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PORTFOLIO DIVERSIFICATION ACROSS CURRENCIES

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Portfolio Diversification Across Currencies

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Introduction and Summary

This paper is concerned with the optimal portfolio rule for investors with an international horizon, in the sense of economic agents who consume bundles of goods from several countries and whose wealth is generated by the real return on assets denominated in several currencies. The development of an international financial market in the last twenty years, and particularly in the recent period of great volatility in major bilateral exchange rates, has naturally raised the question of the relative attractiveness of assets denominated in different currencies, when real returns are highly uncertain due to divergent rates of inflation and unanticipated changes in the relative prices of the different currencies.

A major problem is finding an index of value appropriate for the international investor. This is done in Section I where the concept of the purchasing power of a currency, introduced in a two-good two-country framework by Kouri (1975) and generalized to N goods and N countries in Kouri-Macedo (1978),^{1/} is explicated and explicitly derived from utility maximization in the case where there are N countries and M_1 goods in each country. Some further applications of the concept are stressed, namely its relationship with the increasingly popular indices of real effective exchange rates. Then monthly indices are computed and interpreted for a world consisting of eight major countries, during the floating rate period, using various weighting schemes in the line of the exercises of Rhomberg (1976) with nominal effective exchange rates.

Armed with an index of value to find the relevant real return for

^{1/}McKinnon (1979, p. 122) discusses these indices and also presents annual values computed by Kawai.

the international investor, we proceed in Section II to derive an optimal portfolio rule in that environment. The model is an extension of the theory of intertemporally efficient consumption and savings decisions, the continuous time formulation of which is due to Merton (1971), who derived efficient consumption and portfolio decisions for an investor holding N short bonds denominated in the same currency and consuming one good, when prices were generated by a stochastic process of the Ito type.^{2/} The Merton framework was particularly helpful because it included the Brownian motion special case, where the results of the classic mean-variance framework of Markowitz (1958) and Tobin (1965) are upheld without the awkward assumptions about the normality of returns and the quadratic utility function. It was then extended to a 'weak' international investor whose safe asset has to be domestic by Solnik (1973) and used as the best analytical framework for analyzing the portfolio problem of two investors with different preferences and a two country international horizon by Kouri (1975).^{3/} In Kouri-Macedo (1978) the construction of an index of value for the investor with a N country N good international horizon allows the derivation of an optimal portfolio rule for the Brownian motion case, and its computation for the Brownian investor in a five country world during the floating rate period.

^{2/} The discrete time formulation of the problem is in Samuelson (1969) and Merton (1969) has a continuous time formulation for the Brownian motion case.

^{3/} Chapter 5. In Chapter 4, reproduced in Kouri (1977), the portfolio problem of investors holding bonds and equity from N different countries in a world of purchasing power parity and known inflation in the N th country is analyzed in the Brownian motion case.

Other strands of the finance literature are not discussed here. Thus Grauer-Litchzenberger-Stehle (1977) derive an optimal portfolio for the international investor using a two period state preference approach and Long (1977) and Ross-Ingellson-Cox (1977) analyze the term structure of interest rates in a variant of the Merton (1973) framework with production.

In Section II of this paper various derivations of the optimal portfolio rule for the Brownian motion case are presented, emphasizing in turn the changes in the purchasing power of the currencies, the changes in prices and exchange rates and the changes in the real exchange rates. The special cases of no unanticipated inflation and purchasing power parity are also discussed in this framework. The results are retrieved from an explicit intertemporal consumption-savings model with N assets and N goods in Appendix 1 and from a Brownian motion model with $\sum_{i=1}^N M_i$ goods in Appendix 2.

We then use the actual values of the monthly indices constructed in Section 1 to compute optimal portfolios for the Bernouilli investor in an eight country world during the floating rate period, comparing the optimal portfolio proportions derived from different weighting schemes.

Measurement difficulties dictated the emphasis on the time-invariant portfolio rule for the investor with a unitary risk aversion. Even in that case, explicit tests of the theory are not presented. Data on actual portfolio proportions of international investors is not available, and even for the case of central banks, it is highly aggregated.^{4/} This suggests the need for a way to aggregate the portfolio rules of different investors, in order to make use of the available data. A simple method is suggested in Appendix 3.

The time invariance assumption, implying that optimal portfolio are only restructured when there are changes in real returns, does not, however, seem to be contrary to the behavior of central banks, where the currency

^{4/} Annual data on the currency composition of the reserves of a sample of 53 central banks (and a moving sample of 76) between 1970 and 1976 was made available in Heller-Knight (1978), distinguishing U.S. dollars, pounds, D. marks and French francs.

composition has remained fairly stable from 1970 to 1976, despite the decline in the pound and the increase in the Deutsche-mark and "other currencies" (Swiss francs, Japanese yen).

Given our weights, in any event, the optimal proportion of the dollar in the total portfolio is far smaller than the actual share for central banks, which is about 80%. The conveniences of the "vehicle currency" are, of course, beyond a no-transactions-costs continuous time model. Nevertheless the total portfolios include between 50% and 60% in dollars, which is significantly higher than the expenditure share (between 25% and 43%) and the negative real return (6% differential with Germany and 10% with Switzerland) would have led to believe.

I. Indices of purchasing power

1. A long debate has surrounded the possibility of constructing "ideal" indices of value. The only case where a single number can satisfy the spirit of Fisher's tests is however, the homothetic case. Only if income elasticities are one will the price index be invariant under change in the level of living.^{5/} The assumption of constant expenditure shares is, of course, an important drawback in the application of such exact indices. However, we will take it as adequate for fairly short periods of time. In this section we will derive and compute indices of the purchasing power of a currency in a world of

^{5/} A recent review of the economic theory of index numbers is found in Samuelson-Swamy (1974) and Samuelson (1974) has a modern discussion of the international comparison of real incomes. Diewert (1977) shows that almost exact, or superlative, indices can be constructed for homogeneous quadratic mean of order r aggregator functions. The quadratic utility function ($r=2$) for which Fisher's "ideal" index is exact is in that class.

N-1 floating exchange rates, extending the analysis in Kouri (1975) of the purchasing power of a currency in a single exchange rate world. After the explicit general derivation of the index of purchasing power of a currency over goods produced in N countries proposed in Kouri-Macedo (1978), some further implications are stressed and new indices and weighting schemes presented and interpreted.

Consider the consumption problem in an environment where there are N countries and where M_i goods are produced in country i. There are then $\sum_i M_i$ goods and the utility function of a typical consumer in country k is characterized by constant expenditure shares α_i and β_j^i such that

$$\sum_i \alpha_i = 1 \quad \text{and} \quad \sum_j \beta_j^i = 1$$

and can be written as

$$(1) \quad U^k = (1/\gamma_k) \prod_{i=1}^N \prod_{j=1}^{M_i} \alpha_i \beta_j^i \gamma_k X_{ij}$$

where X_{ij} is the quantity of good ij or good j, $j = 1, \dots, M_i$ produced in country i, $i = 1, \dots, N$

and γ_k depends on the risk aversion of the consumer.

Nominal expenditure in domestic currency E_k is defined as

$$(2) \quad E_k = \sum_i \sum_j P_{ij} (S_k/S_i) X_{ij}$$

where P_{ij} is the price of good ij and S_i is the price of currency i in terms of currency N

The $\sum_i M_i$ first order conditions from the problem of maximizing (1) subject to (2) are

$$(3) \quad \frac{\partial U^k}{\partial X_{ij}} = \alpha_i \beta_j^i U^k X_{ij}^{-1} - \lambda (S_k/S_i) P_{ij} = 0$$

Summing over all i and j we get the marginal utility of income of consumer k

$$(4) \quad \lambda = U^k / E_k$$

The demand functions are of the usual form, namely

$$(5) \quad X_{ij}^* = \alpha_i \beta_j^1 E_k P_{ij}^{-1}$$

and the indirect utility function, separable between expenditure and prices, is

$$(6) \quad V^k = E_k \prod_{i=1}^N \prod_{j=1}^{M_i} \left| \frac{\alpha_i \beta_j^1}{P_{ij} (S_k / S_i)} \right| \alpha_i \beta_j^1$$

The exact index of the utility of an extra unit of currency k for expenditure by individual k , or the index of the purchasing power of currency k over $\sum_{i=1}^N M_i$ goods located in N countries, denoted by Q_k , is given by^{6/}

$$(7) \quad Q_k = \prod_{i=1}^N \prod_{j=1}^{M_i} [P_{ij} (S_k / S_i)]^{-\alpha_i \beta_j^1}$$

In fact, substituting for Q_k in (6) and differentiating with respect to expenditure, we get

$$(8) \quad \frac{\partial \hat{V}^k}{\partial E_k} = Q_k$$

where $\hat{V}^k = V^k \prod_{i=1}^N \prod_{j=1}^{M_i} \alpha_i \beta_j^{-\alpha_i \beta_j^1}$ is the same index of ordinal utility as V^k

Furthermore, since the exchange rates do not depend on j , we can decompose Q_k as

$$(9) \quad Q_k = \prod_{i=1}^N (S_k / S_i)^{-\alpha_i} \prod_{i=1}^N \prod_{j=1}^{M_i} P_{ij}^{-\alpha_i \beta_j^1}$$

^{6/} The purchasing power of one unit of currency over a particular good with price P is just $1/P$.

Note that an effective exchange rate index for currency k using weights α_i can be defined as

$$(10) \quad S_k^e = \prod_{i=1}^N (S_k/S_i)^{-\alpha_i}$$

Also, because there are only $N-1$ exchange rates,

$$(11) \quad \prod_k S_k^e = \prod_k \left(\prod_{i=1}^N (S_k/S_i)^{-\alpha_i} \right)^{\alpha_k} = 1$$

Note further that the world price level P can be defined as

$$(12) \quad P = \prod_{i=1}^N \prod_{j=1}^M P_{ij}^{\alpha_i \beta_j^1}$$

and that the purchasing power of one unit of "world" money is just $1/P$. The purchasing power of world money is equal to the average of the purchasing powers of the N currencies using the same weights. In fact, just like the bilateral exchange rate of currency N is one, world money has an effective exchange rate of one and therefore, using (11) and (12) we can write

$$(13) \quad Q = 1/P = \prod_k Q_k^{\alpha_k}$$

We can thus define the purchasing power of currency k , given weights α_i and β_j as the product of the purchasing power of world money and the effective exchange rate of currency k

$$(14) \quad Q_k = Q(\alpha_i, \beta_j^1) S_k^e(\alpha_i)$$

2. It is often convenient to assume that the β_j^1 are given by the weights of some available national price index, and then compare the purchasing powers of various national currencies for alternative weighting schemes. Thus not only the export price indices of various countries or the wholesale price indices of these countries can be used as proxies for the purchasing power of

their currencies. Price indices with a share of non-traded goods like the consumer price index or the value added deflator can also yield significant insights in a world where international financial intermediation opens a given economy to international investors who attach a weight to non-traded goods produced in various countries. By the same token wages can be used to compare the purchasing power of labor of country i over goods produced in N countries.

Also perhaps most appropriately in a world of greater international financial intermediation than international labor mobility, any asset i the domestic currency price of which is K_i can be expressed in purchasing power units as $K_i Q_i$. Uses for the purchasing power of bonds, of equity, of gold and raw materials come readily to mind.

Similarly, it may be of interest to define the purchasing power of a currency, the weight of which for the international investor is negligible. This can be done by taking that weight to be zero and defining its purchasing power as the purchasing power of the N th currency divided by the price of the domestic currency in terms of the N th currency.^{7/}

Now a real effective exchange rate for currency k , using weights α_i and taking domestic outputs to be composite goods, would be written as^{8/}

$$(15) \quad \tilde{S}_k^e = \prod_i^N (P_i S_{ik} / S_i P_k)^{\alpha_i}$$

^{7/} Thus $Q_k = Q_N / S_k$ where $\alpha_k = 0$. All of the analysis in section 2 applies in the case where there are more exchange rates than prices because of the zero weight of the respective country in the world price level.

