

ECONOMIC GROWTH CENTER

YALE UNIVERSITY

Box 1987, Yale Station
New Haven, Connecticut

CENTER DISCUSSION PAPER NO. 306

EQUITY ORIENTED FISCAL PROGRAMS

John C. H. Fei

January 1979

Notes: Center Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. References in publications to Discussion Papers should be cleared with the author to protect the tentative character of these papers.

Equity Oriented Fiscal Programs

Let $Y=(Y_1, Y_2, \dots, Y_n) \geq 0$ be an income distribution pattern to n - "income receiving units" which may be n -persons, n -families, n -states of the same country or n -countries. Abstractly, a fiscal program is a course of action, undertaken by social consensus, under which portions of the incomes of certain receiving units are transferred to other receiving units to render the income distribution more equitable. The most familiar example of such a fiscal program is the collection of taxes from individuals (or individual families) with the revenue being paid out as welfare payments by the government. As another example, the Federal government may collect taxes from the states only to give some of the revenue back to the states under a "revenue sharing" program. An international consortium or the World Bank may work out a formula under which contributions will be solicited from the wealthy countries or "donors" to provide foreign aid or make concessionary loans to the poor countries. This paper is concerned with the principles governing the design of such equity oriented fiscal programs.

The first general principle concerns the "rationality" of the fiscal program. Suppose the income level of "i" is higher than that of "j". On the one hand, a principle of "minimally progressive" suggests that, in case "i" and "j" are taxpayers, "i" should pay no less taxes than "j" and, in case "i" and "j" are recipients of welfare payments "i" should receive no more than "j". On the other hand, a principle of "incentive preservation" suggests that the disposable income of "i" should be no less than that of "j"--i.e. the fiscal program clearly should not reverse their relative income ranks in order to preserve the incentive for the individuals

to earn a higher income. A rational fiscal program should be both "minimally progressive" and "incentive preserving" which are, indeed, very reasonable and mild requirements from the equity point of view.

A second general principle concerns the overall size of the total budget B (i.e. the total taxes collected or total welfare payments). For when B is higher, the fiscal program can in general collect more taxes from more taxpayers and distribute larger welfare benefits to more recipients. Intuitively, a foreign aid program which operates with a larger total budget B (e.g. brought about by requiring all wealthy aid giving countries to donate a higher percentage of their GNP as foreign aid contributions) can benefit the aid receiving countries more. Similarly, a domestic social welfare program can help the poor more with a larger total budget B . In all cases, it is clear that a social consensus on the total budget size B is a prerequisite for the design of any equity oriented fiscal program.

When an initial income distribution pattern Y is given, the operation of a fiscal program G_Y on Y leads to a pattern of disposable income $D(G_Y) = (D_1, D_2, \dots, D_n)$ to the n -receiving units. The third general principle centers on the choice of a reasonable index of inequality $I(Y)$ so that, when the budget size B is given, the optimum fiscal program G_Y^0 can be chosen to minimize $I(D(G_Y))$ (i.e. minimizing the inequality of the disposable income) under the budget constraint. There are currently many familiar indices of inequality--e.g. the Gini coefficient, the Theil index, the coefficient of variation and the Atkinson index--¹ which, when used for this purpose, leads to formulation of a non-linear

¹See Atkinson, A.B. [2], Theil, H. [14].

programming problem the solution of which may determine the optimum fiscal program G_Y^O uniquely.

The basic theorem that will be proven in this paper is that the unique solution G_Y^O not only exists (and can be quite readily calculated) but is, in fact independent of the index of inequality $I(Y)$ chosen for this purpose--provided these indices are "reasonable". In this context, the "reasonableness" is ensured by the Daltons' "principle of transfer" which is, in fact, satisfied by all the indices of inequality mentioned above. Thus, when indices of inequalities are used as a policy guide¹ (i.e. to construct an optimum G_Y^O to modify an initial income distribution pattern Y from the equity stand point), the basic theorem implies that the Dalton's principle of transfer is sufficient; and hence, the search for a specific (e.g. an ideal) index of inequality is unnecessary and superfluous.

We shall first define the rational fiscal programs in Section I. These programs will be classified in Section II where the method for the computation of the optimum feasible solution G_Y^O will be introduced. In Section III, we will state the basic theorem the proof of which is relegated to the appendix. This theorem will be generalized in Section IV where we will assume that total government revenue will be spent not only as welfare (i.e. transfer) payments but also for "productive" (e.g. national defense) purposes.

¹See Dalton, H. [4].

Section I

Let Ω be the set of all non-negative n-dimension vector $x = (x_1, x_2, \dots, x_n) \geq 0$. An income distribution pattern $Y = (Y_1, Y_2, \dots, Y_n)$ is a point in Ω (i.e. $Y \in \Omega$). A fiscal program operating on Y is defined as:

Definition: A Fiscal program G_Y operating on Y is a vector $G_Y = (G_1, G_2, \dots, G_n)$ satisfying the following conditions (1.1abc):

$$1.1a) \quad G_1 + G_2 + \dots + G_n = 0 \quad \text{Balanced Budget}$$

$$b) \quad G_i \leq Y_i \quad \text{for } i=1, 2, \dots, n \quad \text{Feasibility}$$

$$c) \quad B(G_Y) = \sum_{G_i > 0} G_i = \left(\frac{1}{2}\right) \sum_{i=1}^n |G_i| > 0 \quad \text{Positive Budget}$$

A disposable income pattern $D(G_Y)$ of G_Y is defined as

$$d) \quad D(G_Y) = (D_1, D_2, \dots, D_n) = Y - G_Y \geq 0 \quad (\text{i.e. } D(G_Y) \neq G_Y \text{ and } D(G_Y) \in \Omega)$$

In the above definition, the i-th person will be referred to as a tax-payer, welfare (payment) recipient or unaffected as $G_i > 0$, $G_i < 0$ or $G_i = 0$. Condition 1.1a) states that the fiscal program has a balanced budget i.e. total tax revenue equals total welfare payments.¹ Condition 1.1b) implies that the fiscal program is feasible as a tax payment must not exceed the income of any taxpayer. In 1.1c) $B(G_Y)$ is the total taxes collected which will be referred to as the budget size. Condition 1.1c) implies that there is at least one taxpayer and hence the fiscal program is non-trivial. Notice that (1.1ac) implies that the disposable income pattern $D(G_Y) \in \Omega$ and $D(G_Y) \neq Y$.

¹This condition will be relaxed in Section IV below.

The balanced budget condition (1.1a) implies that a fiscal program does not effect the per capita income:

$$1.2) \bar{Y} \equiv (Y_1 + Y_2 + \dots + Y_n)/n = (D_1 + D_2 + \dots + D_n)/n \equiv \bar{D}(G_Y)$$

A fiscal program is "minimally progressive", "incentive preserving" or "rational" according to the following definitions:

Definition: A fiscal program G_Y is

- i) minimally progressive when for all i, j , $Y_i \geq Y_j$ implies $G_i \geq G_j$
- ii) incentive preserving when for all i, j , $Y_i \geq Y_j$ implies $D_i \geq D_j$
- iii) rational when it has the M-P (minimally progressive) and the I-P (incentive preserving) properties.

The intuitive explanations for these concepts are given in the introduction. There is no loss of generality if we assume that Y is monotonically non-decreasing i.e. $Y_1 \leq Y_2 \leq \dots \leq Y_n$. We can then state without proof:

Lemma one: When Y is monotonically non-decreasing

- a) G_Y is monotonically non-decreasing if and only if the M-P property is satisfied.
- b) $D(G_Y)$ is monotonically non-decreasing if and only if the I-P property is satisfied.

