow f(Z^*)$. Q.E.D.

We now raise the question as to what property R must possess if the function $f(X)$ of Theorem 6.3 is to be continuous everywhere.¹

¹Let $\Gamma f(X)$ be the function $f(X)$ of Theorem 6.3 restricted to the interior of C . Whether or not $\Gamma f(X)$ can be extended to a function $F(X)$ which is continuous everywhere on C including the boundary is given by the following theorem of elementary topology: Let S and T be metric spaces, T complete. Suppose A is a subset of S such that the closure of A is S , and that $\phi: A \rightarrow T$ is a continuous mapping. Then there exists a continuous extension Φ of ϕ from A to S if and only if the oscillation of ϕ is zero at every point of S . [For example, see Hall and Spencer (1955).] It is this theorem which motivates the definition of oscillations in the text.

To state this property, let $h(T)$ denote the image of a subset $T \subset C$ under the choice function $h(X)$. In Figure 3.b, suppose b is a base point. We can construct a descending sequence of neighborhoods about b

$$(6.1) \quad S(b, 1/1) \supset S(b, 1/2) \supset \dots S(b, 1/m) \supset \dots$$

and take their image on the extreme base ray AW

$$(6.2) \quad h(S(b, 1/1)) \supset h(S(b, 1/2)) \supset \dots \supset h(S(b, 1/m)) \supset \dots$$

As m increases, the neighborhoods shrink toward b and their images form a descending sequence on the extreme ray AW . Notice that $h(b)$ is in every $h(S(b, 1/m))$ so

$$(6.3) \quad h(b) \in K_b \equiv \bigcap_m h(S(b, 1/m)) \subset AW$$

where K_b is the intersection of all sets in the descending sequence defined in (6.2). We now state the following definition:

Definition. A continuously ideally-centered complete pre-ordering R is non-oscillating at a base point b when $K_b = \bigcap_m h(S(b, 1/m)) = \{h(b)\}$. R is non-oscillating on the base if it is non-oscillating at every base point.

Since K_b contains the point $h(b)$, non-oscillation at the base point b requires that K_b contain no other points. Intuitively, within a neighborhood $S(b, 1/m)$, there is a most equal and least equal point and the oscillation within that neighborhood refers to the gap between the two. The extreme base ray provides a "scale" for measuring this

gap. If the ordering is non-oscillating at the base point b , the gap measured on the scale shrinks to zero as the neighborhood shrinks to a point

We are now able to state the following indexability theorem:

Theorem 6.4. A complete pre-ordering R on a convex set C which

- (a) is continuously ideally centered at A ,
- (b) has a worst point W ,
- (c) has the local domination property,
- (d) is non-oscillating on the boundary

is indexable by a real-valued function which is continuous everywhere in C .

This theorem is implied by the following lemma:

Lemma 6.5. The real-valued function $f(X)$ of Theorem 6.3 is continuous everywhere in C if and only if R is non-oscillating on the boundary.

This lemma is proven in the appendix.

Observe that, as in Section 3, the indexability theorems of this section are applicable quite generally. We apply them to the special case of Ω_0 in the next section.

7. Indexability Under A1-A4'

The general results of the last section can now be applied to our axiomatic system. We have shown that under A1-A3, R is ideally centered in Ω_0 . To prove that R is indexable by a real-valued function $f(X)$ which is continuous at the non-base points, we have to show R has the local domination property (Theorem 6.3). In addition, to show that $f(X)$ is continuous everywhere, we must establish that R also has the non-oscillation property (Theorem 6.4). We begin with the local domination property.

Lemma 7.1. Under A1-A3, if Q lies between P and the ideal point Φ_i in the rank-preserving subset Ω_i , there exists a real number $\delta > 0$ such that if $c \in S(Q, \delta)$ and $d \in S(P, \delta) \cap \Omega_i$ then $c > d$.