^{8/} Indices of real effective exchange rates have become popular in policy discussion. See Dornbusch's statement to the U.S. House of Representatives (1978), World Financial Markets, May 1978 and April 1979 and International Financial Statistics from January of 1979.

and it is easily seen that

$$(16) \quad \tilde{S}_k^e = P_k Q_k$$

The real effective rate is thus defined as the price of domestic output in terms of the purchasing power of the domestic currency, or the real world price of domestic output.^{9/}

The real effective exchange rate of currency k can thus be seen as the ratio of the purchasing power of currency k for a consumer-investor with an international consumption basket and the purchasing power of currency k for a consumer-investor with a national currency basket defined by $\alpha_k = 1$. The comparison is of course not rigorous because preferences are different between the two consumer-investors but it does give an idea of the attractiveness of currency k relative to the international and the domestic consumer-investor.

An intermediate case is thus one of the investor who is located in country k but only consumes goods of country i so that $\alpha_i = 1$. The purchasing power of currency k for that investor, denoted by Q_k^i , is simply the purchasing power of the currency for the domestic consumer z country i , $1/P_i$, times the bilateral exchange rate, or

$$(17) \quad Q_k^i = S_i / S_k P_i$$

^{9/} If there is only one nominal exchange S , then $\tilde{S}$ is defined as

$$\tilde{S} = P / S P^*$$

where P is the price of the domestic good

P^* is the price of the foreign good

Therefore when the real exchange rate increases, there is a real appreciation of the domestic currency and the terms of trade, $1/\tilde{S}$, deteriorate. Sometimes the real exchange rate is defined as the real domestic price of foreign currency (the terms of trade) and then an increase means a real depreciation.

The ratio of the purchasing powers of a pair of currencies is always the bilateral exchange rate, independently of the weighting scheme, because the price term cancels. This is evident from (14) for the international investor. For the "foreign" investor, we have

$$(18) \quad \frac{Q_k^i}{Q_m^i} = \frac{(P_i S_k / S_i)^{-1}}{(P_i S_m / S_i)^{-1}} = \frac{S_m}{S_k} = S_{km}$$

The ratio of the purchasing powers of the same currency for investors located in two different countries (including the domestic) is the real bilateral exchange rate

$$(19) \quad \frac{Q_i^m}{Q_i^k} = \frac{P_k S_m}{P_m S_k} = S_{km}$$

3. We now compute indices of purchasing power of major currencies for the floating rate period, April 1973 to April 1978. The international investor is assumed to be confined to eight of the major industrial countries, Canada, France, Germany, Italy, Japan, Switzerland, the United Kingdom, and the United States. Furthermore, the results reported below use the national consumer price indices to construct the index of the world price level, with base = 1 in April 1973. End of month exchange rates against the dollar are also converted into indices with the same base for the construction of the effective exchange rate index. The choice of an end of period rather than a period average exchange rate is dictated by the fact that the investor is assumed to be concerned with the actual contract in the foreign exchange market.

The problem of choosing an appropriate set of weights has been discussed with reference to nominal effective exchange rates by Rhomberg. He distinguished between bilateral and global weights, suggesting that the choice

depends "on the particular policy objective"^{10/} The concept of an international investor clearly requires that a rigorous comparison between purchasing powers be possible. It thus rules out bilateral weighting schemes^{11/}

The seven global weighting schemes presented in Table 1 fall into three categories: trade weights (columns 1 to 4), income weights (column 5) and MERM weights (columns 6 and 7).

The trade weights in column 1 are adapted from the shares of the countries indicated^{12/} in total exports of manufactures in 1975.^{13/} In columns 2 and 3 average dollar exports and imports respectively of the eight countries during the period 1973-1977 are used. In column 4 exports plus imports are used.

Income weights are derived from gross domestic product from 1973 to 1977 in 1975 prices converted into dollars at the average exchange rate for the year and averaged for the four year period.

The MERM weights in columns 6 and 7 are derived from the total balance effects in dollars, at the scale of world trade in 1977, of a ten per cent exchange rate depreciation in the domestic currency, as reported in Artus-McGuirk (1978).

^{10/}Rhomberg (1976), p. 89. See also World Financial Markets, May 1979.

^{11/}Consider a N by N matrix of export flows from country i to country j x_{ij} , denoted as X. Construct an export shares matrix A by postmultiplying X by the inverse of a diagonal matrix of total exports from country i $Z_{ii} = \sum_j x_{ij}$ or $A = Z^{-1}X$. Then whereas we can operate on the rows of A and $Ae = e$ (where e is a vector of ones) the elements in each column cannot be compared except through another vector of weights y such that $y'e = 1$ since $y'Ae = 1$. See, however, Appendix 3.

^{12/}The countries in the sample are abbreviated by the first two initials with the exception of Switzerland, which is denoted by SZ.

^{13/}See Deppler (1978).

Table 1

Alternative Weighting Schemes							
	1	2	3	4	5	6	7
CA	0.077	0.082	0.078	0.080	0.050	0.058	0.053
FR	0.124	0.120	0.124	0.122	0.096	0.174	0.159
GE	0.271	0.216	0.174	0.194	0.133	0.153	0.167
IT	0.092	0.078	0.087	0.082	0.050	0.078	0.079
JA	0.099	0.138	0.130	0.134	0.159	0.085	0.101
SZ	0.042	0.031	0.032	0.032	0.017	0.019	0.023
UK	0.097	0.102	0.119	0.111	0.067	0.060	0.072
US	0.198	0.234	0.256	0.245	0.427	0.373	0.347

Note:

- 1 IMF World Trade Model exports
- 2 Exports
- 3 Imports
- 4 Total trade
- 5 Income
- 6 IMF Multilateral Exchange Rate Model
- 7 MERN high elasticity

These total effects, are used to derive bilateral shares for the purpose of constructing effective exchange rates only after adjustment for the initial trade imbalance is made. Here they will simply give a ranking of the eight countries according to "sensitivity" in some sense.^{14/}

The difference between column 6, the "control solution," and column 7 is that price elasticities of finished and semi-finished manufactures are increased by between 25% and 50% depending on the countries.^{15/}

Three of these seven weighting schemes will be used below.^{16/} Whereas "normal" MERM weights in column 6 give a measure of marginal openness, or sensitivity, for a given size of the external sector, total trade weights in column 4 give a measure of the average openness of the economy. Aside from the U.S., only in the case of France is the former measure larger than the latter.

Income weights are, at best, a short hand for relative size, since they include goods that may not be consumed by the international investor. They are however, relatively easier to compare with the national price indices used.

^{14/}The sensitivity/openness issue was first addressed in Cooper (1968) and has become a popular topic in international political economy. See Keohane-Nye (1977) and Macedo (1978) for example.

^{15/}For example the "control" elasticities for finished manufactures are taken to be -1.00 for the U.K., -1.25 for CA, GE, IT and SZ and -2.00 for FR, JA and the U.S. The higher elasticities are obtained by adding -.5. See Artus-McGuirk (1978), tables 2, 4 and 5. Another simulation involves higher feedback from wages into prices, and from prices into return to capital and the tax rate. The associated weights, however, do not differ from the control solution at the third decimal point and are thus not reported.

^{16/}Another interesting weighting scheme attempts to capture directly the relative importance of currencies in international financial markets. When advocating a few years ago "the commercial use of SDR's" World Financial Markets proposed a simplified SDR basket consisting of U.S. dollar, D. marks, Japanese yen, pound sterling and French francs. It was used in the computation of the "optimal portfolio" in Kouri-Macedo (1978). See section II below.

Only the U.S. and Japan rank highest on income weights, whereas in the case of the U.K., the income share is larger than the MERM share. For Canada, Germany, Italy and Switzerland, the ranking is total trade, MERM and income weights, as one would expect of smaller open economies.

4. We can use as illustration indices of the purchasing power of these currencies, where weights are attached to the national consumer price indices. The first point is that there is hardly any difference in the aggregate from the use of the three types of weighting schemes. In fact, the values of the indices of world money computed from total trade weights (column 4 in Table 1), income weights (column 5) and MERM weights (column 6) were in January 1970, 120.1; 119.0 and 118.9 respectively and in April 1978, 62.8; 63.9 and 63.8 respectively for a base value = 100 in April 1973. Obviously the purchasing powers of the individual currencies show a wider variation depending on the weights, but it is not dramatic.^{17/}

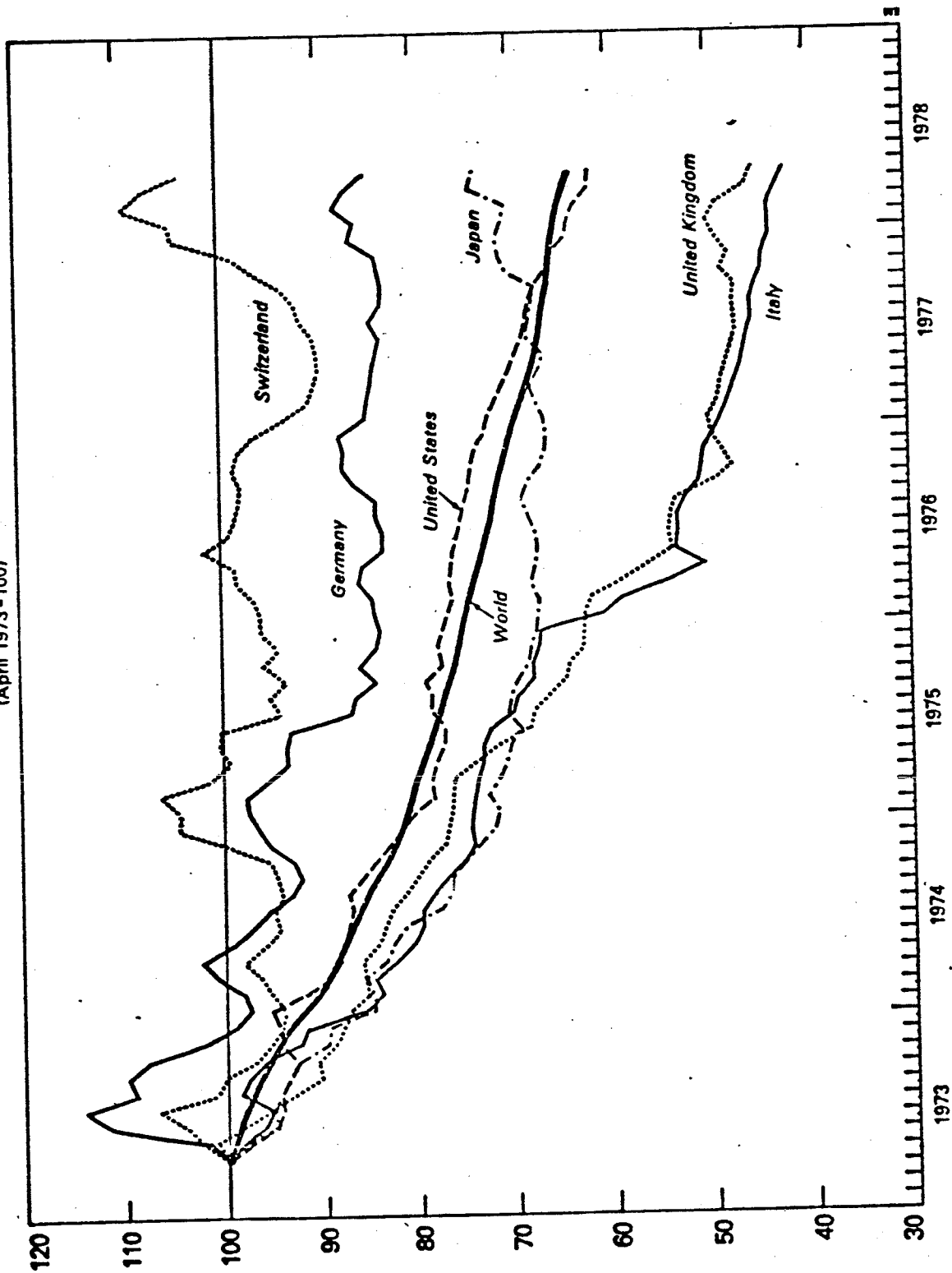
In Figure 1 we show the evolution of the purchasing power of six major currencies and world money, from the base period to April 1978. During these five years the only currency to have kept its purchasing power was the Swiss franc, for which it had increased by 3.7%. The largest decline in purchasing power was the one of the Italian lira (57.8%), closely followed by the British pound (54.5%). The Deutsche mark had lost 15.1% of its purchasing power.^{18/}

The downward trend in world money is particularly noteworthy. The variations of the purchasing powers of the various currencies thus follow

^{17/} This can be seen from Table 2 in the next section.