For two persons with the same income ($Y_i = Y_j$), we have,

Lemma two: When G_Y has the M-P property or the I-P property $Y_i = Y_j$ implies $D_i = D_j$.

Any rational program possesses the following property:

Lemma three: For any rational program, the disposable income of a welfare recipient is not higher than that of any taxpayer.

Proof: Let the j -th person be a welfare recipient and the i -th person be a taxpayer. Then $G_j < 0 < G_i$ which implies $Y_j \leq Y_i$ by the M-P property. This in turn implies $D_j \leq D_i$ by the I-P property. QED.

This lemma suggests that a rational fiscal program is fair and equitable. For any monotonically non-decreasing X and Y in Ω with $\sum X_i = \sum Y_i$, " X Lorenze Dominates Y ", in notation $L_X \geq L_Y$, is defined as:

$$1.3a) X_1 + X_2 + \dots + X_i \geq Y_1 + Y_2 + \dots + Y_i \text{ for } i = 1, 2, \dots, n$$

$$b) X_1 + X_2 + \dots + X_i > Y_1 + Y_2 + \dots + Y_i \text{ for some } i$$

We have the following theorem which states that rationality ensures $D(G_Y)$ Lorenze Dominates Y :

Theorem one: For any rational fiscal program G_Y , we have $L_{D(G_Y)} \geq L_Y$

Proof: Let Y be monotonically non-decreasing, then $D(G_Y)$ is monotonically non-decreasing by the I-P property (Lemma one b). Let $s_i = \sum_{k=1}^i D_k - \sum_{k=1}^i Y_k$

$$= \sum_{k=1}^i (Y_k - G_k - Y_k) = - \sum_{k=1}^i G_k. \quad \text{We want to show all } s_i \text{ are non-negative and at}$$

least one $s_i > 0$. Notice that by lemma one a, $s_1 = -G_1 > 0$ because the poorest person is a welfare recipient. If the first $q > 0$ persons are all the welfare recipients $s_q = B(G_Y) > 0$, by 1.1c). Then $s_1, s_2, \dots, s_q$ monotonically increase to s_q and hence non-negative. Since $s_n = 0$, the sequence $s_q, s_{q+1}, s_{q+2}, \dots, s_n$ monotonically decreases to zero and hence also non-negative. QED.

This theorem is another indication that a rational fiscal program is an equity oriented one.¹ Notice that $L_{D(G_Y)} \geq L_Y$ does not necessarily imply that G_Y is rational as can be seen from $Y = (3,5,9)$ and $D(G_Y) = (3,6,8)$. (For $G_Y = Y - D(G_Y) = (0,-1,1)$ which violates the M-P property).

Section II

We want to identify two special types of rational fiscal programs. The first type is the unique "mean deviation program" G_Y^1 given in the following definition:

Definition: The mean deviation program of Y is $G_Y^1 = (Y_1 - \bar{Y}, Y_2 - \bar{Y} \dots Y_n - \bar{Y})$. The budget size $B(G_Y^1) = B_M$ will be referred to as the maximum rational budget.

It is obvious that G_Y^1 is a rational fiscal program that completely equalizes disposable income $D(G_Y)$ at $\bar{Y}$ for everyone. It is the only fiscal program that completely equalized disposable income as

Lemma four: If $D(G_Y) = (u, u, \dots, u) > 0$ then $G_Y = G_Y^1$

(Proof: by 1.2) $\bar{Y} = \bar{D}(G_Y) = u$. Then $G_Y = Y - (\bar{Y}, \bar{Y} \dots \bar{Y}) = G_Y^1$. QED.)

The fact that B_M is referred to as the maximum rational budget is readily seen from the following lemma:

¹Thus the Lorenze curve of $D(G_Y)$ lies everywhere "above" the Lorenze curve of Y . See Atkinson [2], Rothschild and Stiglitz [10], Dasgupta, Sen and Starrett [5] on the relation between Lorenze Domination and inequality comparison (i.e. on the fact that $D(G_Y)$ can be regarded as more equally distributed than Y).

Lemma five: If the budget size $B(G_Y)$ of any fiscal program G_Y is not less than B_M , (i.e. $B(G_Y) \geq B_M$); then

- i) the disposable income of at least one tax payer is not more than the mean income $\bar{Y}$.
- ii) the disposable income of at least one welfare recipient is not less than the mean income $\bar{Y}$.

(proof: obvious)

which leads directly to the following theorem,

Theorem two: For any rational fiscal program $G_Y \neq G_Y^1$, we have $0 < B(G_Y) < B_M$.

Proof: Suppose $B(G_Y) \geq B_M$. Lemma five (ii) and lemma three imply that under G_Y , the disposable income of all taxpayers are not less than $\bar{Y}$. This, in turn implies that (i) the income Y_i of all such taxpayers are not less than $\bar{Y}$ and, in fact, must be strictly more than $\bar{Y}$ and (ii) the maximum amount of taxes collected from every taxpayer is the mean deviation $Y_i - \bar{Y}$. Thus $Y_i - \bar{Y}$ must be collected from every $Y_i > \bar{Y}$ if the entire amount B_M is to be collected. Similarly, under G_Y , $\bar{Y} - Y_j$ must be paid to all welfare recipients with $Y_j < \bar{Y}$. Thus $G_Y = G_Y^1$ which is a contradiction. QED.

Thus, with the exception of the mean deviation program G_Y^1 , the budget size of all rational fiscal programs is strictly less than the maximum rational budget B_M . A second type of rational fiscal program, to be referred to as two-valued fiscal programs, is given by the following definition:

Definition: A fiscal program G_Y is two-valued when there exist two critical values $0 < M_* \leq M^*$ (i.e. a floor value M_* and a ceiling value M^*) such that the i -th person

- i) is a taxpayer and taxed by the amount $Y_i - M^* > 0$ if and only if $Y_i > M^*$.
- ii) is a welfare recipient and receives a welfare payment $M^* - Y_i > 0$ if and only if $Y_i < M_*$.

Thus, under a two-valued program G_Y the disposable income of all taxpayers (welfare recipients) becomes M^* (M_*) while the disposable income of all unaffected persons Y_i lies between M^* and M_* (i.e. $M_* \leq Y_i = D_i \leq M^*$). When Y is monotonically non-decreasing, both G_Y and $D(G_Y)$ are monotonically non-decreasing and hence lemma one implies that a two-valued program is rational. The ceiling value M^* cannot be less than the mean income $\bar{Y}$, for $M^* < \bar{Y}$ implies that total taxes collected will be greater than B_M contradicting theorem two. Similarly $M_* \leq \bar{Y}$. If $M^* = \bar{Y}$, total tax collected will be B_M . Theorem two implies $G_Y = G_Y^1$ and $M^* = M_* = \bar{Y}$ as the two-valued program becomes the mean deviation program. These results may be summarized as:

Theorem three: A two value program G_Y is rational and

- a) $M_* = M^*$ implies $G_Y = G_Y^1$
- b) If $G_Y \neq G_Y^1$ then $M_* < \bar{Y} < M^*$

A two-valued program, much like a negative income tax proposal, guarantees that all individuals receive at least the floor income M_* . It also enforces a rule under which no individual can receive more than the ceiling income M^* . When a total budget size $B > 0$ is specified, to construct a two valued program collect as much tax as possible from the wealthiest person until his income is lowered to the level of the second wealthiest. Then collect the same amount of taxes from the two wealthiest persons, as much as possible, until their income level is lowered to the third wealthiest. Proceed in this way until the entire amount B is collected, and, in this way determine the ceiling value M^* . Similarly the distribution of the welfare benefits starts with the poorest person. Thus, when B is given, not only can we determine a two-valued program uniquely but G_Y is also seen to be the most "equitable" fiscal program G_Y under the given budget constraint B . When B is raised, G_Y will collect higher taxes from more taxpayers and distribute higher welfare benefits to more welfare recipients. Let " t " be the number of taxpayers and " w " be the number of welfare recipients. Formally, we have the following theorem:

Theorem four: A two valued program G_Y is uniquely determined by its budget size B (i.e. $0 < B < B_M$) where

- i) the ceiling value $M^*(B)$ is a strictly monotonically decreasing function and where the floor value $M_*(B)$ is a strictly monotonically increasing function of B .
- ii) the number of taxpayers $t(B)$ and the number of welfare recipients $w(B)$ are non-decreasing functions of B .