Proof: Since P and Q are in Ω_i , $Q_1 = Q_2 = \dots = Q_i = P_1 = P_2 = \dots = P_i = 0$. We define $V_t = \sum_{k=1}^t (Q_k - P_k)$. Then $V_1 = V_2 = \dots = V_i = 0$ and $V_t > 0$ for $i+1 \leq t \leq n-1$ because the Lorenz curve of Q strictly dominates that of P in Ω_i . Pick $\delta = \epsilon/3n$ where $\epsilon = \min_{1 \leq t \leq n} (V_t)$. Since c is in $S(Q, \delta)$, $\sum_{i=1}^n (c_i - Q_i)^2 \leq \delta^2$. Thus $(c_i - Q_i)^2 = |c_i - Q_i|^2 \leq \delta^2$ and hence $|c_i - Q_i| \leq \delta$ for $i = 1, 2, \dots, n$. Since d is in $S(P, \delta) \cap \Omega_i$, then $d_1 = d_2 = \dots = d_i = 0$ and $|d_i - P_i| \leq \delta$ for $i = 1, 2, \dots, n$. We want to show the Lorenz curve of c dominates that of d . Define $W_t = \sum_{k=1}^t (c_k - d_k)$. For $t \leq i$, $W_t = \sum_{k=1}^t c_k = 0$. Thus it remains to show $W_t > 0$ for $t > i$. We have $W_t = \sum_{k=1}^t c_k - \sum_{k=1}^t Q_k + \sum_{k=1}^t Q_k - \sum_{k=1}^t P_k +$

$$\begin{aligned}
& + \sum_{k=1}^t P_k - \sum_{k=1}^t d_k \text{ and for } t > i, W_t \geq u + \epsilon \text{ where } u = \sum_{k=1}^t (c_k - Q_k) \\
& + \sum_{k=1}^t (P_k - d_k) \text{ with } |u| \leq \sum_{k=1}^t |c_k - Q_k| + \sum_{k=1}^t |P_k - d_k| \leq 2n\delta = 2\epsilon/3.
\end{aligned}$$

Thus $W_t > 0$. Q.E.D.

In Lemma 7.1, if we let $\Omega_1 = \Omega_0$, then $S(P, \delta) \cap \Omega_1 = S(P, \delta)$.

Thus

Corollary 7.2. Under A1-A3, the complete pre-ordering R has the local domination property.

We have shown that under A1-A3, the complete pre-ordering has an extreme base ray ΦU (2.8). Furthermore, under the weak axiom of continuity A4, R is continuously ideally centered at Φ . The above Corollary and Theorem 6.3 imply the following weak indexability theorem:

Theorem 7.3. A complete pre-ordering satisfying A1-A4 is indexable by a real-valued function which is continuous everywhere in the monotonic rank-preserving space Ω_0 except possibly at the base.

This indexability theorem is labeled "weak" because the weak axiom of continuity which is used guarantees indexability by a real-valued function which may not be continuous at a base point. That this can happen is shown by the following example.

Example 7. Let us refer to the complete pre-ordering depicted in Figure 4 (see Example 5), which satisfies A1-A4. A continuous monotonic real-valued function $g(X)$ is first defined on the extreme base ray ΦU and extended to become $f(X) = g(h(X))$ on the triangle $\Phi \Phi_1 U$ above the base line $\Phi_1 U$. Then at b , $f(X)$ fluctuates between $f(a)$ and $f(c)$ no matter how small a neighborhood $S(b, \delta)$ one constructs about b . Thus, $f(X)$ cannot be continuous at b .

If the strong axiom of continuity A4' is used in place of the weak axiom A4, we have the following lemma:

Lemma 7.4. A1-A4' imply that R is non-oscillating on the boundary.

Proof: Let b be a base point. We want to show that R is non-oscillating at b . Construct the descending sequences $S(b, 1/m)$ (6.1) and their images $h(S(b, 1/m))$ (6.2). Suppose, contrary to the lemma, that K_b (6.3) has another point $Y \neq h(b)$. Then in every $S(b, 1/m)$ there exists a point X_m such that

- (i) $X_m \in S(b, 1/m)$,
- (ii) $h(X_m) = Y$, hence $X_m \approx Y$,
- (iii) $Y \neq h(b)$ is a point on the extreme base ray ΦU of Ω_0 .

There are two cases: $Y > h(b)$ and $Y < h(b)$.