^{18/} The remaining values were CA = 54.9; FR = 61.3; JA = 73.7 and U.S. = 61.9.

FIGURE 1
PURCHASING POWERS OF SELECTED CURRENCIES APR. 1973-APR. 1978
(April 1973=100)



closely the ones of their nominal effective exchange rates, with built-in depreciation of about 7.25% p.a. over the eight years from 1970.

Thus the drastic increase in the German discount rate soon after the beginning of generalized floated, which increased the effective D. mark rate substantially can be seen in Figure 1, where the purchasing power index for that currency was 114 in July. Similarly the lifting of restrictions on the access of non residents to the German capital markets in the fall of 1974 appreciated the effective mark and led to an increase in the purchasing power from 92 in September to 98 in February of 1975. The decline in domestic inflation since the first semester of 1976, on the other hand, propelled a continued appreciation of the effective mark which in March 1978 was 35% above its value in five years earlier. This was just enough to keep the purchasing power index in the 84-87 range, whilst world money had declined from 74 to 64 since April 1976.

The slowing down of the decline in the purchasing power of the British pound in late 1976 at a value of around half of the base period can also be related to the increase of 8% in the effective pound rate after the IMF stabilization plan and the increase in oil exports.^{19/} At the end of the sample period, on the other hand, the effective dollar rate was at its April 1973 low level, having appreciated slightly until late 1977. In terms of purchasing power, however, the main feature in Figure 1 is that world money has but smoothed slightly the dollar trend, whose weight is about 43%.

In Figure 2 we show the real effective exchange rates of the same

^{19/} Between 1976; 11 and 1978; 4. We have been quoting the World Financial Markets nominal effective rates.

currencies, and the decline in the dollar rate becomes evident, whilst the Japanese yen shows a much higher appreciation than the purchasing power would have led to believe, given that the in world money weight is the second largest (16%). The real appreciation of the yen is of course much smaller when only traded goods prices are included.^{20/}

Finally, Figure 3 displays trends in the relative attractiveness of the U.S. dollar by comparing the increase of the U.S. consumer price index (line labelled "United States") the purchasing power of the U.S. dollar for the international investor (line labelled "World") and the purchasing power of the U.S. dollar for investors located in Italy, the U.K., Germany and Switzerland. These four lines are simply the increase of the dollar value of the domestic price level, so that the decline in the purchasing power of the dollar is naturally lowest for Switzerland and highest for Italy. Once again, these are merely suggestive because the investors have different preferences.

^{20/} Thus in May 15, 1978 the real effective rate of the yen with 1976 bilateral manufactures trade shares as weights and March 1973 as base was 109.1 using wholesale prices of manufactures and 132.7 using consumer prices. Using export unit values leads to a real depreciation of 9%. See World Financial Markets, May 1978. The IMF values are 107.7 for relative wholesale prices in 1978 relative to 1973 and 123.6 for relative value added deflators during the same interval.

FIGURE 2
REAL EFFECTIVE EXCHANGE RATES APR. 1973-APR. 1978
(April 1973 = 100)

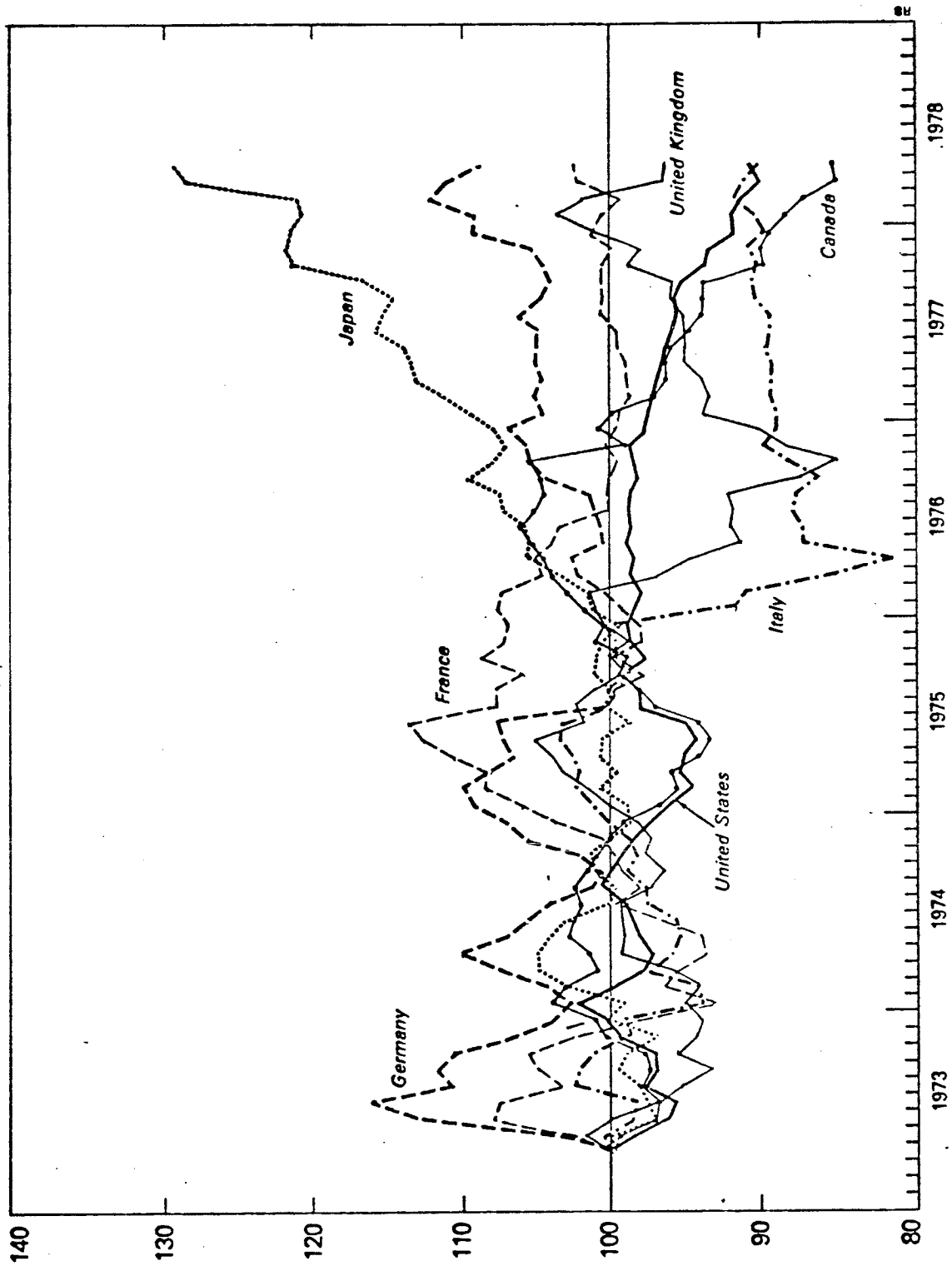
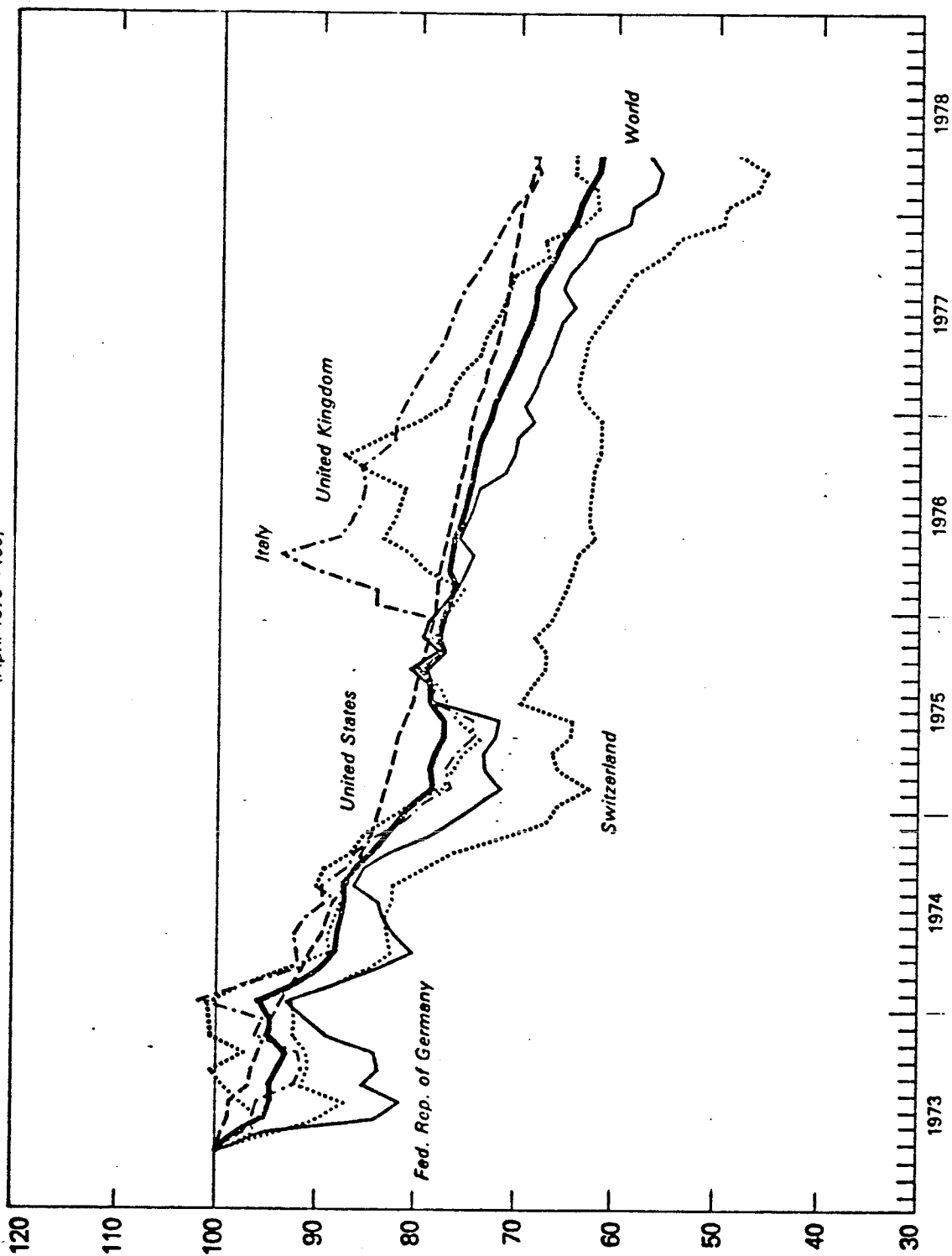


FIGURE 3
THE PURCHASING POWER OF THE U.S. DOLLAR APR. 1973 - APR. 1978
(April 1973 = 100)



II. Optimal portfolio diversification

1. Independently of how suggestive they are for describing developments in the international economy, the indices of purchasing power constructed in Section I can be used to find and compare real returns to assets denominated in different currencies. In this section, we take the simple case where the purchasing powers, or their components, are stationary and log normally distributed, thus allowing the separation between the consumption and the portfolio decision as in the traditional mean variance framework of Markowitz (1958) and Tobin (1965). The optimal portfolio rule is then time invariant and the only portfolio reshuffling that takes place is due to changes in nominal returns. This assumption is brutal in light of the deterioration in the purchasing powers of most currencies that we have just discussed, but it is empirically implementable and provides a benchmark against which other rules can be assessed. It is, in fact, simple to exclude anticipated inflation by means of autogressive transformations of the changes in purchasing powers, but it is also ultimately arbitrary and therefore vindicates the benchmark of the optimal portfolios of the Bernouilli investor in Brownian motion assets computed below. Some exercises will, however, be conducted to make the rule more "realistic," drawing on the optimal rules in the non Brownian motion case derived in Appendix 1. Empirical computations on the rule when national outputs are not composite goods, derived for the Brownian motion case in Appendix 2, also suggest themselves. Most of the derivations included here are implicit in Kouri-Macedo (1978) where optimal portfolio are also computed.