The proof of this theorem is a constructive one that proceeds to determine M^* and M_* . For any M^* in the open interval $(\bar{Y}, \infty)$ and for any M_* in the open interval $(0, \bar{Y})$, the total tax revenue $R(M^*)$ and total welfare payments $E(M_*)$ for any two-valued program are:

$$2.1 \text{ a) Total tax revenues: } R(M^*) = \sum_{Y_i > M^*} (Y_i - M^*) \text{ for } \bar{Y} < M^*$$

$$\text{b) Total welfare payment: } E(M_*) = \sum_{Y_i < M_*} (M_* - Y_i) \text{ for } 0 < M_* < \bar{Y}$$

Notice that as real valued functions $R(M^*)$ is strictly monotonically decreasing and $E(M_*)$ strictly monotonically increasing. Furthermore,

$$2.2 \text{ a) } \lim_{M^* \rightarrow \bar{Y}} R(M^*) = B_M \text{ and } \lim_{M^* \rightarrow \infty} R(M^*) = 0$$

$$\text{b) } \lim_{M_* \rightarrow \bar{Y}} E(M_*) = B_M \text{ and } \lim_{M_* \rightarrow 0} E(M_*) = 0$$

Since $R(M^*)$ and $E(M_*)$ are continuous, the inverse functions exist

$$2.3 \text{ a) } M^*(B) = R^{-1}(B) \quad 0 < B < B_M \quad (M^*(B) \text{ monotonically decreasing})$$

$$\text{b) } M_*(B) = E^{-1}(B) \quad 0 < B < B_M \quad (M_*(B) \text{ monotonically increasing})$$

When the budget size B is given in the range $0 < B < B_M$ we can determine a pair of critical values uniquely by 2.3. That both $t(B)$ and $w(B)$ are non-decreasing is obvious. This proves theorem four.

Section III

An index of inequality $I(Y)$ is a real valued function defined on Ω which contains all disposable income patterns $D(G_Y)$ (1.1d). When such an index is given and when there is a social consensus on the "maximum welfare budget" $\bar{B}$ (see introduction), it is natural to formulate the following non-linear programming problem:

- 3.1) To minimize $I(D(G_Y))$ for all fiscal programs G_Y that satisfy

$$B(G_Y) \leq \bar{B} < B_M.$$

The solution G_Y^0 is the optimum fiscal program which minimizes the degree of inequality of the disposable income. Notice that in case $\bar{B} \geq B_M$, the problem becomes trivial as the mean deviation program G_Y^1 is obviously the unique solution. Thus the condition $\bar{B} < B_M$ is added to render the problem non-trivial. The basic theorem of this paper states that the optimum solution is a two-valued program (with a budget size $\bar{B}$) provided the index of inequality is a "reasonable" one. Heuristically, the theorem states:

Theorem five: For all "reasonable" indices of inequality, the unique solution to (3.1) is the two valued program with a budget size $\bar{B}$.

By this surprisingly strong theorem, we do not need the algebraic form of $I(Y)$ to compute the optimum program G_Y^0 -- as we have shown in the last section.¹ What constitutes a "reasonable" index $I(Y)$ is obviously a crucial matter. We shall require that a "reasonable index" satisfies the following two

¹For recent discussions on axiomatic approaches to inequality comparison see Champenowne [3], Kondor [9], Sen [11], Szal and Robinson [13], Fields and Fei [6]. In view of the fact that the "axioms" are "incomplete", much current research effort (mostly futile) is directed at searching for new axioms to determine the "ideal" index uniquely. We have shown that the search is unnecessary.

conditions:

3.2 a) Anonymity: $I(Y_1, Y_2, \dots, Y_n) = I(Y_{i_1}, Y_{i_2}, \dots, Y_{i_n})$ if $(i_1, i_2, \dots, i_n)$ is a permutation of $(1, 2, \dots, n)$.¹

b) Dalton's Transfer Property When X and Y are monotonically non-decreasingly arranged, $I(X) < I(Y)$ when the following conditions are satisfied:

for some i, j ($i < j$) and $h > 0$

- i) $X_k = Y_k$ for $k \neq i, j$
- ii) $X_i = Y_i + h$ and $X_j = Y_j - h$ where
 - if $j = i+1$, $h \leq (1/2) (Y_j - Y_i)$
 - if $j > i+1$, $h \leq \min [(Y_{i+1} - Y_i), (Y_j - Y_{j-1})]$

Precedent for 3.2b) dates back at least half a century to Dalton who called this "principle of transfer".² i.e. the transfer of a positive amount (h) of money from a wealthy (j th) to a poor person, (i.e. i th) without effecting the income rank of all individuals will lower the index of inequality. The fact that a) and b) are mild and reasonable is testified by the fact that many well known indices of inequality--e.g. the Gini coefficient, Atkinsons index, Theil Index, coefficient of variation--satisfy these conditions.³ Theorem five is valid when any one of these indices is used. When the Dalton's principle of

¹We wish to emphasize a distinction that 3.2a (alternatively referred to the principle of "symmetry" in the literature) is less controversial than 3.2b and, for this reason, will not be referred to again in this paper. See Sen [11] for the view point that 3.2a may also be controversial because of its conflict with a Benthanmite utilitarian approach.

²The formulation of the Dalton's Transfer property in 3.2b emphasized the "rank non-reversal" specification. See Fields and Fei, page 307.[6]

³For a proof see Fields and Fei, pages 313-314. [6]

transfer (3.2b) is satisfied, it is well known that¹:

3.3) $L_X \geq L_Y$ implies $I(X) < I(Y)$

In other words, Lorenze Domination implies that all the familiar indices agree on the relative inequality of X and Y .

Making use of (3.3), theorem five can be proved through the following three lemmas:

Lemma six: If G_Y and G'_Y are two "two-valued programs" and if G_Y has a larger budget size than G'_Y (i.e. $B(G_Y) > B(G'_Y)$) then

$$L_D(G_Y) \geq L_D(G'_Y).$$

Lemma seven: The disposable income $D(G'_Y)$ of any fiscal program G'_Y which is not a two-valued program and which satisfies the I-P property is Lorenze dominated by the disposable income $D(G_Y)$ of the two valued program G_Y with the same budget size (i.e. $B(G'_Y) = B(G_Y)$).

Lemma eight: If G_Y is a fiscal program which does not satisfy the I-P property, there exists a fiscal program G'_Y which satisfies the following conditions:

- i) G'_Y satisfies the I-P property
- ii) The elements in $D(G'_Y)$ are a permutation of the elements in $D(G_Y)$
- iii) $B(G'_Y) \leq B(G_Y)$

¹The proof of 3.3 centers in showing $L_X \geq L_Y$ implies the existence of a finite sequence of Dalton's transfer with rank preservation (Fields and Fei [6] or with or without rank preservation (Rothschild and Stiglitz [10])).