Case one: $Y > h(b) \approx b$. By the continuously ideally centered property there exists a point Y^* lying between b and Φ . Apply the local domination property to b and Y^* . We can find neighborhoods $S(b, \delta)$ and $S(Y^*, \delta)$ such that every point in the first neighborhood is inferior to every point in the second. We can let m^* be large enough so that $S(b, 1/m^*) \subset S(b, \delta)$. Thus $Y^* \approx Y$ is superior to every point in $S(b, 1/m^*)$. This contradicts (i) and (ii) above.

Case two: $Y < h(b)$. The point b cannot be the worst point. Hence there exists a rank-preserving subspace Ω_i such that $b \in \Omega_i$ and $b \notin \Omega_{i+1}$. Then b lies on a base ray $b_i \Phi_i$ of Ω_i and $b > b_i$ (i.e., $b_i \in \Omega_{i+1}$ and b lies between Φ_i and b_i in Ω_i). There are then two sub-cases: $b > Y \geq b_i$ or $b > b_i > Y$. In the first sub-case, by A4' there exists a point Y^* on $b_i \Phi_i$ such that $Y^* \approx Y$. Since Y^* lies between b and b_i , we can apply Lemma 7.1 to the

pair of points (b, Y^*) to produce $S(Y^*, \delta)$ and $S(b, \delta) \cap \Omega_1$. When the integer m^* is large enough, $S(b, 1/m^*) \subset S(b, \delta)$. Thus, $X_{m^*} < Y^* \simeq Y$ which is a contradiction. In the second sub-case, apply Lemma 7.1 to the pair of points (b, b_1) and choose the m^* large enough so that $S(b, 1/m^*) \subset S(b, \delta)$. Then $X_{m^*} > b_1 > Y$ which is a contradiction. Q.E.D.

Since the strong axiom $A4'$ implies the weak axiom $A4$, by Theorem 6.4 and Lemma 7.4, we now have the following indexability theorem:

Theorem 7.5. Under $A1-A4'$ (the strong axiom of continuity) R is continuously indexable (i.e., there is a continuous real-valued function $f(X)$ which induces R on Ω_0).

We have yet to show that $f(X)$ in the above theorem can be extended from Ω_0 to a function $F(X)$ which induces R over the entire income distribution space Ω^+ . If $Y = (Y_1 \ Y_2 \ \dots \ Y_n)$ is a point of Ω^+ , we define

$$(7.1) \quad (a) \ N(Y) = (\theta_1 \ \theta_2 \ \dots \ \theta_n) \text{ where } \theta_i = Y_i / (Y_1 + Y_2 + \dots + Y_n),$$

$$(b) \ N(Y)^* = (\theta_{i_1} \ \theta_{i_2} \ \dots \ \theta_{i_n}) \in \Omega_0 \text{ (i.e., } i_1 \ i_2 \ \dots \ i_n \text{ is a permutation of } 1, 2, \dots, n \text{)}.$$

Then $A1$ and $A2$ imply $N(Y)^* \simeq N(Y) \simeq Y$. Hence the extension of $f(X)$ is defined by $F(X) = f(N(X)^*)$ and we have the following result:

Lemma 7.6. If $f(X)$ is continuous in Ω_0 , then $F(X)$ is continuous in Ω^+ .

Proof: Let the sequence $X_i \rightarrow X_0$ in Ω^+ . We want to prove $F(X_i) \rightarrow F(X_0)$. Since Ω^+ is the union of a finite number $(n!)$ of rank-preserving subsets $C(i)$ (1.5), at least one of these subsets will contain infinitely many X_i . Let the number of such subsets be denoted by r and denote these subsets by $C^r(i)$, $i = 1, \dots, r$. Since the other subsets contain only a finite number of X_i , we can assume X_i are contained only in the first $C^r(i)$. The X_i in $C^r(1)$ constitute a subsequence $X_{i_j} \rightarrow X_0$. Since $C^r(1)$ is a closed set, X_0 is in $C^r(1)$, and the sub-sequence $N(X_{i_j}) \rightarrow N(X_0)$ in $\Omega^r(1) = C^r(1) \cap \Omega^c$ (1.6). Since $\Omega^r(1)$ is a closed set, it contains X_0 . Since $F(X)$ is continuous in $\Omega^r(1)$, given $\epsilon > 0$, we can find a $\delta_1 > 0$ such that $u \in S(X_0, \delta_1) \cap \Omega^r(1) \Rightarrow |F(u) - F(X_0)| < \epsilon$. Thus there exists an integer j_1 such that $i_j > j_1$ implies $|F(N(X_{i_j})) - F(N(X_0))| = |F(X_{i_j}) - F(X_0)| < \epsilon$. Apply the same argument to the remaining $C^r(i)$ to produce $\delta_2, \dots, \delta_r$ and j_2, \dots, j_r . Let $\delta = \min \delta_i$ and $j = \max j_i$. Then all except finitely many X_i satisfy the condition $|F(X_{i_j}) - F(X_0)| < \epsilon$. Q.E.D.