2. The well known problem of maximizing a utility function with constant relative risk aversion $1-b$, which is linear in mean and variance of real returns, is solved here in the context of a portfolio of short bonds denominated in N currencies, the changes in the purchasing power of which are random.

Define the proportion of wealth invested in currency i as x_i , and the mean real rate of return of a bond denominated in currency i as r_i .

Then the maximization problem can be written as

$$(1) \quad \text{Max } U = x'r - \frac{1}{2} bV(x'r)$$

subject to $x'e = 1$

where $x = (x_1 \quad x_N)'$

$r = (r_1 \quad r_N)'$

e is a N column vector of ones and V denotes variance

Now if nominal returns R_i are known the variance of mean return can be written as

$$(2) \quad V(x'r) = E(x'rr'x) = x'\Omega x$$

where $r_i = R_i + \pi_i$

π_i is the mean percentage change in the purchasing power of currency i

$\Omega = [\omega_{ij}]$ is the N by N variance covariance matrix of changes in purchasing powers

Since π_i and ω_{ij} are given we can find the optimal portfolio by simply differentiating (1) with respect to the decision variables x_i , imposing the constraint and accepting short sales so that x_i can be negative. We form the Lagrangean

$$L = x'r - 1/2 b x'\Omega x + \lambda (x'e - 1)$$

Differentiating L with respect to the vector of instruments we find the $N+1$ first order conditions as^{21/}

²¹ We will not differentiate the notation between vectors and scalars since it is obvious by the context. Thus in (4a) 0 is a vector and in (4b) a scalar.

$$(3a) \quad \frac{\partial L}{\partial x} = r - b\Omega x + \lambda e = 0$$

$$(3b) \quad \frac{\partial L}{\partial \lambda} = x'e - 1 = 0$$

The variance covariance matrix being non singular we can solve for x and, using (3b) to eliminate λ we have the solution vector as ^{22/}

$$(4) \quad x = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{1}{b} \Omega^{-1} [I - e \frac{e'\Omega^{-1}}{e'\Omega^{-1}e}] r$$

²²The derivation is as follows. From (3a)

$$x = \frac{\Omega^{-1}}{b} r + \frac{\lambda}{b} e$$

using the constraint

$$\lambda = b - e' \Omega^{-1} r$$

so

$$r - b\Omega x + (b\Omega - e' \Omega^{-1} r) e = 0$$

or

$$x = \Omega^{-1} e + \frac{\Omega^{-1}}{b} r - \frac{1}{b} e' \Omega^{-1} r \Omega^{-1} e$$

using the constraint again to substitute for $\Omega^{-1} e$

$$1 = e' \Omega^{-1} e (1 - e' \Omega^{-1} r) + \frac{1}{b} e' \Omega^{-1} r$$

$$x = \frac{\Omega^{-1}}{b} r + \Omega^{-1} e + \left[\frac{1}{e' \Omega^{-1} e} - \frac{1}{b} \frac{e' \Omega^{-1} r}{e' \Omega^{-1} e} - 1 \right] \Omega^{-1} e$$

or (4)

We see that there is a separation of the portfolio between one which minimizes the variance of real returns^{23/} and does not depend on risk and returns and another which depends on risk aversion and also on the real returns relative to the returns of the minimum variance portfolio. Specifically we define the minimum variance portfolio as

$$(5) \quad x^m = \frac{\Omega^{-1} e}{e' \Omega^{-1} e}$$

noting that $e' x^m = 1$

and the speculative portfolio^{24/} as

$$(6) \quad x^s = \frac{1}{b} \Omega^{-1} [I - e x^{m'}] r$$

noting that $e' x^s = x^{s'} e = 0$ ^{25/}

^{23/} To see this, solve

Min $x' \Omega x$

subject to $x' e = 1$

which implies $\lambda = \frac{1}{e' \Omega^{-1} e}$

and $x^m = \frac{\Omega^{-1} e}{e' \Omega^{-1} e}$ is the solution

^{24/} To see that this speculative portfolio could be obtained by simply maximizing returns computed with reference to the return on the minimum variance portfolio

Max $x' r - \frac{1}{2} b x' \Omega x$

subject to $x' e = 0$

The solution being

$$x^s = \frac{1}{b} \Omega^{-1} r + \frac{1}{b} \Omega^{-1} e \frac{e' \Omega^{-1} r}{e' \Omega^{-1} e} = \frac{1}{b} \Omega^{-1} r^*$$

where $r^* = r - e x^{m'}$

^{25/}

Since $e' \Omega^{-1} [I - \frac{e e' \Omega^{-1}}{e' \Omega^{-1} e}] = 0$

and $[I - \frac{e e' \Omega^{-1}}{e' \Omega^{-1} e}]' \Omega^{-1} e = 0$

Now define

$$(7) \quad \Sigma = \Omega^{-1} [I - e x^m']$$

and note that it can be partitioned as

$$\begin{bmatrix} \Delta & | & -\delta' \\ \hline -\delta & | & \delta_N \end{bmatrix}$$

$$\underline{e}' \Delta = -\delta'$$

$$\underline{e}' \delta = \delta_N$$

$\underline{e}$ is a N-1 column vector of ones

so that, substituting (5), (6) and (7) into (4) we can write the optimal portfolio rule in compact form as

$$(8) \quad x = x^m + \frac{1}{b} \Sigma r$$

3. The structure of x^m and Σ , and the role therein of preferences, as captured by the expenditure shares, can be further analyzed by transforming the maximization problem (1) into an unconstrained one. This is done by substituting for the proportion held in the non-currency, say the dollar, and using the properties of the indices of purchasing power described in Section I.

Denote by q_1, s_1^e, s_1 the unanticipated proportionate changes of Q_1, S_1^e and S_1 over the holding period of the nominal return, say a quarter.

Then

$$(9) \quad \sum_{i=1}^N x_i q_i = \sum_{i=1}^{N-1} x_i (q_i - q_N) + q_N$$

But we know that

$$q_1 - q_N = s_1^e - s_N^e = -s_1$$

$$\text{and that } q_N = \sum_{j=1}^N \alpha_j P_j^N$$

where $P_j^N = P_j - s_j$ are the unanticipated changes in dollar prices in country j .

Then, the variance of mean real return can be written as

$$(10) \quad V = \left(-\sum_{i=1}^{N-1} x_i s_i - \sum_{j=1}^N \alpha_j p_j \right)^2$$

or, in matrix notation,

$$(10') \quad V = \underline{x}' S \underline{x} + \alpha' P^N \alpha + 2 \underline{x}' \Pi \alpha$$

where $\underline{x} = (x_1 \quad x_{N-1})$

$\alpha = (\alpha_1 \quad \alpha_N)$

$S = E(ss')$ is the $N-1$ by $N-1$ variance covariance matrix of exchange rate changes, since $s = [s' \quad s_{N-1}]'$

$P^N = E(p^N p^{N'})$ is the N by N variance covariance matrix of dollar price changes since $p^N = [p_1^N \quad p_{N-1}^N \quad p_N^N]'$

$\Pi = -E(s')$ is minus the $N-1$ by N covariance matrix between dollar exchange rates and dollar price changes

Denoting by $\hat{r}$ the vector of relative real returns

$$F = [r_1 - r_N \quad r_{N-1} - r_N]'$$

we can write the unconstrained maximization problem as

$$(11) \quad \text{Max } F = \underline{x}' \hat{r} + r_N - \frac{1}{2} b (\underline{x}' S \underline{x} + \alpha - 2 \underline{x}' \Pi \alpha)$$

so that the first order condition becomes

$$(12) \quad \frac{\partial F}{\partial \underline{x}} = \hat{r} - b S \underline{x} + b \Pi \alpha = 0$$

solving for $\underline{x}$ we obtain

$$(13) \quad \underline{x} = S^{-1} \Pi \alpha + \frac{1}{b} S^{-1} \hat{r}$$

Now define a $N-1$ by N matrix Γ_1 and a N by N matrix Γ as

$$(14) \quad \Gamma_1 = S^{-1} \Pi \quad \Gamma = [\Gamma_1 \mid e' - e' \Gamma_1]'$$

So that the complete system can be written as

$$(15) \quad x = \begin{bmatrix} S^{-1} \Pi \\ \dots\dots\dots \\ e' - \underline{e}' S^{-1} \Pi \end{bmatrix} \alpha + \frac{1}{b} \begin{bmatrix} S^{-1} & : \\ \dots\dots\dots & \dots\dots\dots \\ -\underline{e}' S^{-1} & : \underline{e}' S^{-1} \underline{e} \end{bmatrix} r$$

or more completely

$$(15') \quad x = \Gamma \alpha + \frac{1}{b} \Sigma r$$

From (15') it is clear that preferences as captured by the expenditure shares only affect the minimum variance portfolio. These are transformed by a matrix Γ which measures the covariances of exchange rates and dollar prices relative to the covariance of exchange rates.

Also, since the bordered $N-1$ by $N-1$ Δ matrix implied by the structure of the Σ matrix in (7) is nothing but the inverse of the variance covariance matrix of exchange rate changes, it is clear that the speculative portfolio does not depend on the expenditure shares.

4. We now introduce the special cases of purchasing power parity and anticipated inflation.

If purchasing power parity holds, we have

$$(16) \quad p_N = p_i - s_i \quad i = 1, \dots, N-1$$

so that the matrix Π reduces to its last column, denoted by μ post multiplied by a N row vector of ones. $\mu = [\text{cov}(s_i p_N) \quad \text{cov}(s_{N-1} p_N)]'$

When we post multiply that matrix by the α vector, we are left with μ , since the weights sum to one. Therefore, the minimum variance portfolio is independent of preferences

$$(17) \quad x_{PPP}^m = S^{-1} \mu$$

If there is no unanticipated inflation, on the other hand, the Γ matrix is

just minus the expectation of the outer product of the vector of changes in the exchange rates since by assumption $p_i = 0$ for all i . In that case the Γ_1 matrix in (14) reduces to the identity matrix augmented by a column of zeros and the $e' - e'\Gamma_1$ vector just has a one in its Nth column. Thus the gamma matrix becomes the identity matrix and the minimum variance portfolio is given by the expenditure shares. The divergence between the vector of expenditure shares and the minimum variance portfolio is, of course, a measure of unanticipated inflation.

These two polar cases can be brought out in a third derivation which uses the constraint on the expenditure shares and thus expresses the variance of the mean real relative returns given in (10) above in terms of changes in the real exchange rates $\tilde{s}_i$ ^{26/} or:

$$(18) \quad V = E \left(\sum_{i=1}^{N-1} \alpha_i \tilde{s}_i - \sum_{i=1}^{N-1} x_i s_i - p_N \right)^2$$

In the same matrix notation, this is

$$(18') \quad V = \underline{x}' S \underline{x} + \underline{\alpha}' \bar{S} \underline{\alpha} + \sigma_{PN}^2 - 2\underline{x}' \phi \underline{\alpha} + 2\underline{x}' \mu - 2\underline{\alpha}' \theta$$

where $\bar{S}$ is the variance covariance matrix of real exchange rate changes

ϕ is the covariance matrix between changes in nominal and real exchange rates

μ is the covariance vector between changes in dollar prices and in nominal exchange rates

θ is the covariance vector between changes in dollar prices and the real exchange rates

²⁶

The term $\sum_{j=1}^N \alpha_j (p_j - s_j)$ in (10) above can then be written as

$$\sum_{i=1}^{N-1} \alpha_i (p_i - s_i - p_N) + p_N$$

or defining the changes in the dollar real exchange rate of country i

as $\tilde{s}_i = s_i - p_i + p_N$ we get $-\sum \alpha_i \tilde{s}_i + p_N$

If we now perform the unconstrained maximization as before, we get instead of (12):

$$(19) \quad \hat{r} - bS\underline{x} + b \underline{\phi}\alpha - b\underline{\mu} = 0$$

$$\text{Note that } \phi_{ij} = \pi_{ij} - \pi_{iN} \text{ and } \mu_i = \pi_{iN}$$

With PPP the ϕ matrix vanishes, since the real exchange rates are known and the minimum variance portfolio is like in (17). When inflation rates are known $\phi = S$ and μ is a zero vector^{27/} An intermediate case of interest is when the N'th country inflation rate is known. Then μ is a zero vector and $\phi = \tilde{\Pi}$ where $\tilde{\Pi}$ is the Π matrix with the last column truncated.