The condition of anonymity (3.2a) implies $I(D(G_Y^1)) = I(D(G_Y))$ in lemma eight. Condition (iii) in this lemma implies that the search for the optimum solution of 3.1 can be restricted to the set of all programs that satisfy the I-P property. Lemma seven and 3.3) imply that the research can be further restricted to the set of all two-valued programs. Finally Lemma six and 3.3) imply $I(D(G_Y)) < I(D(G_Y^1))$ and hence the two-valued program with a larger budget size is always more equitable. These lemmas, which imply theorem five, will be proven in the appendix.

As a summary, when Y is given, let the following subsets of Ω be defined:

- 3.4a) $P^2 = \{ D(G_Y) \mid G_Y \text{ is a two valued fiscal program} \}$
- b) $R = \{ D(G_Y) \mid G_Y \text{ is a rational fiscal program} \}$
- c) $L = \{ X \mid L_X \geq L_Y; X_1 + X_2 + \dots + X_n = Y_1 + Y_2 + \dots + Y_n \}$
- d) $F = \{ D(G_Y) \mid G_Y \text{ is a fiscal program} \}$

It is obvious that $P^2 \subseteq R \subseteq L \subseteq F$ form an ascending sequence of proper inclusions. Notice that the "policy space" is F which contains the disposable income patterns of all conceivable balanced-budget transfer programs. If rationality (i.e. the "minimally progressive" and the "incentive preserving" properties) is accepted as a reasonable requirement, policy choices are restricted to R. If, in addition, a "reasonable" index of inequality $I(X)$ is accepted, then every point in R is unambiguously more equitable than Y due to the following corollary:

Corollary: For any reasonable index of inequality $I(X)$ satisfying
3.2ab $I(D(G_Y)) < I(Y)$ for any rational fiscal program G_Y .

(Proof: by theorem one and 3.3)

However, now even a rational fiscal program may not be an optimum choice as theorem five shows that optimum fiscal program should be restricted further to P^2 . Since R is a proper subset of L our analysis suggests that "rationality" is a less controversial concept than the Dalton's principle of transfer.

Section IV

The model of "fiscal programs" (1.1abc) is adequate for the analysis of equity-oriented policies as long as the role of the "central authority" is restricted to income transfers--e.g. the transfers of foreign aid funds from the donors to aid receiving countries. (see introduction). Whether or not the aid giving countries are to provide "technical assistance" (a production related service) is clearly an independent issue. However, the central government of a country does have a "production budget" $P > 0$ (e.g. total expenditures on national defense, maintenance of laws and order, road construction) which is usually much larger than its welfare budget $W \geq 0$ (i.e. total transfer payments). The total budget B (i.e. total government revenue) is the sum: $B = P + W$. A "full" fiscal program may thus be defined as follows:

Definition: A full fiscal program $G_Y = (G_1, G_2, \dots, G_n)$ satisfies (1.1bc) and

$$4.1) \quad P(G_Y) = G_1 + G_2 + \dots + G_n \geq 0$$

This condition 4.1) replaces (1.1a). The number $P(G_Y)$ is now interpreted as the total production budget. Denoting the sum of the negative entries of G_Y by $-W(G_Y)$, we have:

$$4.2a) \quad B(G_Y) = P(G_Y) + W(G_Y) \quad \text{where}$$

$$b) \quad W(G_Y) = -\sum_{G_i < 0} G_i \geq 0$$

which shows that the total budget (or total government revenue) $B(G_Y)$ (1.1c) has two components: a production budget $P(G_Y)$ and a welfare budget $W(G_Y)$. The definition of the rationality of G_Y remains unchanged and lemma one, two and three are still valid.

The per capita income $\bar{Y}$ and the per capita disposable income $\bar{D}(G_Y)$ are now related as follows:

$$4.3a) \quad \bar{Y} = \bar{D}(G_Y) + P(G_Y)/n \quad (\text{by 1.1d and 4.1}) \text{ or}$$

$$b) \quad \bar{D}(G_Y) = \bar{Y} - P(G_Y)/n$$

Thus, when Y is given, the per capita disposable income $\bar{D}(G_Y)$ is completely determined by the size of production budget. Since $\bar{D}(G_Y)$ is non-negative, 4.3) implies

$$4.4) \quad 0 \leq P(G_Y) \leq n\bar{Y}$$

which shows the obvious fact that the size of the production budget cannot exceed national income ($n\bar{Y}$).

Our purpose is to generalize theorem five in the last section. As in Section II, we shall first construct the optimum solution. There are two special types of full fiscal programs. When the production budget is zero, we have the special case of (1.1). When the welfare budget is zero, we have the special case of a "taxation program", for "production" purposes only. A special type of "taxation program", to be referred to as one valued taxation program is defined as follows:

Definition: A full fiscal program $T^1 = (T_1, T_2, \dots, T_n)$ is a one valued taxation program when there exist a critical value $M^{**} > 0$

such that $T_i = \text{Max}(0, Y_i - M^{**})$ for $i=1,2,\dots,n$.

Thus $T^1 \geq 0$ and $T_i = Y_i - M^{**} > 0$ if and only if $Y_i > M^{**}$. The disposable income of all taxpayers in $D(T^1) \geq 0$ is M^{**} which is the maximum level of disposable incomes for everyone. When a production budget $\bar{P}$ ($0 < \bar{P} \leq n\bar{Y}$) is specified, the following lemma states that a one valued taxation program $T^1(\bar{P})$ is uniquely determined such that $B(T^1(\bar{P})) = \bar{P}$.

Lemma nine : For any production budget $\bar{P}$ satisfying $0 < \bar{P} \leq n\bar{Y}$, a one valued taxation program $T^1(\bar{P})$ is uniquely determined such that $B(T^1(\bar{P})) = P(T^1(\bar{P})) = \bar{P}$ (i.e. $W(T^1(\bar{P})) = 0$).

The proof is a construction one in which the determination of $M^{**}(\bar{P})$ is exactly the same as the determination of the upper critical value $M^*(\bar{B})$ in (2.3a) when the domain of definition of $R(M^*)$ in 2.1a) is extended for all $M^* \geq 0$. Since $D(T^1(\bar{P})) \geq 0$, let $G^1 = G^1(D(T^1(\bar{P})))$ be the mean deviation program defined for $D(T^1(\bar{P}))$ and let $B_{\bar{P}} = B(G^1) = B(G^1(D(T^1(\bar{P}))))$ be the maximum rational budget of $D(T^1(\bar{P}))$. When a welfare budget $\bar{W}$ is specified to satisfy the following condition:

$$4.6) \quad 0 \leq \bar{W} \leq B_{\bar{P}} = B(G^1(D(T^1(\bar{P}))))$$

theorem four implies we can determine a two value fiscal program $G_Y^2(\bar{W})$ operating on $D(T^1(\bar{P}))$ satisfying:

$$4.7a) \quad B(G_Y^2(\bar{W})) = \bar{W} \text{ and}$$

$$b) \quad \bar{W} = B_{\bar{P}} \text{ implies } G_Y^2(\bar{W}) = G^1(D(T^1(\bar{P}))).$$

Notice that the concept of a two-valued program, as formally defined in Section II, is still applicable for full fiscal programs. The following theorem is a direct generalization of theorem four:

Theorem six: A unique two valued program G_Y is determined when the production budget ($\bar{P}$) and the welfare budget ($\bar{W}$) are specified to satisfy the following conditions:

$$4.8a) \quad 0 \leq \bar{P} \leq n\bar{Y}$$

$$b) \quad 0 \leq \bar{W} \leq B_{\bar{P}} = B(G^1(D(T^1(\bar{P}))))$$

The uniqueness is obvious. Consider $G_Y = T^1(\bar{P}) + G_Y^2(\bar{W})$ as constructed from lemma nine and 4.7a). Then $D(G_Y) = Y - (T^1(\bar{P}) + G_Y^2(\bar{W})) = D(T^1(\bar{P})) - G_Y^2(\bar{W})$ which is the disposable income pattern with two critical values when $G_Y^2(\bar{W})$ operates on $D(T^1(\bar{P}))$. Thus G_Y is a two value program. We want to prove:

$$4.9a) \quad B(G_Y) = \bar{P} + \bar{W} \quad \text{and}$$

$$b) \quad W(G_Y) = \bar{W}$$

Let $G_Y^2(\bar{W}) = (G_1, G_2, \dots, G_n)$. Then $T_i > 0$ implies $G_i \geq 0$ because the i -th person receives the maximum income M^{**} in $D(T^1(\bar{P}))$ and hence cannot be a welfare recipient in $G_Y^2(\bar{W})$. Then $G_i < 0$ implies $T_i = 0$. Thus $W(G_Y) = W(G_Y(\bar{W})) = \bar{W}$ by 4.7a. Also $G_i + T_i > 0$ implies $G_i \geq 0$. Since $T_i \geq 0$, then $B(G_Y) = B(G_Y^2(\bar{W})) + B(T^1(\bar{P})) = \bar{W} + \bar{P}$ by 4.7a) and lemma nine. This proves 4.9a). QED

In order to generalize the non-linear programming problem, let us assume that the voters agree on a production budget $\bar{P}$ and a maximum welfare budget $\bar{W}$. The optimization problem is:

4.10) To minimize $I(D(G_Y))$ for all full fiscal programs that satisfy:

$$a) \quad P(G_Y) = \bar{P} \quad (\bar{P} \geq 0)$$

$$b) \quad W(G_Y) \leq \bar{W} \quad (\bar{W} \geq 0)$$

Notice that when $\bar{P} > n\bar{Y}$, condition 4.4) implies that there is no solution. When $\bar{P} \leq n\bar{Y}$ and $\bar{W} = B_{\bar{P}}$ of 4.6, condition 4.7b) implies the existence of a solution that leads to a complete equalization of disposable income in $D(G_Y)$. Thus for the non-trivial cases, we can add 4.8ab) as constraints for the parameters $\bar{P}$ and $\bar{W}$. Then a feasible solution always exists by theorem six. Let the two valued fiscal program determined in theorem six be denoted by $G_Y(\bar{P}, \bar{W})$. The following theorem is a direct generalization of theorem five:

Theorem seven: For all "reasonable" indices of inequality, the unique solution to 4.10 is $G_Y(\bar{P}, \bar{W})$.

Notice that for full fiscal programs the total disposal income $n\bar{D}(G_Y)$ is no longer a constant by 4.3). Thus the following "scale irrelevant"¹ property must be postulated:

$$4.11) \quad I(kY) = I(Y) \text{ for all } k > 0$$

in addition to 3.2ab, as a requirement of a "reasonable" index of

¹Notice that 4.11) is not required for our analysis in Section III. On the economic significance of 4.11) see Hirschman, Rothschild, [8], Atkinson [2], Rothschild and Stiglitz [10], and Dasgupta, Sen and Starrett [5].

inequality which includes, as special cases all the familiar indices mentioned in the last section. (See footnote '2' to 3.2ab). With a modification of the proof given in the appendix, theorem seven can be proved when 4.11) is added.

When the parameters $\bar{P}$ and $\bar{W}$ vary, we have the following theorem of parametric programming which will be proved in the appendix.

Theorem eight: The index value of the optimum solution of 4.10 satisfies the following conditions:

$$4.12a) \ I(D(G_Y(\bar{P}, \bar{W}))) < I(D(G_Y(\bar{P}', \bar{W}')))) \text{ when } (\bar{P}, \bar{W}) \succeq (\bar{P}', \bar{W}')$$

$$b) \ I(D(G_Y(\bar{P}, \bar{W}))) < I(D(G_Y(\bar{P}', \bar{W}')))) \text{ when } \bar{W} + \bar{P} = \bar{W}' + \bar{P}' \text{ and } \bar{W} > \bar{W}'$$

Condition 4.12a) states that a program becomes more equitable when the size of the production and/or the welfare budget strictly increases as long as the size of the other budget does not decrease. Condition 4.12b) states that when the total budget size is constant, an increase in the welfare budget (i.e. substituting the production budget) lowers the index of inequality. As a popular application suppose via a "tax revolt", the voters seek to reduce the total government expenditure $P_0 + W_0$ to a lower level $P' + W'$ with a drastic cut in the welfare budget to $W' \leq W_0$. One can readily show by 4.12:

$$4.13) \ P_0 + W_0 > P' + W' \text{ and } W' \leq W_0 \text{ imply } I(D(G_Y(P_0, W_0))) < I(D(G_Y(P', W'))))$$

and hence the fiscal reform is a "conservative one" as it always leads to higher income distribution inequality. With a reduction in total

government expenditure, the welfare budget must be increased absolutely if the reform is not to lead to inequity.

Conclusion

When the measurement of income distribution inequality is treated as a policy oriented issue, we have shown that the algebraic form of the index is unimportant. In future research, the reasonableness of the Dalton's principle of transfer, usually taken for granted, should be reexamined from the view point of work incentive. The fact that an optimum fiscal program is always two-valued, which implies that families in the middle income range should never be taxed or subsidized, hardly squares with empirical reality. A part of the tax payment by every tax payer amounts to the purchase of a product (e.g. services of firemen and garbage collectors) of public enterprises.¹ When these "purchases" are subtracted from both government revenue and expenditures the "imputed" fiscal program should not only be rational but, in fact should be a two-valued one. For if this is not the case, the conclusion is inevitable that either the Dalton's transfer principle is unreasonable from the work incentive point of view or the legislative procedure is less than perfect. These remarks suggest the possibility of empirical research in the future.

¹In this paper we have neglected the income distribution impact due to an imputation of the benefits of government "production expenditures" $P(G_Y)$ to individual families. See Aaron and McGuire [1], and Gillespie [7], for issues of this type which should be integrated with the model of this paper in future research.

Appendix

To prove lemmas six, seven, and eight in the text, we shall make use of the following lemma:

Al) Lemma: If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ satisfy the following conditions:

A1.1a) $x \neq y$

- b) $x \geq 0, y \geq 0$ (i.e. x and y are non-negative)
- c) x and y are monotonically non-decreasing
- d) $x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n$

then " x Lorenze dominates y " (in notation $L_x \geq L_y$) when
there exists an integer q ($1 \leq q < n$) such that the vector
 $f = x - y = (f_1, f_2, \dots, f_n)$ can be partitioned into two sub-
vectors $f = (f^1, f^2)$ which satisfy the following
 $l_{xq} \leq l_{yq}$ and $l_{x(n-q)} \leq l_{y(n-q)}$
conditions:

A1.2a) $f^1 \geq 0$ (i.e. f^1 is non-negative)

b) $f^2 \leq 0$ (i.e. f^2 is non-positive)

Proof: Define $e_i = \sum_{k=1}^i f_k = \sum_{k=1}^i x_k - \sum_{k=1}^i y_k$. We want to show

$$\text{Al.3a) } e_i \geq 0 \text{ for } i = 1, 2, \dots, n$$

b) $e_i > 0$ for at least one i

Notice that

$$A1.4a) \quad e_{i+1} - e_i = x_{i+1} - y_{i+1} = f_{i+1} \quad \text{or}$$

b) $e_{i+1} = e_i + f_{i+1}$ for $i = 1, 2, \dots, n-1$

Thus by Al.2a) $e_1, e_2, \dots, e_q$ monotonically non-decrease from $e_1 = f_1 \geq 0$ and hence are all non-negative. By Al.2b) the sequence $e_q, e_{q+1}, e_{q+2}, \dots, e_n$ monotonically non-increases from $e_q \geq 0$ to $e_n = 0$ (by Al.1d). Thus $e_i \geq 0$ for all i and Al.3a is proven. If $e_2 = e_3 = \dots = e_n = 0$ then Al.4b) implies $f_2 = f_3 = \dots = f_n = 0$ which implies $x_i = y_i$ for $i = 2, 3, \dots, n$. Then Al.1a) implies $x_1 \neq y_1$ which, in turn contradicts Al.1d. Thus $e_i > 0$ for some i satisfying $2 \leq i \leq n$ QED.