Combining Theorem 7.5 and Lemma 7.6, we have the basic indexability theorem of our paper:

Theorem 7.7. Under A1-A4', a complete pre-ordering is continuously indexable over the entire income distribution space Ω^+ .

8. Implications of Indexability

The significance of the indexability of a complete pre-ordering R may now be examined from the point of view of empirical and theoretical research. For empirically collected statistical data, a real-valued index is needed to calculate the degree of income distribution inequality. The processing of empirical data would be hampered if a complete pre-ordering R (no matter how reasonable and how ideally defined) could not be expressed by an index formula (i.e., not be indexable). Our four axioms which ensure that R is indexable in principle meet this requirement of empirical research. Usually, $f(X)$ is transformed by

$$(8.1) \quad F(X) = (f(X) - f(A)) / (f(W) - f(A))$$

so that $F(A) = 0$ at the ideal point and $F(W) = 1$ at the worst point. All the familiar indices mentioned in Section 5 are of this type.

For purposes of building a positive theory of the determination of the distribution of income, an indexable pre-ordering offers certain advantages. One is that the degree of inequality $I = F(X)$ can be treated as an endogenous variable of formal economic models. At present, we have only an embryonic understanding of the determinants of income distribution, especially in a development context. It would be desirable to represent ordinal judgments about the overall degree of inequality by an index. We know from our indexability theorem that this can be done.

Furthermore, the fact that the indexing function is continuous allows us to deduce properties of R which prove to be indispensable

for such a positive theory.¹ For example, we derive

Theorem 8.1. Under $A1-A4'$,

- (a) Inferior sets, superior sets, and indifference sets are closed sets,
- (b) If two points P and Q , $P > Q$, are connected by a simple arc, then if $P > Y > Q$, there exists a point Z on the arc such that $Z \simeq Y$.

Proof: (a) follows from the fact that the inverse images (under $F(X)$) of the half lines are the superior and inferior sets. (b) follows from the fact that on the arc (which is a compact set), $F(X)$ takes on any value r satisfying $F(P) \leq r \leq F(Q)$. Q.E.D.

As another example, Theorem 8.1.b suggests that we define continuity of a complete pre-ordering as follows:

Definition. A complete pre-ordering R of a convex set C is continuous if for $P > Y > Q$ in C , then any simple arc connecting P and Q contains a point Z such that $Z \simeq Y$.

Notice that an R which is continuous according to this definition implies the properties of the strong and weak axioms ($A4$ and $A4'$).

¹ Consider again the analogy with models of consumer behavior. It is well known that cardinality of the utility function $U(Y)$ is unnecessary since all that is required for the analysis of consumer behavior is the ordering induced by $U(Y)$. Nevertheless, it is assumed implicitly that the ordinal ranking is indexable so that the cardinal utility is represented by an endogenous variable which can be maximized. It is hard to imagine how familiar notions such as the income and substitution effects could be deduced without the indexability property.

Hence Theorem 8.1.b implies

Theorem 8.2. Under A1-A4', R is continuous.

Finally, for optimization problems, the continuously indexable function may also be differentiable, in which case we could treat the maximization problem with conventional mathematical methods such as ordinary differential calculus. This is very convenient and may prove to be quite important for helping to integrate inequality considerations into models of optimal growth.