²⁷ Alternatively, if we express the process of dollar prices relative to the processes of domestic currency prices and the dollar exchange rate, we are able to decompose

$$- \Pi = [S \cdot 0] + \Psi$$

where Ψ is the covariance between dollar exchange rates and domestic prices

If $\tilde{S}$ is S augmented by a column of zeros (since it is summed over the α 's) we have the minimum variance portfolio as

$$\underline{x}^m = S^{-1} (\tilde{S} + \Psi)\alpha = (\tilde{I} + S^{-1}\Psi)\alpha$$

so that we can write the N equations as

$$\underline{x}^m = (I + \Theta)\alpha$$

$$\text{where } \Theta = \begin{bmatrix} S^{-1}\Psi & \dots & \dots \\ \vdots & \ddots & \vdots \\ e' - e'S^{-1}\Psi \end{bmatrix}$$

and I is N by N

Note that $e'\Theta = e'$

If inflation rates are known, Θ is a zero matrix

5. We now compute the optimal portfolio proportions that would obtain if in April 1978 an international investor with expenditure shares α_i $i=1, \dots, N$, and a unitary relative risk aversion would use the portfolio rule derived in (5) above.

This rule involves computing a minimum variance portfolio, x^m , independent of mean real returns and a speculative portfolio, x^s , dependent on mean real returns, where the quarterly change in the purchasing power of the currencies is used to deflate nominal returns on short bonds denominated in different currencies.

Table 2 indicates the mean quarterly rate of change in these purchasing powers over the period. It appears that the income and MERM weights are closer together than they are to the total trade weights. The ranking of the variances is in fact the same for income and MERM weights, with the exception of France, who comes before Japan and Canada for the income weights and after for the MERM weights. The ranking of total trade weights also interchange the relative ranking of Germany and Italy, brings Canada to the fourth largest variance and thus coincides with the MERM ranking.

In Table 3, mean and standard deviation of nominal returns on short bonds are reported, together with the mean real returns corresponding to the three different weighting schemes used in Table 2. The short term rate has been used but for Italy and Switzerland, where this series was not readily available, the medium term government bond yield and the government bond yield respectively were used.

The standard deviation of nominal returns is ignored in the computation since it is assumed that these are known with certainty. The ranking of real returns is the same independently of the weighting scheme, except that with income weights the U.K. rather than Italy ranks last.

Table 2

Mean and Standard Deviation of Quarterly Changes in
Purchasing Powers (in % p.a.) Apr 1973-Apr 1978

Weights	Mean			s.d.		
	Y	MERM	XM	Y	MERM	XM
CA	-11.97	-11.64	-12.20	11.57	12.95	13.88
FR	-9.41	-9.18	-9.78	14.05	12.29	12.20
GE	-2.36	-2.12	-2.77	17.91	16.74	15.46
IT	-17.01	-16.76	-17.33	16.19	15.48	15.74
JA	-5.76	-5.44	-6.05	12.67	13.45	12.84
SZ	1.56	1.83	1.17	18.81	17.99	16.87
UK	-14.67	-14.38	-15.00	14.62	14.55	13.60
US	-9.74	-9.42	-9.98	8.37	9.72	11.11

Table 3
Rates of Return Apr 1973-1978

	Nominal		Mean Real		with weights
	mean	s.d.	Y	XM	MERM
CA	7.55	1.12	-4.42	-4.09	-4.65
FR	9.56	2.10	.15	.39	-.21
GE	6.10	3.22	3.74	3.99	3.33
IT*	11.14	2.80	-5.87	-5.61	-6.18
JA	8.51	2.91	2.75	3.07	2.46
SZ*	5.52	1.24	7.08	7.34	6.69
UK	9.78	2.28	-4.89	-4.60	-5.22
US	7.13	2.36	-2.61	-2.28	-2.84

* Government bond yield. Others are money market rates.

Furthermore, the price differences in real returns wash out when differential rates of return with the U.S. are computed. Since

$\hat{r} = R_1 - R_N - s_1$ the speculative portfolio is independent of preferences. ^{28/}

The augmented S^{-1} matrix of own and cross effects of rate of return changes on the speculative portfolio, which we defined in (7) and (15') above as Σ is displayed in Table 4, after division by 400 to make it compatible with annual percentage rates of return.

The large own effect of the U.S. dollar, and to a lesser degree of the Canadian dollar, are noteworthy. The Canadian dollar rate exhibits a much smaller variance than any of the other N-1 rates and has therefore a larger own effect. The large value for the U.S. dollar comes from the sum of the cross effects of the dollar and the remaining currencies, namely the Canadian dollar. The importance of own and cross effects as given by the Σ matrix in Table 4 is larger than the difference in real rates of return in Table 2, so that the speculative portfolio for these three weighting schemes is the same at two decimal points.

In Table 5, the ranking of the currency holdings in decreasing order are reported, together with the ranking of real returns from Table 3 above for the total trade and MERM weights.

^{28/} Empirically the differential rates obtained from Table 2 are not equal to the change in the dollar exchange rate, since subtracting r_N from r_1 would involve calculating

$$\frac{Q_{1t} - Q_{1t-3}}{Q_{1t-3}} - \frac{Q_{Nt} - Q_{Nt-3}}{Q_{Nt-3}} \text{ which is different from } \frac{Q_{1t}/Q_{Nt} - Q_{1t-3}/Q_{Nt-3}}{Q_{1t-3}/Q_{Nt-3}}$$

and only the latter is the equivalent to $\frac{1/S_{1t} - 1/S_{1t-3}}{1/S_{1t-3}}$ in discrete time.

Table 4
The Sigma Matrix

CA	9.76							
FR	1.65	2.98						
GE	-1.10	-1.38	3.23					
IT	1.08	-.60	-.11	1.48				
JA	1.00	.25	-.84	-.08	2.60			
SZ	-.98	-.36	-1.64	.12	-.20	2.61		
UK	.06	-.26	.06	-.31	-.40	-.42	1.66	
US	-11.46	-2.28	1.78	-1.58	-2.34	.88	-.40	15.39

Table 5

Rankings of Returns and Speculative Portfolio

	x^S	r
US =	.28	SZ
SZ =	.15	GE
JA =	.07	JA
GE =	-.02	FR
FR =	-.03	US
IT =	-.08	CA
UK =	-.09	UK
CA =	-.28	IT

Thus the international investor would speculate on the appreciation of the U.S. dollar, the Swiss franc and the Japanese yen, and would sell Canadian dollars and, to a lesser extent, pounds, lire, French francs and D. marks.

Although we started out by computing the speculative portfolio, because it was independent of preferences, we know that the "capital position" of the international investor is given by the portfolio proportions which minimize the variance of the return, and which, being independent of real returns, are the equivalent of the "risk-free" asset in domestic finance.

Nevertheless, the minimum variance portfolio proportions need not be all positive, so that there is a certain amount of risk that the international member has to bear, even when following the 'optimal rule.'

Note, furthermore, that the relative weight of the speculative portfolio depends on the risk aversion of the investor and that the assumption of a Bernouilli investor is somewhat strong even for large organizations. The other extreme is infinite risk aversion, where only the minimum variance portfolio is held.

We show in Table 6 the gamma matrix which, together with the weighting scheme, determines the minimum variance portfolio. As noted above, it is also independent of preferences, since it is obtained by multiplying the $N-1$ by N covariance matrix between changes in dollar exchange rates and in dollar prices by the inverse of the variance covariance matrix of dollar exchange rate changes. Its columns can be interpreted as giving the proportions of the $N-1$ currencies for an investor which would only consume domestic goods

Table 6

The gamma matrix

CA	1.010	0.077	0.111	-0.213	0.408	0.070	0.079	0.111
FR	0.027	1.001	0.071	0.004	-0.038	0.049	0.240	0.016
GE	-0.030	0.019	0.987	0.084	0.081	-0.046	-0.026	-0.023
IT	0.038	0.012	-0.055	0.850	-0.024	0.010	-0.025	0.056
JA	-0.045	-0.082	-0.032	-0.145	0.792	-0.203	-0.045	-0.048
SZ	0.012	0.006	-0.027	0.069	-0.049	0.992	-0.131	-0.011
UK	-0.002	0.001	0.041	0.016	0.081	0.066	0.950	0.016
US	-0.010	-0.033	-0.096	0.336	-0.250	0.063	-0.042	0.882

since they represent the minimum variance portfolio of investor 1 when $\alpha_1 = 1$. To give an example consider the minimum variance portfolio of an investor who only consumes Japanese goods. It will be 79% in yen, 41% in Canadian dollars and 8% in D. marks and British pounds, and it will be short 25% in U.S. dollars, 5% in Swiss francs and 2% in Italian lire. Japan is a case where the "preferred habitat" hypothesis seems to be refuted. This hypothesis would imply a much higher share of yen as in the case for Canada and France (over 100%) and Germany and Switzerland (99%), and, to a lesser degree, for the UK (95%). The hypothesis is also in doubt for the U.S. (88%) and Italy (85%).

We have shown above that the gamma matrix can be decomposed into the sum of the identity matrix and the phi matrix, obtained by multiplying S^{-1} by the covariance matrix of exchange rates and domestic currency prices. The ϕ matrix, whose columns sum to zero, is then a measure of unanticipated inflation, since when inflation is perfectly anticipated the minimum variance portfolio is just α . This matrix is reported in percentage terms in Table 7, where the last row reports the column sums. The size of the diagonal elements is larger for Italy, Japan and the U.K., showing that these are the countries with higher unanticipated inflation. In the case of Japan the use of consumer prices probably explains the low share in the minimum variance portfolio, whereas for the two high inflation countries it shows that in this framework such inflation was largely unanticipated. It is also interesting to note that the Swiss and the Italian investors go substantially short in yen, but less than the Japanese investor, and that the U.K. investor goes short in Swiss francs. We already noted that the Japanese investor goes short in U.S. dollars.

Consider now departures from PPP. In that case the Γ matrix is given by the last column pre-multiplied by an N row vector of ones, so that the minimum variance portfolio is independent of preferences and includes 88% in U.S. dollars and 11% in Canadian dollars, whereas it is short in all the "strong"

Table 7

The phi matrix

	CA	FR	GE	IT	JA	SZ	UK	US
CA	1.0	7.7	11.1	-21.3	40.8	7.0	7.9	11.1
FR	2.7	.1	7.1	.4	-3.8	4.9	24.0	1.6
GE	-3.0	1.9	-1.3	8.4	8.1	-4.6	-2.6	-2.3
IT	3.8	1.2	-5.5	-15.0	-2.4	1.0	-2.5	5.6
JA	-4.5	-8.2	-3.2	-14.5	-20.8	-20.3	-4.5	-4.8
SZ	1.2	.6	-2.7	6.9	-4.9	-.8	-13.1	-1.1
UK	-.2	.1	4.1	1.6	8.1	6.6	-5.0	1.6
US	-1.0	-3.3	-9.6	33.6	-25.0	6.3	4.2	-11.8
Σ	0	.1	0	.1	.1	0	0	-.1

Data from Table 6.

currencies, with a positive correlation with the U.S. price index (.23 for Japan, .11 for Germany and .09 for Switzerland). The negative covariance with Italy is larger than with Canada but the much higher variance of the lira depresses its share to half of Canada's.^{29/}

Returning to the ϕ matrix, another way to obtain the relative importance of the departure from the no inflation case is to compare the proportions of the minimum variance portfolio with the expenditure shares, their difference being given by $\phi\alpha$. In Table 8 we include the minimum variance portfolios corresponding to the weighting schemes on Table 1 above. In Table 9 we give $x^m - \alpha$ in percent for the income weights and then the difference between that value and the corresponding value for the total trade weights and the MERM weights. Aside from computing that the weighting scheme does not make too big a difference, Table 9 shows that the largest differences between x^m and α occur for Canada, the U.S. and Japan, or the low variance exchange rate countries with a large covariance (in absolute value) with the Nth country's price level.^{30/}

^{29/} Canada's exchange rate with the U.S. dollar has by far the lowest variance. This well-known fact has traditionally been a source of difficulties in the testing of flexible exchange rate theories. There again, it gives Canada a much larger share in the portfolio of all national investors, as is evident from the first row of Γ in Table 6. The positive relation with the Japanese yen comes from the fact that the yen has the second lowest variance (about double of the Canadian dollar's) and a positive covariance with the U.S. price level, as pointed out in the text.