To prove lemma six in the text let us assume that Y is monotonically non-decreasing. Let $x = D(G_Y)$ and $y = D(G'_Y)$ then Al.1abcd) are satisfied by 1.1d), lemma one b, 1.2) and theorem four. Let (M_*, M^*) and $(M'_*, M^{*'})$ be the critical value of G_Y and G'_Y respectively. Then by theorem four:

$$M'_* < M_* \leq M^* < M^{*'} \dots (i)$$

Let $q+1$ be the poorest (i.e. the first) taxpayer under G_Y and consider the vector $f = x - y = D(G_Y) - D(G'_Y) = (f^1, f^2)$ as partitioned in lemma Al. $1 \times q \quad 1 \times (n-q)$
We want to prove Al.2ab). Claim $f^1 \geq 0$. For $i \leq q$, under G_Y the i -th person is not a taxpayer. There are two subcases. Subcase one: the i th person is unaffected under G_Y . By theorem four, the i th person is also unaffected under G'_Y . Thus $f_i = Y_i - Y_i = 0$. Subcase two: the i th person is a welfare recipient under G_Y . Then the disposable income of the i th person under G_Y is $M_* > \max(M'_*, Y_i)$ by (i), and $f_i = M_* - D'_i$ where D'_i is the disposable income of the i th person under G'_Y . By (i), the i th person is not a taxpayer under G'_Y . If he is unaffected under G'_Y , then $G'_i = Y_i$ and $f_i = M_* - Y_i > 0$. If he is a welfare recipient under G'_Y then $G'_i = M'_*$ and $f_i = M_* - M'_* > 0$. This proves Al.2a). Claim $f^2 \leq 0$. For $i > q$, $f_i = M^* - D'_i$. By (i), the i th person is not a welfare recipient under G'_Y . If he is unaffected under G'_Y then $f_i = M^* - Y_i < 0$ because the i th person is a taxpayer under G_Y .

If he is a taxpayer under G_Y' then $f_i = M^* - M_i^* < 0$ by (i). This proves A1.2b) QED.

To prove lemma seven in the text, let $x = D(G_Y)$ and $y = D(G_Y')$. When Y is monotonically non-decreasing, condition A1.1c) is satisfied by the I-P property and lemma one b. Since y is not a two-valued program and x is a two valued program condition A1.1a is satisfied. Conditions A1.1bd are also satisfied by 1.1d and 1.2. Thus lemma A1 can be applied. Let (M_*, M^*) be the critical values of G_Y . Let $D(G_Y) = (D_1, D_2, \dots, D_n)$ and let $D(G_Y') = (D_1', D_2', \dots, D_n')$. Since G_Y is rational, the M-P property implies that G_Y is monotonically non-decreasing (by lemma one b). Let there be $q > 0$ welfare recipients under G_Y and $s \geq q$ persons be welfare recipients or unaffected under G_Y then

$$D_i = M_* > Y_i \quad \text{for } i = 1, 2, \dots, q \quad \dots(i)$$

$$D_i = Y_i \quad \text{for } i = q+1, q+2, \dots, s \quad \dots(ii)$$

$$B(G_Y) = (M_* - Y_1) + (M^* - Y_2) + \dots + (M^* - Y_q) \quad \dots(iii)$$

$$= (Y_{s+1} - M^*) + (Y_{s+2} - M^*) + \dots + (Y_n - M^*) \dots(iv)$$

Two cases will be proven separately. For the first case we assume $D_1' \geq M_*$

Claim: (1) $D_i' = D_i = M_*$ for $i \leq q$. Since $D(G_Y')$ is monotonically non-decreasing, $D_i' \geq M_* > Y_i$ for $i \leq q$ (by i). Under G_Y' , $i \leq q$ implies that the i th person is a welfare recipient and receives a welfare payment $D_i' - Y_i \geq M_* - Y_i > 0$. Since $B(G_Y') = B(G_Y)$, the welfare payments to the i th person ($i \geq q$) under G_Y' must not exceed $M_* - Y_i$ by (iii). Thus

$D'_i - Y_i = M^* - Y_i$ or $D'_i = M^*$ which proves claim (1). Thus under G'_Y , the total subsidy payments to the first q person is $B(G'_Y) = B(G_Y)$,

hence:

$$D'_i \leq Y_i \text{ for all } i > q \dots (v).$$

Claim (2): There exist $t > s$ such that $D'_t > M^*$. For in case $D'_i \leq M^*$ for all $i > s$, then for every $i > s$ at least as much taxes is collected under G'_Y from the i th person as under G_Y . Thus under G'_Y at least $B(G_Y)$ will be collected as taxes from the $s+1, s+2, \dots, n$ persons. This implies $D'_i = Y_i$ for all i satisfying $q < i \leq s$ (by v) as no taxes can be collected from these persons because $G(G_Y) = B(G'_Y)$. Thus exactly $B(G_Y) = B(G'_Y)$ must be collected from the $s+1, s+2, \dots, n$ persons and hence $D'_i = M^*$ for all $i > s$. Claim (1) then implies that $G_Y = G'_Y$ and $D(G_Y) = D(G'_Y)$ which contradicts A1.1a). This proves claim (2).

Let D'_{t_0} be the first element in $(D'_{s+1}, D'_{s+2}, \dots, D'_n)$ such that $D'_{t_0} > M^*$.

Then, since $D(G'_Y)$ is monotonically non-decreasing, we have

$$D'_i > M^* = D_i \quad i = t_0, t_0 + 1, t_0 + 2, \dots, n \quad \dots (vi)$$

$$D'_i \leq M^* = D_i \quad i = s+1, s+2, \dots, t_0 - 1 \quad \dots (vii)$$

Consider the partition $f = (f^1, f^2)$ of $f = (f_1, f_2, \dots, f_n) = D(G_Y) - D(G'_Y)$ where f^1 contains the first $t_0 - 1$ elements of f .

Claim 3: $f_1^1 \geq 0$. For $i \leq q$, $f_i = M_* - M_* = 0$ by claim 1. For $q < i \leq s$, $f_i = D_i - D_i' = Y_i - D_i' \geq 0$ by (v). For $s < i \leq t_0 - 1$, $f_i = M_* - D_i' \geq 0$ by (vii). This proves claim 3.