9. Directions for Future Research

In this paper, we have shown that when a pre-ordering R defined on an arbitrary convex set C is continuously indexable when R (a) is continuously ideally centered, (b) has an extreme base ray, (c) has the local domination property, (d) is non-oscillating on the boundary. Furthermore, we have shown that under four reasonable axioms for inequality, these conditions are met.

Future research on inequality of income distribution is likely to move in three directions: empirical measurement, theoretical research on the determinants of inequality, and the design of better inequality indices. We have already addressed the first two points in Section 8. We now add some concluding remarks on the last issue.

The four axioms, as a set, are incomplete in that they cannot uniquely determine R .¹ A feasible direction for future research is to investigate what additional axioms can be postulated in order to complete the axiomatic system, or failing that to reduce further the zones of ambiguity. In this respect, three points may be raised by our approach.

First, the additional axioms may be imposed on Ω_0 , the monotonic rank preserving set, which is simpler than the entire income distribution space not only because it is smaller (i.e., a subset) but also because it possesses the intrinsic merit of rank preservation. If we can compare pairs of points in Ω_0 , we can in fact rank pairs

¹As we have seen, many indices of inequality satisfy the four axioms.

in Ω^+ . (Theorem 1.1). Conversely if we can not compare pairs of points in Ω_0 it is even more difficult to compare those in Ω^+ in which we have to face the additional problem of rank reversals, a troublesome issue in a status conscious society.

Second, we have shown that there are n rank-preserving subspaces in Ω_0 . It can easily be shown that when $n = 2$, our axiomatic system is complete. However, all the problems of inequality comparisons are found when there are three or more families. Thus, future work on the search for new axioms may be conducted for the case $n = 3$. Whatever reasonable properties one may derive can then be extended inductively to the general (n -family) case.

Finally, in the search for new axioms on Ω_0 , our paper shows that we can uniquely determine R once we have specified the choice function $h(X)$, i.e., a rule for comparing any point with a point on the extreme base ray. This suggests that future research may concentrate on the specification of reasonable properties for the choice function.

APPENDIX

In this appendix, we prove Lemma 6.5 in the text. We work with a complete pre-ordering R on a convex set C which is continuously ideally centered at A , has a worst point U , and also has the local domination property. Then R is induced by a real-valued function $f(X)$ which is (i) continuous at all non-base points (i.e., all points in $C - B(C,A)$, which includes the interior of C), and (ii) continuous along any base ray bA including the base point b (see Lemma 6.2 and Theorem 6.3). Let P and Q ($P \neq Q$) be two points of C . Let $\alpha(P,Q)$ be a simple arc connecting P and Q (i.e., $\alpha(P,Q)$ is homeomorphic to a real closed interval $[p,q]$). Then

Lemma A.1. Let P, Q ($P \neq Q$) be points of $S(b,\delta)$, an open neighborhood of a base point $b \in B(C,A)$. Then there exists a simple arc $\alpha(P,Q) \subset S(b,\delta)$ on which $f(X)$ is continuous.

Proof: If P and Q are non-base points, from the convexity of C , the line segment PQ is such an arc $\alpha(P,Q)$ which lies within $S(b,\delta)$ and which contains no base points. Hence Lemma A.1 holds.

Suppose now both P and Q are base points lying within $S(b,\delta)$. Then consider the base rays PA and QA . We can take a point $p \neq P$ on PA (and $q \neq Q$ on QA) which is close enough to P (Q) that p (q) is in $S(b,\delta)$. The arc $\alpha(P,Q)$ is formed by the line segments pP and qQ and the line connecting p and q . When only one of P or Q is a base point, the construction is similar. Q.E.D.

Now suppose $P > Q$, and hence $f(P) < f(Q)$. If r is any

real number satisfying $f(P) < r < f(Q)$, then since $\alpha(P, Q)$ is compact there exists a point Z on $\alpha(P, Q)$ such that $f(Z) = r$, or $P > Z > Q$. Thus we have:

Lemma A.2. If P, Q ($P \neq Q$) are two points in $S(b, \delta)$ such that $P > Y > Q$, then there exists a point Z in $S(b, \delta)$ such that $Z \simeq Y$.