^{30/} Another comparison of interest is between the ranking in the variance in the purchasing power (Table 3 above) and in the minimum variance portfolio. For income weights the ranking is identical except for Japan and Germany, whose relative position is interchanged. In forms of variance alone U.S. is first, Canada second, Japan third and Germany seventh (last is Switzerland). Including the covariance between purchasing powers--to arrive at the minimum variance portfolio $x^m = \Omega^{-1}e/e'\Omega^{-1}e$ --makes Germany third and Japan seventh. With MERM weights Japan starts fourth, in terms of variance and there are minor changes in the rankings of the other currencies. With total trade weights, Japan's relative positions change in the same way or with income weights, but Germany ranks highest in the minimum variance portfolio, followed by Canada (up from 5th).

Table 8

Minimum variance portfolios for alternative weighting schemes

	1	2	3	4	5	6	7
CA	0.171	0.191	0.182	0.186	0.181	0.154	0.154
FR	0.171	0.162	0.169	0.165	0.125	0.205	0.193
GE	0.274	0.221	0.179	0.199	0.137	0.155	0.170
IT	0.074	0.066	0.077	0.072	0.057	0.080	0.077
JA	0.020	0.055	0.047	0.051	0.078	0.010	0.023
SZ	0.023	0.010	0.010	0.010	-0.004	0.006	0.007
UK	0.119	0.124	0.139	0.132	0.091	0.079	0.092
US	0.148	0.170	0.198	0.184	0.335	0.311	0.284

Table 9

Departures of the minimum variance portfolio
from anticipated inflation (%)
difference with

	income weights (5)	total trade weights (4)	MERM weights (6)
CA	13	2	3
FR	3	-1	0
GE	0	-1	0
IT	1	2	1
JA	-8	0	0
SZ	-2	0	-1
UK	2	0	0
US	-11	-3	-3

Note: values have been rounded.

In Table 10 the total portfolios from the three main weighting schemes are reported, by simply adding the respective values of Tables (4) and (8), since we take relative risk aversion to be unity. Both the effect of the weighting scheme and of the speculative portfolio can be seen in the share of the U.S. dollar, whereas the proportion of the D.mark comes wholly from the minimum variance portfolio. The weight of the French franc is another noteworthy feature, together with the limited role of the Japanese yen, which was held positively in the speculative portfolio.

Table 11 compares total portfolio and expenditure shares. The U.S. and Canada continue to be the largest difference, just like in Table 9, but the signs are reversed because of the speculative attractiveness of the U.S. dollar during the sample period. In fact, except for Japan, the signs are always reversed relative to Table 9. Furthermore, in the total portfolio weights do make a difference because of the very large role of the Swiss franc in the speculative portfolio, second highest using income weights and third highest using the other systems, whereas it always has the lowest share in expenditure.

6. Even though the framework presented here was not explicitly tested, in part because of the lack of data on actual portfolio proportions, a great deal of insight on the international financial system was gained by the use of the indices of purchasing power and the optimal portfolio propositions. Old features, like the special role of the Canadian dollar, were found to apply in this approach as well, whereas the importance of the U.S. dollar in a non-transactions-costs model came as a surprise, together with the limited attractiveness of the Japanese yen.

Significant departures from purchasing power parity, and the related presence of significant unanticipated inflation in high inflation countries were other findings of interest. Both these and the share of the yen should

Table 10

Total Portfolio

	X+M	Y	MERM
CA	-.09	-.10	-.12
FR	.13	.09	.17
GE	.18	.12	.14
IT	-.00	-.02	.00
JA	.12	.14	.08
SZ	.16	.15	.16
UK	.04	-.00	-.01
US	.46	.61	.59

Table 11

Total portfolio and
expenditure shares (%)

	income weights (5)	difference with total trade weights (4)	MERM weights (6)
CA	-15	2	3
FR	-1	2	-1
GE	-1	0	0
IT	-7	1	1
JA	-2	-1	-1
SZ	3	-10	-11
UK	-7	0	0
US	18	-3	-4

certainly be examined with indices using traded goods prices, but, subject to the usual caveats about real growth, purchasing power parity, if it is a useful hypothesis, should obtain with factor mobility for non-traded goods as well as for traded goods.

Other uses could be cited for the framework developed here, like the computation of the real forward premiums taking the cross effects given by the Σ matrix into account, or the investigation of a weighting scheme that would make a given portfolio optimal.

To sum up, the framework developed in this paper shows the importance of the microeconomic foundations of international financial intermediation and provides a guide for currency diversification policy on the part of large organizations, in particular central banks, a policy that single exchange rate models have to neglect entirely.

Appendix 1

The optimal portfolio and consumption rule in the general case

1. In this Appendix we derive the optimal portfolio rule and consumption in the intertemporal consumption-savings framework that is required when the stochastic processes generating purchasing powers, exchange rates or prices are not Brownian motion, or stationary and log-normally distributed. Imposing this assumption leads to the time invariant rules described in the text so that the explicit intertemporal optimization can be done without and the result derived from an instantaneous maximization. The derivation of this general case is based in Merton (1971), extended to N countries and N consumption goods.

Consider the intertemporal consumption-savings problem of an individual j who consumes amounts Z_i of goods produced in N countries and holds N_i short term bonds denominated in the N currencies of these countries. The bond of country i is defined as having a domestic price P_i of unity and a certain nominal return R_i .

We want to solve this problem over the individual's lifetime (assuming for simplicity that there is no "bequest" function), when the price of the bond is generated by a particular class of continuous-time Markov process which are functions of Gauss-Wiener Brownian motions called Ito Processes, and are defined by the stochastic integral

$$P_i(t) = P_i(0) + \int_0^t F_i(P,s)ds + \int_0^t g_i(P,s)dz_i$$

where $P = [P_1 \quad P_N]$

and z_i is a standard normal random variable, (so that dz_i is a Wiener process or white noise, or Brownian motion). Ito's Lemma states the stochastic differential equation obtained from a time dependent random variable F which is a function of the vector P . It is

$$dF = \sum_{i=1}^N \frac{\partial F}{\partial P_i} dP_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 F}{\partial P_i \partial P_j} dP_i dP_j$$

where the product of the differentials in the variance covariance term is defined by the multiplication rule

$$dz_i dz_j = P_{ij} dt$$

$$dz_i dt = 0$$

where P_{ij} is the instantaneous correlation coefficient between the Wiener processes dz_i and dz_j .^{1/}

In our problem wealth will be a function of Ito processes. We first define real wealth at time t as the purchasing power of the portfolio, which in discrete time, denoting a time interval, however short, as h , can be written as:

$$(1) W(t) = \sum_{i=1}^N N_i(t-h) Q_i(t)$$

Similarly, real consumption expenditure ^{2/} net of interest payments is

$$(2) E(t) = \sum_{i=1}^N Z_i(t) P_i(t) Q_i(t) - \sum_{i=1}^N N_i(t) Q_i(t) R_i(t)$$

Real expenditure during period h is equal to the change in real wealth

$$-E(t+h)h = \sum_{i=1}^N [N_i(t+h) - N_i(t)] Q_i(t+h)$$

^{1/} See Kushner (1971) and Merton (1971) for further details.

^{2/} Note that $E = E_k Q_k$ for all k where E_k is nominal consumption expenditure denominated in currency k .

$$= \sum_{i=1}^N [N_i(t+h) - N_i(t)] [Q_i(t+h) - Q_i(t)] + \sum_{i=1}^N [N_i(t+h) - N_i(t)] Q_i(t)$$

which, taking the limit as $h \rightarrow 0$, becomes

$$(3) \quad -E dt = \sum_{i=1}^N dN_i dQ_i + \sum_{i=1}^N dN_i Q_i$$

whereas in continuous time (1) becomes

$$(4) \quad W(t) = \sum_{i=1}^N N_i(t) Q_i(t)$$

Since Q is a stochastic integral, to obtain the change in wealth we use Ito's Lemma to obtain

$$(5) \quad dW = \sum_{i=1}^N N_i dQ_i + \sum_{i=1}^N dN_i Q_i + \sum_{i=1}^N dN_i dQ_i$$

This change has a capital gains component ($\sum N_i dQ_i$) and a portfolio reshuffling component which is equal by (3) to real expenditure. Using the definition of real expenditure net of interest payments in (2), and substituting in (5) we obtain:

$$dW = \sum_{i=1}^N N_i dQ_i + (\sum_{i=1}^N N_i Q_i R_i - \sum_{i=1}^N Z_i P_i Q_i) dt$$

Defining the portfolio proportions x_i , we have

$$(6) \quad dW = W \sum_{i=1}^N x_i (R_i dt + \frac{dQ_i}{Q_i}) - \sum_{i=1}^N Z_i P_i Q_i dt$$

$$\text{where } x_i = N_i Q_i / W$$

$$\text{and } \sum_{i=1}^N x_i = 1$$

The objective function is the expected utility of consumption over an interval $[0, T]$, which can be written as

$$(7) \quad U = E_0 \int_0^T \frac{1}{\gamma} \prod_{i=1}^N Z_i^{\alpha_i \gamma}(t) dt$$

where γ depends on the risk aversion so that the problem is to maximize

(8) subject to (6). Assuming that terminal wealth is zero (a "bequest"

function concave in Z_i would not change the result), the technique of stochastic dynamic programming (see sketch in Merton, 1971) involves defining a function, J , of the state variables of the system which is equal to the maximum of the expectation of the maximand with respect to the control variables, where the expectation is conditional on the steady state values of the state variables.

Bellman's theorem can then be extended to stochastic programming problems by maximizing with respect to the controls the "stochastic Hamiltonian," which is the sum of the utility function at time t and the expected mean proportional rate of change of the J function, given by the Dinkin operator, L . Thus, we find the differential of J by Ito's Lemma, take the conditional expectation and "divide" by dt .

We now proceed to derive the solution of intertemporal maximization problem under two alternative assumptions. First we assume that the purchasing power follow an Ito process and then we define Ito processes for the prices and exchange rates rather than for the purchasing powers.

2. Assume that the change in the purchasing power of currency i is generated by the following Ito process

$$(8) \quad \hat{q}_i = \frac{dQ_i}{Q_i} = r_i(Q_i, t)dt + \sigma_i(Q_i, t)dz_i$$

where r_i is the instantaneous conditional expected proportional change in purchasing power per unit of time

σ_i^2 is the instantaneous conditional variance per unit of time

dz_i is a Wiener process as before

If r_i^1 and σ_i^2 were constant, q_i would follow a Brownian motion, or white noise, as in the text, and the only state variable of the system would be wealth. In this case however we have $N + 1$ state variables W and Q_i and

$N + 1$ controls x_1 and C , where for simplicity we aggregate all N consumption goods and make the nominal return zero.

Define

$$(9) \quad J(W, Q_1) = \max_{c, x_1} E_t \int_0^T U(C, s) ds$$

where the expectation is conditional on

$$W(t) = W$$

$$Q_1(t) = Q_1$$

and

$$(10) \quad \phi(x_1, C, W, Q_1) = U(C) + L(J)$$

where L is the Dynkin operator.