Claim 4: $f_1^2 \leq 0$. For $s < i \leq n$, $f_i = M_* - D_i' < 0$ by (vi). This proves claim 4. Thus A1.2ab) is proved, which completes the proof of case one ($D_1' \geq M_*$). When $D_1' < M_*$, define

$$e_i = (D_1 + D_2 + \dots + D_i) - (D_1' + D_2' + \dots + D_i') \text{ for } i = 1, 2, \dots, n \quad \dots \text{(viii)}$$

$$= -(G_1 + G_2 + \dots + G_i) + (G_1' + G_2' + \dots + G_i') \text{ for } i = 1, 2, \dots, n \quad \dots \text{(ix)}$$

Since $e_1 = D_1 - D_1' = M_* - D_1' > 0$, to prove $L_{D(Y)} \geq L_{D(G_Y')}$, it is sufficient to show

$$e_i \geq 0 \text{ for } i = 1, 2, \dots, n \quad \dots \text{(x)}$$

by 1.3ab). Notice that:

$$\text{For all } i, -(G_1' + G_2' + \dots + G_i') \leq B(G_Y') \quad \dots \text{(xi)}$$

$$\text{For } i \text{ satisfying: } q \leq i \leq s, (G_1 + G_2 + \dots + G_i) = B(G_Y) = B(G_Y') \quad \dots \text{(xii)}$$

$$\text{Thus (ix), (xi) and (xii) imply } e_i \geq 0 \text{ for all } i \text{ satisfying } q \leq i \leq s. \quad \dots \text{(xiii)}$$

Claim 5: $e_i \geq 0$ for $i \leq q$. If $D_i' \leq M_*$ for all $i \leq q$, then (viii) implies for all $i \leq q$, $e_i = (M_* - D_1') + (M_* - D_2') + \dots + (M_* - D_i') \geq 0$. Thus we may assume there exists an r satisfying $1 < r < q$ such that

$$D_i' \leq M_* \text{ for } i \leq r \quad \dots \text{(xiv)}$$

$$D_i' > M_* \text{ for } i \text{ satisfying } r < i \leq q \quad \dots \text{(xv)}$$

because $D(G_Y')$ is monotonically non-decreasing and $D_1' < M_*$. Then

$e_1 > 0$, $e_2, e_3, \dots, e_r$ form a monotonically non-decreasing sequence by

(xiv) and hence $e_i \geq 0$ for $i \leq r$. Moreover, $e_{r+1}, e_{r+2}, \dots, e_q$ form a monotonically non-increasing sequence by (xv) which decreases to $e_q \geq 0$ by (xiii), thus $e_i \geq 0$ for i satisfies $r < i \leq q$. Thus claim 5 is proved. Claim 6 $e_i \geq 0$ for all i satisfying $s < i$, we have, by (viii),

$$e_i = e_s + (M^* - D'_{s+1}) + (M^* - D'_{s+2}) + \dots + (M^* - D'_i) \dots (xvi)$$
where $e_s \geq 0$ by (xiii). If $D'_i \leq M^*$ for all $i > s$, then (xvi) implies claim 6. Otherwise, since $D(G'_Y)$ is monotonically non-decreasing, there exists an r satisfying $s < r \leq n$ such that

$$D'_i \leq M^* \text{ for } i = s+1, s+2, \dots, r \dots (xvii)$$

$$D'_i > M^* \text{ for } i = r+1, r+2, \dots, n \dots (xviii)$$
Conditions (xvi) and (xvii) imply $e_s, e_{s+1}, e_{s+2}, \dots, e_r$ form a monotonically non-decreasing sequence from $e_s \geq 0$. Thus $e_i \geq 0$ for $i \leq r$. Conditions (xvi) and (xviii) imply $e_{r+1}, e_{r+2}, \dots, e_n$ form a monotonically non-increasing sequence which decreases to $e_n = 0$. Thus $e_i \geq 0$ for $i > s$ and claim 6 is proved. Thus $e_i \geq 0$ for all i and (x) is proved. QED.

To prove lemma eight in the text, given $G_Y = (G_1, G_2, \dots, G_n)$ and $D(G_Y) = (D_1, D_2, \dots, D_n)$ which is not monotonically non-decreasing let $D(G'_Y) = (D_{i_1}, D_{i_2}, \dots, D_{i_n})$ be defined such that $i_1, i_2, \dots, i_n$ is a permutation of $1, 2, \dots, n$ rendering $D(G'_Y)$ monotonically non-decreasing. Now construct $G'_Y = Y - D(G'_Y) = (G'_1, G'_2, \dots, G'_n)$ and conditions (i) and (ii) of lemma eight are satisfied. It remains to show $B(G'_Y) \leq B(G_Y)$. The reordering of $D(G_Y)$ into $D(G'_Y)$ can be accomplished in a finite number of steps of interchanging two adjacent $D_i > D_{i+1}$. We may assume Y is monotonically non-decreasing then, the following conditions are satisfied:

$$\theta = Y_{i+1} - Y_i \geq 0 \quad \dots(i)$$

$$\emptyset = D_i - D_{i+1} > 0 \quad \dots(ii)$$

$$G_i = Y_i - D_i \quad \dots(iii)$$

$$G_{i+1} = Y_{i+1} - D_{i+1} \quad \dots(iv)$$

$$G'_i = Y_i - D_{i+1} \quad \dots(v)$$

$$G'_{i+1} = Y_{i+1} - D_i \quad \dots(vi)$$

Then we have:

$$G'_i = G_i + \emptyset = G_{i+1} - \theta \quad \dots(vii) \text{ implying } G'_i \leq G_{i+1} \quad \dots(viii) \text{ and}$$

$$G'_i > G_i \quad \dots(ix)$$

$$G'_{i+1} = G_i + \theta = G_{i+1} - \emptyset \quad \dots(x) \text{ implying } G'_{i+1} \geq G_i \quad \dots(xi) \text{ and}$$

$$G'_{i+1} < G_{i+1} \quad \dots(xii)$$

$$G'_{i+1} + G'_i = G_{i+1} + G_i \quad \dots(xiii)$$

Let $v(u)$ be the total amount of taxes collected from the i th and the $(i+1)$ -th person after (before) the interchange of D_i and D_{i+1} . It is sufficient to prove:

$$u \geq v \quad \dots(xiv) \text{ where}$$

$$u = \text{Max}(0, G_i) + \text{Max}(0, G_{i+1}) \geq 0 \quad \dots(xv) \text{ and}$$

$$v = \text{Max}(0, G'_i) + \text{Max}(0, G'_{i+1}) \geq 0 \quad \dots(xvi)$$

Condition (xiv) can be proved for the following four cases separately:

case one: $G'_i \leq 0$, case two: $G_{i+1} \leq 0$, case three: $G_{i+1} > 0$, $G'_i > 0$, $G_i \geq 0$.

case four: $G_{i+1} > 0$, $G'_i > 0$, $G_i < 0$: subcase one $G'_{i+1} < 0$,

subcase two $G'_{i+1} \geq 0$.

In case one, (ix) implies $G_i < 0$. $u = \text{Max}(0, G_{i+1})$ and $v = \text{Max}(0, G'_{i+1})$

then $u \geq v$ by (xii). In case two (viii) implies $G'_i \leq 0$

which is case one. In case three (xi) implies $G'_{i+1} \geq 0$. Then

$u = G_i + G_{i+1}$ and $v = G'_i + G'_{i+1}$ and $u = v$ by (xiii). In subcase one of case four $u = G_{i+1}$ and $v = G'_i$ and $v \leq u$ by (viii). In subcase two of case four $v = G'_i + G'_{i+1} = G_i + G_{i+1}$ by (xiii) and $u = G_{i+1}$. Thus $v = G_i + u$ and $v < u$ because $G_i < 0$. QED.