Let b be a base point. Then for the image $h(S(b, 1/m))$ in (6.2), we have

Lemma A.3. The set $h(S(b, 1/m))$ is convex.

Proof: Suppose $h(Q) < h(P)$ are points in $h(S(b, 1/m))$ included in the extreme base ray AW . If Y lies between $h(P)$ and $h(Q)$, by the ideal centered property of R , $h(P) > Y > h(Q)$ or $P > Y > Q$. Lemma A.2 implies there exists a point Z in $S(b, 1/m)$ satisfying $Z \simeq Y$, and hence $Y = h(Z) \in h(S(b, 1/m))$. Q.E.D.

Since $h(S(b, 1/m))$ is a convex set on a line (i.e., the base ray AW), it is an interval (u, v) . Thus the descending sequence $h(S(b, 1/m))$ can be written as

$$(A.1) \quad h(S(b, 1/m)) = (u_m, v_m), \quad m = 1, 2, \dots$$

The fact that $h(b)$ is in all such intervals and that these intervals are descending can be written as

$$(A.2) \quad (a) \quad u_m \geq h(b) \geq v_m \quad \text{or} \quad f(u_m) \leq f(h(b)) \leq f(v_m)$$

$$(b) \quad f(u_m) \leq f(u_{m+1}), \quad f(v_m) \geq f(v_{m+1}).$$

Thus $f(u_m)$ is monotonically non-decreasing (and $f(v_m)$ monotonically non-increasing) and is bounded from above (below) by $f(h(b))$. Let the least upper bound LUB (GLB) of $f(u_m)$ ($f(v_m)$) be denoted by u^* (v^*). We have:

$$(A.3) \quad (a) \quad f(u_m) \rightarrow u^*, \quad f(v_m) \rightarrow v^* \\ (b) \quad u^* \leq f(b) \leq v^* .$$

When R is non-oscillating at b , we readily see

$$(A.4) \quad u^* = f(b) = v^*$$

for otherwise the set

$$(A.5) \quad K_m = \bigcap_m (u_m, v_m)$$

would contain a point other than $h(b)$. These may be summarized as

Lemma A.4. If R is non-oscillating at b , then given any open neighborhoods (s,t) of $h(b)$ (i.e., (s,t) is an open line segment on the extreme base ray containing $h(S(b, \delta))$, almost all $h(S(b, 1/m))$ are in (s,t) .

We then have the following result [Lemma 6.5 in text]:

Theorem A.5. The real-valued function $f(X)$ that induces R is continuous at a base point b if and only if R is non-oscillating at b .

Proof: Suppose R is oscillating at b . Then K_b contains a point $Y \neq h(b)$, i.e., $f(Y) \neq f(h(b)) = f(b)$. There then exists

a point Z_m in $S(b, 1/m)$ such that $f(Z_m) = f(Y) \neq f(b)$. Hence $f(X)$ cannot be continuous at b . Conversely, suppose R is non-oscillating at b . We want to show $f(X)$ is continuous at b . Suppose this is not true. Then there exists an $\epsilon > 0$ and Z_m in $S(b, 1/m)$ such that $|f(b) - f(Z_m)| > \epsilon$. Thus $h(S(b, 1/m))$ contains a point $h(Z_m)$ satisfying $|f(h(b)) - f(h(Z_m))| > \epsilon$. Since by assumption $f(X)$ is continuous at $h(b)$ along the base ray, given $\epsilon > 0$, there exists (s, t) containing $h(b)$ such that $|f(Y) - f(b)| < \epsilon$ for all Y in (s, t) . Lemma A.4 then implies a contradiction. Q.E.D.

Notice that in our paper oscillation has been defined in the ordering sense. The ordinary meaning of oscillation (as used in the theorem quoted in the second footnote of Section 6) is defined in a topological sense. The following theorem immediately establishes a relation between the two meanings of oscillation:

Theorem A.6. The restriction $\Gamma f(X)$ of the real-valued function $f(X)$ on the non-base points is non-oscillating in the topological sense if and only if R is non-oscillating in the ordering sense.

The topological theorem in the text and Theorem A.5 immediately imply this theorem.

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