By Ito's Lemma we find that

$$(11) \quad dJ = \frac{\partial J}{\partial W} dW + \sum_i \frac{\partial J}{\partial Q_i} dQ_i + \frac{1}{2} \frac{\partial^2 J (dW)^2}{\partial W^2} \\ + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 J}{\partial Q_i \partial Q_j} dQ_i dQ_j + \sum_i \sum_j \frac{\partial^2 J}{\partial Q_i \partial W} dQ_i dW$$

and by definition

$$L(J) = \frac{1}{dt} E_t (dJ)$$

so that

$$(12) \quad L(J) = \frac{\partial J}{\partial W} (\sum_i w_i r_i W - C) + \sum_i \frac{\partial J}{\partial Q_i} r_i Q_i + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} W^2 \sum_i \sum_j x_i x_j \sigma_{ij} + \\ \frac{1}{2} \sum_i \sum_j \frac{\partial^2 J}{\partial Q_i \partial Q_j} \sigma_{ij} Q_i Q_j + W \sum_i \sum_j Q_i x_j \sigma_{ij} \frac{\partial^2 J}{\partial Q_i \partial W}$$

If U is strictly concave in C then, by Belman's Theorem, there exist

optimal control rules x_1^* and C^* such that

$$(13) \quad 0 = \phi(x_1^*, C^*, W^*, Q_1^*)$$

Now define the Lagrangian

$$(14) \quad M = \phi + \lambda(1 - \sum x_i)$$

The first order conditions are

$$(15) \quad \frac{\partial M}{\partial C} = \frac{\partial U^*}{\partial C}(C^*) - \frac{\partial J}{\partial W} = 0$$

$$(16) \quad \frac{\partial M}{\partial x_k} = -\lambda + \frac{\partial J}{\partial W} r_k^W + \frac{\partial^2 J}{\partial W^2} \sum_1^N \sigma_{ki} x_i^* W^2 + \sum_1^N \frac{\partial^2 J}{\partial Q_i \partial W} \sigma_{ik} Q_i^W = 0$$

$$(17) \quad \frac{\partial M}{\partial \lambda} = 1 - \sum x_i^*$$

The consumption rule is independent of the portfolio rule and is given by

$$(18) \quad C^* = G \frac{\partial J}{\partial W} \text{ where } G = \left[\frac{\partial U}{\partial C} \right]^{-1}$$

Since (15) is linear in x_i^* we eliminate λ and solve for x_i^* . We can rewrite (15) in matrix form as

$$(19) \quad J_{W\sim}^W r_{\sim} + J_{WW}^W \Omega_{\sim}^2 x_{\sim}^* + \Omega_{\sim}^J Q_W = \lambda e$$

$$\text{where } e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad r_{\sim} = \begin{bmatrix} r_1 \\ r_N \end{bmatrix} \quad x_{\sim}^* = \begin{bmatrix} x_1^* \\ x_N^* \end{bmatrix} \quad J_{\sim}^{QW} = \begin{bmatrix} J_{Q1W}^{Q1W} \\ : \\ J_{QNW}^{QNW} \end{bmatrix} \quad \text{and } \Omega = \begin{bmatrix} \sigma_{11} & \sigma_{1N} \\ \sigma_{N1} & \sigma_{NN} \end{bmatrix}$$

if Ω is non-singular we can express $x_{\sim}^*$ as

$$(20) \quad x_{\sim}^* = \frac{\Omega^{-1}}{J_{WW}^W} \left[\lambda e - J_{W\sim}^W r_{\sim} - \Omega_{\sim}^J Q_W \right]$$

using (17) we have

$$(21) \quad \lambda = \frac{1}{e' \Omega^{-1} e} \left[J_{WW} W^2 + J_W W e' \Omega^{-1} r + e' J_{\sim QW} \right]$$

Substituting for λ in (20) we get

$$(22) \quad x^* = \frac{\Omega^{-1} e}{e' \Omega^{-1} e} + \frac{J_W}{J_{WW} W} \frac{e' \Omega^{-1} \chi \Omega^{-1} e - \Omega^{-1} r}{e' \Omega^{-1} e} + \frac{1}{J_{WW} W^2} \frac{e' J_{\sim QW} \Omega^{-1} e - J_{\sim QW}}{e' \Omega^{-1} e}$$

Define the minimum variance portfolio as

$$(23) \quad x_{\sim}^{**} = \frac{\Omega^{-1} e}{e' \Omega^{-1} e} \text{ and note } e' x_{\sim}^{**} = 1$$

and rewrite (22) as

$$(24) \quad x_{\sim}^{**} = \frac{1}{1 - \gamma} (x_{\sim}^{**'} r \Omega^{-1} e - \Omega^{-1} r + \frac{1}{J_{WW} W^2} (e' J_{\sim QW} x_{\sim}^{**} - J_{\sim QW}))$$

where

$$\frac{1}{1 - \gamma} = \frac{-J_W}{J_{WW} W^2}$$

and γ is the Arrow-Pratt measure of relative risk aversion.

Note that $x_{\sim}^{**'}$ is the return on the minimum variance portfolio and that

$$(25) \quad \Omega^{-1} (r - x_{\sim}^{**'} r e) = \Omega^{-1} (I - e x_{\sim}^{**'}) r = \Omega^{-1} K r = \Sigma r$$

we can then write the optional portfolio rule as

$$(26) \quad x^* = x_{\sim}^{**} + \frac{\Sigma r}{1 - \gamma} = \frac{K J_{\sim QW}}{J_{WW} W^2}$$

If dQ_1/Q_1 is Brownian motion $J_{\sim QW} = 0$ and we get

$$(27) \quad \underset{\sim}{x}^* = \underset{\sim}{x}^{**} + \frac{\Sigma r}{1 - \gamma}$$

which is just equation (6) in the text.

3. We now derive the optimal portfolio rule in terms of the determinants of the purchasing powers.

If we eliminate x_N from (6) above we have

$$(28) \quad dW = W \left(\sum_{i=1}^N \underset{\sim}{x}_i \hat{\underset{\sim}{R}}_i + \frac{dT_1}{T_1} + \frac{dQ_N}{Q_N} \right) - \sum_{i=1}^N Z_i P_i Q_i dt$$

where $T_i = Q_i/Q_N$ and $\hat{\underset{\sim}{R}}_i = \underset{\sim}{R}_i - \underset{\sim}{R}_N$

Using Itô's lemma, we find that,

$$(29) \quad \frac{dT_i}{T_i} = -\frac{dS_i}{S_i} + \left(\frac{dS_i}{S_i}\right)^2$$

and, taking national outputs as composite goods, that,

$$(30) \quad \frac{dQ_N}{Q_N} = -\sum \alpha_i \frac{dP_i^N}{P_i^N} + \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j \frac{dP_i^N}{P_i^N} \frac{dP_j^N}{P_j^N}$$

where $P_i^N = P_i/S_i$

We now assume that exchange rates and dollar prices $\frac{1}{P_i}$ are determined by the following stochastic processes:

$\frac{1}{P_i}$ Note that if $\frac{dP_i}{P_i} = v_i dt + \phi_i du_i$, then $\mu_i = v_i - \pi_i + \sigma_i^2 - \psi_{ii}$ and

$\delta_i du_i = (\phi_i - \sigma_i) du_i$ where $\psi_{ii} = \rho_{ii} \sigma_i \phi_i$ and ρ_{ii} is the correlation coefficient between domestic prices and exchange rates.

$$(31) \quad s_i = \frac{dS_i}{S_i} = \pi_i dt + \sigma_i dz_i$$

$$(32) \quad p_i = \frac{dP_i^N}{S_i} = \mu_i dt + \delta_i du_i$$

where π_i and μ_i are the respective mean changes and σ_i and δ_i the standard deviations. So that the wealth constraint becomes

$$(33) \quad dW = [W(\sum_{i=1}^{N-1} x_i (\hat{R}_i - \pi_i + \sigma_i^2) - \sum_{i=1}^N \alpha_i \mu_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \delta_{ij}) - \sum_{i=1}^N z_i P_i Q_i] dt - \sum_{i=1}^{N-1} x_i \sigma_i dz_i - \sum_{i=1}^N \alpha_i \delta_i du_i$$

The problem is now to maximize (7) subject to (33). We define

$$(34) \quad J(W, S_i, P_j) = \max_j E_t \int_t^T \frac{1}{Y} \prod_{i=1}^N z_i^{\alpha_i \gamma}(z) dt$$

where the expectation is conditional on

$$W(t) = W$$

$$S_i(t) = S_i \quad i=1, \dots, N-1$$

$$P_j(t) = P_j \quad j=1, \dots, N$$

and

$$(35) \quad \phi(x_i, z_j, W, S_i, P_j) = U + L(J)$$

The differential of J is now

$$(36) \quad dJ = \frac{\partial J}{\partial W} dW + \sum_{i=1}^{N-1} \frac{\partial J}{\partial S_i} dS_i + \sum_{j=1}^N \frac{\partial J}{\partial P_j} dP_j + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} dW^2 + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \frac{\partial^2 J}{\partial S_i \partial S_j} dS_i dS_j + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 J}{\partial P_i^N \partial P_j^N} dP_i^N dP_j^N + \sum_{i=1}^{N-1} \sum_{j=1}^N \frac{\partial^2 J}{\partial S_i \partial P_j^N} dS_i dP_j^N + \sum_{i=1}^{N-1} \frac{\partial^2 J}{\partial S_i \partial W} dS_i dW + \sum_{j=1}^N \frac{\partial^2 J}{\partial P_j^N \partial W} dP_j^N dW$$

Substituting from (28), (31) and (32) we find the Dynkin operator as

$$\begin{aligned}
 (37) \quad L(J) = & J_w [W(\sum_{i=1}^{N-1} x_i (\hat{R}_i - \pi_i + \sigma_i^2) - \sum_{i=1}^N \alpha_i \mu_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \delta_{ij}) - \\
 & \sum_{i=1}^N Z_i P_i Q_i] + \sum_{i=1}^{N-1} J_{S_i} S_i \pi_i + \sum_{i=1}^N J_{P_i} P_i \mu_i + \frac{1}{2} J_{WW} W^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} x_i x_j \sigma_{ij} + \\
 & \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \delta_{ij} + 2 \sum_{i=1}^{N-1} \sum_{j=1}^N x_i \alpha_j \theta_{ij}] + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^N J_{P_i P_j} P_i P_j \delta_{ij} + \\
 & \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^N J_{S_i S_j} S_i S_j \sigma_{ij} + \sum_{i=1}^{N-1} J_{S_i W} W S_i (\sum_{j=1}^N x_j \sigma_{ij} + \sum_{j=1}^N \alpha_j \theta_{ij}) + \\
 & \sum_{i=1}^N J_{P_i W} W P_i (\sum_{j=1}^N \alpha_j \delta_{ij} + \sum_{j=1}^{N-1} \alpha_j x_j \theta_{ij}) + \sum_{i=1}^N \sum_{j=1}^{N-1} J_{P_i S_j} P_i S_j \theta_{ij}
 \end{aligned}$$

where, to ease the notational burden, partial derivatives of J with respect to the state variables are denoted by subscripts, and $\theta_{ij} = \rho_{ij} \sigma_i \delta_j$ is the covariance between exchange rate i and dollar price j .

Since $\prod_{i=1}^N Z_i^{\alpha_i}$ is concave in Z_i , by Bellman's Theorem there exists optimal x_i^* and Z_i^* such that $\phi^* = 0$.

The first order conditions with respect to Z_k and x_k are, respectively

$$(38) \quad \frac{\partial \phi}{\partial Z_k} = \alpha_k Z_k^{-1} U - \frac{\partial J}{\partial W} = P_k Q_k = 0$$

$$(39) \quad \frac{\partial \phi}{\partial x_k} = J_w W(\hat{R}_k - \pi_k + \sigma_k^2) + J_{WW} W^2 (\sum_{j=1}^{N-1} x_j \sigma_{kj} + \sum_{j=1}^N \alpha_j \theta_{kj}) +$$

$$W \left[\sum_{i=1}^{N-1} J_{S_i W} S_i \sigma_{ik} + \sum_{i=1}^N J_{P_i^N W} P_i^N \theta_{ik} \right] = 0$$

The consumption rule (38) implies that

$$(40) \quad \frac{E}{U} = \frac{\partial J}{\partial W}$$

so that

$$(41) \quad Z_k^* = \frac{\alpha_k E}{P_k Q_k} = \frac{\alpha_k E_k}{P_k}$$

which implies an indirect utility function

$$(42) \quad \tilde{V}_k = a' E_k Q$$

$$\text{where } a' = \prod_{i=1}^N \alpha_i^{\alpha_i}$$

and Q is the purchasing power of world money.