To prove 4.12b) of theorem eight in the text let $M_* \leq M^*$ be the critical values of $G_Y \equiv G_Y(\bar{P}, \bar{W})$ and let $M'_* \leq M'^*$ be the critical values of $G'_Y \equiv G_Y(\bar{P}', \bar{W}')$. The conditions $B(G_Y) = B(G'_Y)$ and $\bar{W} > \bar{W}'$ imply:

$$M^* = M'^* \quad \dots (i)$$

$$M_* > M'_* \quad \dots (ii)$$

Denote $D(G_Y) = (D_1, D_2, \dots, D_n) \geq 0$ and $D(G'_Y) = (D'_1, D'_2, \dots, D'_n) \geq 0$ which are monotonically non-decreasing when Y is assumed to be monotonically non-decreasing. Condition (i) and (ii) implies there exist q' and q satisfying $1 \leq q' \leq q < n$ such that:

$$D_i = M_* \quad i=1, 2, \dots, q \quad \dots (iii)$$

$$D'_i = M'_* \quad i=1, 2, \dots, q' \quad \dots (iv)$$

$$D'_i = M'_* \quad i=q'+1, q'+2, \dots, q \quad \dots (v)$$

$$D'_i = D_i \quad i > q. \quad \dots (vi)$$

Denote the mean value of $D(G_Y)$ and $D(G'_Y)$ by $\bar{D}(G_Y)$ and $\bar{D}(G'_Y)$ respectively and let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be the normalization of $D(G_Y)$ and $D(G'_Y)$ respectively i.e.

$$x_i = D_i / n\bar{D}(G_Y) \quad i=1, 2, \dots, n \quad \dots (vii)$$

$$y_i = D'_i / n\bar{D}(G'_Y) \quad i=1, 2, \dots, n \quad \dots (viii)$$

$$x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n = 1 \quad \dots (ix)$$

Then x and y satisfy A1.1abcd). By the scale irrelevant property (4.11)

$I(x) = I(D(G_Y))$ and $I(y) = I(D(G'_Y))$. It is sufficient to prove

$$L_x \geq L_y \quad \dots (x)$$

Since the production budget of G'_Y is larger than that of G_Y , 4.3b) implies

$$\bar{D}(G_Y) > \bar{D}(G'_Y) \dots (xi)$$

Conditions (xi) and (vi) imply:

$$x_i < y_i \text{ for } i > q \quad \dots (xii)$$

Claim $y_1 < x_1$. For otherwise, since y_i is monotonically non-decreasing

$y_i \geq x_i$ for $i = 1, 2, \dots, q$ because $x_1 = x_2 = \dots = x_q$ by (iii). Then (xii)

implies $y \geq x$ which contradicts (ix). Since y is monotonically non-decreasing

$y_1 < x_1$ and (xii) imply there exists an r satisfying $1 \leq r \leq q$ such that

$$y_i \leq x_i \text{ for } i=1, 2, \dots, r \quad \dots (xiii)$$

$$y_i > x_i \text{ for } i=r+1, r+2, \dots, q \quad \dots (xiv).$$

Consider the partition $f = (f^1, f^2)$ of $f = x - y = (f_1, f_2, \dots, f_n)$ where f^1 contains the first r elements of f . Then (xiii) implies $f^1 \geq 0$ and (xiv) and (xii) imply $f^2 \leq 0$. This proves A1.2ab and hence (x) is proved. QED.

To prove 4.12a) of theorem eight in the text the fact that $B(G_Y) = \bar{P} + \bar{W} >$

$B(G'_Y) = \bar{P}' + \bar{W}'$ and $\bar{W} \geq \bar{W}'$ imply:

$$M^* < M'^* \quad \dots (xv)$$

$$M_* \geq M'_* \quad \dots (xvi)$$

$$M'_* \leq M_* \leq M^* < M'^* \quad \dots (xvii)$$

There exist integers q', q, s and s' satisfying $1 \leq q' \leq q \leq s \leq s' < n$ such that:

$$\text{For } i \leq q': D'_i = M'_* \leq M_* = D_i \quad \dots(\text{xviii})$$

$$\text{For } q' < i \leq q: D'_i = Y_i \leq M_* = D_i \quad \dots(\text{ixx})$$

$$\text{For } q < i \leq s: D'_i = D_i = Y_i \quad \dots(\text{xx})$$

$$\text{For } s < i \leq s': D_i = M_* \leq Y_i = D'_i \leq M_*' \quad \dots(\text{xxi})$$

$$\text{For } s' < i \leq n: D_i = M_* < M_*' = D'_i \quad \dots(\text{xxii})$$

Define x and y as in (vii) and (viii) which satisfy A1.labcd). We wish to prove (x). Since $\bar{P} \geq \bar{P}'$ we have:

$$\bar{D}(G_Y) \leq \bar{D}(G'_Y) \quad \dots(\text{xxiii})$$

which implies

$$y_i \leq x_i \quad \text{for } i=1,2,\dots,s \quad \dots(\text{xxiv})$$

by (xviii), (ixx), and (xx). Claim $y_n > x_n$. Otherwise by (xxi) and (xxii)

$$y_n \leq x_{s+1} = x_{s+2} = \dots = x_n \quad \text{and since } y \text{ is monotonically non-decreasing}$$

$$y_i \leq x_i \quad \text{for } i = s+1, s+2, \dots, n \quad \dots(\text{xxv}).$$

Conditions (xxiv) and (xxv) contradicts (ix). Since y is monotonically non-decreasing (xxiv) and $y_n > x_n$ imply there exists an r satisfying $s < r < n$ such that

$$y_i \leq x_i \quad \text{for } i = s+1, s+2, \dots, r \quad \dots(\text{xxvi})$$

$$y_i > x_i \quad \text{for } i = r+1, r+2, \dots, n \quad \dots(\text{xxvii})$$

Let $f = (f^1, f^2)$ be a partition for $f = x - y = (f_1, f_2, \dots, f_n)$ where f^1 contains the first r elements of f . Conditions (xxiv) and (xxvi) imply

$f^1 \geq 0$ and condition (xxvii) implies $f^2 \leq 0$. This proves A1.2ab) and hence (x) is proved. QED

REFERENCES

1. Aaron, H. and M.C. McGuire, "Public Goods and Income Distribution", Econometrica, Vol. 38, No. 6, (November 1970).
2. Atkinson, A.B., "On the Measurement of Inequality," Journal of Economic Theory, 5 (1970), 244-263.
3. Champnowne, D.G., "A Comparison of Measures of Inequality of Income Distribution," The Economic Journal, 84 (1974), 787-816.
4. Dalton, H., "The Measurement of the Inequality of Incomes," Economic Journal, 30 (1920), 348-361.
5. Dasgupta, Partha, Amartya Sen, and David Starrett, "Notes on the Measurement of Inequality," Journal of Economic Theory, 8 (1973), 180-187.
6. Fields, Gary and J. Fei, "On Inequality Comparisons", Econometrica, (March 1978).
7. Gillespie, W.I., "Effect of Public Expenditures on the Distribution of Income", in R.A. Musgrave, ed., Essays in Fiscal Federalism, Washington, D.C., Brookings Institution, (1965)
8. Hirschman, Albert, "The Changing Tolerance for Income Inequality in the Course of Economic Development," with a mathematical appendix by Michael Rothschild, Quarterly Journal of Economics, 87 (1973) 544-566.
9. Kondor, Yaakov, "Value Judgements Implied by the Use of Various Measures of Income Inequality," Review of Income and Wealth, 21 (1975) 309-321.
10. Rothschild, Michael, and Joseph E. Stiglitz, "Some Further Results on the Measurement of Inequality," Journal of Economic Theory, 8 (1973), 188-204.
11. Sen, Amartya K., On Economic Inequality, New York: Norton, (1973).
12. _____, "Poverty: An Ordinal Approach to Measurement," Econometrica, 44 (1976), 219-231.
13. Szal, Richard, and Sherman Robinson, "Measuring Income Inequality," Princeton University, Brookings Institution, mimeo, (1975).
14. Theil, Henri, Economics and Information Theory, Chicago, Rand McNally, (1967).