Since we maximized with respect to real rather than nominal expenditure, the exchange rates wash out and we are left with Q as an index of value for any consumer-investor with preferences described by α_k . Writing the portfolio rule (39) in matrix form, we have

$$(39) \quad J_{\tilde{W}} \hat{r} + J_{\tilde{W}W} W^2 (\underline{Sx} + \Pi \alpha) + W [\hat{S} \tilde{J}_{\tilde{S}} + \hat{\Pi} \tilde{J}_{\tilde{P}}] = 0$$

$$\text{where } \hat{r} = R - \pi + \sigma^2$$

$$J_{\tilde{S}} = [J_{S_1 W} \quad J_{S_{N-1} W}]$$

$$J_{\tilde{P}} = [J_{P_1^N W} \quad J_{P_N^N W}]$$

$$R = [R_1 - R_{US} \quad R_{N-1} - R_{US}]'$$

$$\pi = [s_1 \quad s_{N-1}]'$$

$$\sigma^2 = [\sigma_{S1}^2 \quad \sigma_{SN}^2]'$$

$$S = \begin{bmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_{N-1} \end{bmatrix}$$

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{1N-1} \\ \sigma_{N-1 1} & \sigma_{N-1 N-1} \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} P_1^N & 0 \\ & \ddots & \\ 0 & & P_N^N \end{bmatrix}$$

$$-\Pi = \begin{bmatrix} \theta_{11} & \theta_{1N} \\ \theta_{N-1 1} & \theta_{N-1 N} \end{bmatrix}$$

$$\alpha = [\alpha_1 \quad \alpha_N]'$$

$$x = [x_1 \quad x_N]'$$

$$\underline{x} = [x_1 \quad x_{N-1}]'$$

So that the optimal portfolio proportions for the N-1 assets are given by

$$(40) \quad \underline{x} = \Gamma_1 \alpha + \frac{1}{1-\gamma} S^{-1} \hat{r} - \frac{1}{J_{WW}} S^{-1} (\hat{S} \hat{J}_{\hat{S}} + \Pi \hat{P} J_{\hat{P}}) = 0$$

where

$$-\frac{J_W}{J_{WW}} = \frac{1}{1-\gamma} \text{ as in (24)}$$

$$\Gamma_1 = S^{-1} \Pi \text{ as in text}$$

and $\underline{x} e = x_N$ by definition

In the Brownian motion case, wealth is the only state variable so that $J_{\hat{S}} = J_{\hat{P}} = 0$ and (40) reduces to equation (13) in the text.

APPENDIX 2

A time-invariant portfolio rule when there are many goods produced in each country

In this Appendix we derive the optimal portfolio rule in the case where there are $\sum_{i=1}^N M_i$ goods, and country 1 produces M_1 goods.

Then the variance of mean real return relative to the Nth country which is given by (10) in Section II, 3 of the text becomes

$$(1) \quad V = E \left| \sum_{i=1}^{N-1} x_i s_i - \sum_{i=1}^N \sum_{j=1}^{M_i} \alpha_{ij} (\beta_{ij} p_{ij} - s_i) \right|^2$$

where p_{ij} is the change in the price of the ij th good.

Defining a $\sum_{i=1}^N M_i$ vector of dollar prices $\tilde{p}_1$ and a $\sum_{i=1}^N M_i$ by N

block diagonal matrix of domestic weights B . We can rewrite (1) as

$$(1') \quad V = E[-\underline{x}'s - \tilde{p}'Ba]^2$$

where x is $N-1$ by 1

α is N by 1

$$B = \begin{bmatrix} \beta' & 0 \\ 0 & \beta^N \end{bmatrix}$$

1/ When $M_i = M$ for all i we have simply

$$\tilde{p} = p - S X e$$

where p is a NM by 1 vector of domestic currency prices

$$s = [s_1 \quad s_{N-1} \quad 0]$$

and e is a M by 1 vector of ones

$$\beta^1 = (\beta_1^1 \dots \beta_{M_1}^1)^1$$

and $e_1' \beta^1 = 1$ where e_1 is an M_1 vector of ones

We now define a $N-1$ by $\Sigma_{M_1}^N$ matrix of covariances between exchange rates and dollar prices

$$\tilde{\gamma} = E[\tilde{sp}'] = \begin{vmatrix} \sigma_{11}^1 & \dots & \sigma_{1M_1}^1 & \sigma_{21}^1 & \dots & \sigma_{2M_2}^1 & \dots & \sigma_{N1}^1 & \dots & \sigma_{NMN}^1 \\ \sigma_{11}^{N-1} & \dots & \sigma_{1M_1}^{N-1} & \sigma_{21}^{N-1} & \dots & \sigma_{2M_2}^{N-1} & \dots & \sigma_{N1}^{N-1} & \dots & \sigma_{NMN}^{N-1} \end{vmatrix}$$

and the usual variance covariance matrix of exchange rate changes

$$S = E[ss']$$

Denoting the vector of relative real returns as $\hat{r}$, we can then write the objective function of our unconstrained maximization problem as

$$(2) \quad F = \underline{x}' \hat{r} + r_N - \frac{1}{2} b [\underline{x}' S \underline{x} + \alpha' B' \tilde{\gamma} B \alpha - 2 \underline{x}' \tilde{\gamma} B \alpha]$$

and the first order condition is

$$(3) \quad \frac{\partial F}{\partial \underline{x}} = \hat{r} - b S \underline{x} + b \tilde{\gamma} B \alpha = 0$$

so that substituting in a $N-1$ by $\Sigma_{M_1}^N \tilde{\gamma}$ matrix and including the constraint on x_N , we get the optimal portfolio rule as

$$(4) \quad \underline{x} = \tilde{\gamma} B \alpha + \frac{1}{b} \Sigma \underline{r}$$

which is the same as (15) but with a N by $\Sigma_{M_1}^N$ gamma matrix, $\tilde{\gamma}$, and a system of weights given by $B \alpha$.

If purchasing power points holds for each good

$$(5) \quad P = P_N e_M^N$$

where e_M^N is a 1 by M_1 vector of ones

also

$$(6) \quad \tilde{\Pi} = \mu e_M^N$$

where $\mu = [\sigma_N', \sigma_N^{N-1}]$ as before

Now, since B is a block diagonal matrix of weight vectors β^1 , we can write

$$(7) \quad e_M^N B = e_N^N$$

where e_N^N is a 1 by N vector of ones

The minimum variance portfolio is thus independent of preferences again, since

$$(8) \quad \tilde{\Pi} B \alpha = \mu$$

If PPP holds for domestic output, we have instead of (5)

$$(9) \quad \tilde{p}' B = P_N e_N^N$$

and the result is the same as before since then PPP constrains relative price changes between countries. When these are ruled out explicitly

$$(10) \quad p_{ij} = p_i \quad \text{for all } j$$

Then

$$(11) \quad \tilde{\Pi} B = \Pi$$

and therefore $\sigma_{ij}^1 = \sigma_j^1$ for all j

Appendix 3

The Aggregation of Portfolios of Different Investors

The crucial role of the vector of expenditure shares in the computation of the optimal portfolio has been pointed out. We talked loosely of home, foreign and international investors and compared purchasing power indices for different investors in Section I. In Section II we compared the gamma matrix to a stacking of the portfolios of "national" investors. We now suggest a simple way to arrive at an international portfolio from a given set of domestic or bilateral weights and an international weighting criterion. Our two tier procedure is a straightforward extension of the earlier one, but it may be less arbitrary given the types of aggregative data that are found.

Thus rewrite (15') for the investor of country i as

$$(1) \quad x^i = \Gamma \alpha^i + \frac{1}{b_i} \sum r$$

Define

$$X = (X^1 \dots X^n)$$

a N by N matrix such that $e'X = e'$

$$A = (\alpha^1 \dots \alpha^n)$$

a N by N matrix such that $e'X = e'$

$$b = (b_1 \dots b_n)$$

and

$$B = Ib$$

The N by N matrix which diagonalizes b

Then we can write the portfolio rules of the N investor in compact form as

$$(2) \quad X = \Gamma_A + \Sigma_B^{-1} r e'$$

Now define an international weighting criterion y such that $e'y = 1$ and we can collapse (2) into

$$(3) \quad z = \Gamma_B + \Sigma_B^{-1} r$$

$$\text{where } z = Xy \quad \text{and} \quad e'z = 1$$

$$\beta = Ay \quad \text{and} \quad e'\beta = 1$$

If all investors are Bernouilli, we are back to (1) with two explicit weighting schemes A and y .

References

- Artus, Jacques R. and McGuirk, Anne K., A Revised Version of the Multilateral Exchange Rate Model, Draft, Research Dept., IMF, 16, May 1978.
- Artus, Jacques R. and R.R. Rhomberg, "A Multilateral Exchange Rate Model," IMF Staff Papers, November 1973.
- Cooper, Richard N., The Economics of Interdependence, New York: McGraw-Hill, 1968.
- Cox, John C., Jonathan Ingersoll and Stephen Ross, "A Theory of the Term Structure of Interest Rates and the Valuation of Interest-dependent Claims," Draft, July 1977.
- Deppler, Michael C., "Weights for the Indicators of Competitiveness," Memorandum, Research Dept., IMF, 6, June 1978.
- Diewert, W.E., "Exact and Superlative Index Numbers," Journal of Econometrics, 4, 1976.
- Dornbusch, Rudiger, Statement Hearings on the Conduct of Monetary Policy Committee on Banking, Finance and Urban Affairs, U.S. House of Representatives, 7, March 1978.
- Grauer, Frederick L.A., Robert H. Litzenberger and Richard E. Stehle, "Sharing Rules and Equilibrium in an International Capital Market Under Uncertainty," Journal of Financial Economics, 1976.
- Heller, H. Robert and Malcolm Knight, Reserve Currencies Preferences of Central Banks, Princeton Essays in International Finance, no. 131, December 1978.
- Keohane, Robert E. and Joseph S. Nye, Power and Interdependence, Boston: Little Brown, 1977.
- Kouri, Pentti J.K., Essays on Flexible Exchange Rates, Ph.D. thesis, MIT, 1975.
- Kouri, Pentti J.K., "International Investment and Interest Rate Linkages Under Flexible Exchange Rates," in R. Aliber, ed., The Political Economy of Monetary Reform, London: MacMillan, 1977.
- Kouri, Pentti J.K. and Jorge B. de Macedo, "Exchange Rates and the International Adjustment Process," Brookings Papers on Economic Activity, 1978.
- Kushner, H.J., Introduction to Stochastic Control, New York: Academic Press, 1971.
- Long Jr., John B., "Stock Prices, Inflation and the Term Structure of Interest Rates," Journal of Financial Economics, 1974, I.
- Macedo, Jorge B. de, Recent developments in international economics. Report to the Rockefeller Foundation, May 1978.
- Markowitz, Harry, Portfolio Selection, New Haven: Yale University Press, 1958.
- McKinnon, Ronald I., Money in International Exchange: The Convertible Currency System, Oxford: Oxford University Press, 1979.

Merton, Robert C., "Lifetime Portfolio Selection Under Uncertainty: the Continuous Time Case," Review of Economics and Statistics, August 1969.

Merton, Robert C., "Optimum Consumption and Portfolio Rules in a Continuous Time Model," Journal of Economic Theory, 1971, 3.

Merton, Robert C., "An Intertemporal Capital Asset Pricing Model," Econometrica, September 1973.

Rhomberg, Rudolf R., "Indices of Effective Exchange Rates," IMF Staff Papers, July 1976.

Samuelson, Paul A., "Lifetime Portfolio Selection by Dynamic Stochastic Programming," Review of Economics and Statistics, August 1969.

Samuelson, Paul A., "Analytical Notes on International Real Income Measures," Economic Journal, September 1974.

Samuelson, P.A. and S. Swamy, "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis," American Economic Review, September 1974.

Solnik, Bruno H., "An Equilibrium Model of the International Market," Journal of Economic Theory, 1974, 8.

Tobin, James, "The Theory of Portfolio Selection," in F. Hahn and J. Brechling (editors), The Theory of Interest Rates, New York: Macmillan, 1965.