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ENDOGENOUS FERTILITY IN AN OVERLAPPING GENERATIONS GROWTH MODEL

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1. Introduction

Kuznets (1966) defines economic growth as a sustained increase in both income per capita and population size. The role of population is central not only in the operational or empirical approach of Kuznets, but also in the neoclassical theory of growth identified with Solow (1958). Phelps (1961, 1966) established the golden rule of capital accumulation and deduced the condition for optimality that the net marginal product of eapital per-capita not be less than the rate of growth in population. Diamond (1965) provided a microeconomic foundation for aggregate capital accumulation in the context of an overlapping generations model from which the identical condition for optimality appeared to result. However, unlike Kuznets's view, economic growth theory has not been concerned with population as an outcome of the process of economic development.

This paper analyzes the equilibria of a competitive economy within an overlapping generations growth model in which fertility is subject to choice at the level of the individual. The model is a straightforward extension of the Samuelson (1958) - Diamond (1965) framework, except that agents are assumed to enjoy parenthood, and offspring are assumed to consume resources before reaching adulthood (Becker, 1960, Willis, 1973). We first investigate the steady state of this economy and compare its characteristics to those of an equivalent social planner problem. Then, in subsequent sections, we characterize possible associations between fertility (the rate of population growth) and income per-capita over the equilibrium time series path of a particular economy and in the cross-section, i.e., at the steady states of different economies. The results demonstrate the flexibility of the model in generating associations between population growth and income per-capita that fit the stylized facts (e.g., see Kuznets).

Samuelson (1975) has discussed exogenously given steady state optimal population growth in Diamond's overlapping generations growth model. Unfortunately, Deardorff (1976) showed that if the production function is unbounded, there does not exist a finite stationary capital-labor ratio or a positive rate of population growth that solve Samuelson's planner problem. Razin and Ben-Zion (1975) considered a model of population growth where the representative agent has an infinite horizon time separable utility function in consumption and "newly born people".[1] Therefore, the long run allocation is similar to that of the social planner allocation in Samuelson and in this paper as well. By including fertility in the utility function, Razin and Ben-Zion partially avoided the indeterminacy problem of Samuelson.[2]

In this paper, each individual derives utility only from own consumption and from own children. We show that the allocation of the

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[1] The motivation for this specification of preferences is that parents care about the utility of their offspring.

[2] In the planner problem the budget constraint is not convex. Hence, there may not exist an interior maximum or a unique solution, even when fertility enters the utility function directly (See Razin and Ben-Zion footnote 12).

social planner in this framework is not characterized by minimizing population growth; given reasonable assumptions about preferences, fertility occurs as an interior solution of the planner problem. Furthermore, as in the case where fertility is exogenous, the competitive equilibrium converges either to the Golden Rule or to a stationary path where the net marginal product of capital per-capita exceeds the rate of population growth. In this regard, results with endogenous population growth are not dissimilar to those from conventional competitive growth models. Moreover, the planner allocation always satisfies the Golden Rule as in the Diamond model. In contrast to the case with exogenous fertility, we prove that even when a paper asset is privately valued, in which case the economy is on the Golden Rule, the competitive allocation is not the same as the planner allocation. Indeed, the competitive allocation is characterized by fewer children and more savings than is the planner allocation. Intuitively, this follows from the fact that each agent can obtain the same rate of return on the paper asset regardless of the desired quantity of own children, and thus without incurring the expense of having own children. Given this, we also demonstrate that a voluntary social security program, in which the size of the transfer to a particular agent is directly tied to own fertility, leads to the planner allocation.[3]

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[3] Willis (1980) conjectured that such a program would induce this equivalence, arguing by analogy from a model of the extended family in which old age security is tied to own fertility.

In the next section, we present a growth model with endogenous fertility. We describe the economy and define the competitive equilibrium. In section 3 we present the problem of the social planner and compare the stationary competitive and planner allocations. In section 4 we demonstrate the propositions of the previous section with an example and provide a brief discussion of optimality issues. In the following section, we explore the positive aspects of the model. The final section summarizes and highlights areas for future research.

2. A Growth Model with Endogenous Fertility

The economy to be studied is a variant of the Samuelson (1958) - Diamond (1965) overlapping generations growth model. In this section we describe the technology, the preferences and the characteristics of the decentralized economy.

The technology is represented by a constant return to scale aggregate production function, $F(K, L)$ where K is capital and L is labor, such that $f(k) = F(\frac{K}{L}, 1)$, $f(0) = 0$, $f'(0) = \infty$, where $k = \frac{K}{L}$. The single good can be either consumed or stored as capital for next period production. We assume that capital depreciates at rate δ in storage and production. Time is discrete and individuals live for three periods; infants, who make no decisions in the first period, working ("young") in the second period, and retired ("old") in the third period. In the second period individuals supply one unit of labor and they decide on their lifetime consumption and savings. We assume that individuals enjoy parenthood and they decide about the quantity of own children in the second period of their life. Each child born at time t consumes e units of the single good.

The representative individuals life-time utility function is

$$V(C_1(t), C_2(t), 1 + n(t + 1)) . \tag{2.1}$$

where $C_i(t)$ is the consumption of a member of generation t at period $i + 1$ of his/her life, $i = 1, 2$, and $1 + n(t + 1)$ is the number of children of each member of generation t. The utility function (2.1) satisfies the usual concavity and differentiability conditions with respect to all variables, as well as the following conditions:

(i) $$\frac{V_1(C_1, C_2, 1+n)}{V_2(C_1, C_2, 1+n)} \to \infty(0) \quad \text{as} \quad C_1(C_2) \to 0$$

(ii) $$V_3(C_1, C_2, 1+n) \to \infty \quad \text{as} \quad n \to \varepsilon - 1 \quad \text{with} \quad 0 < \varepsilon \leq 1$$

where $$V_i = \frac{\partial V}{\partial C_i}, \quad i = 1, 2 \quad \text{and} \quad V_3 = \frac{\partial V}{\partial(1+n)}$$

Condition (i) ensures that C_1 or C_2 is never optimally zero. Condition (ii) states that the marginal utility of children becomes unbounded as the number of children approaches some number less than one.[4]

The economy at date $t=1$ consists of a given number, $N(0)$, of old people and a given number, $N(1)$, of young people, with $N(0)$ and $N(1) > 0$. Each initial old person is endowed with $K(1)$ units of capital and $h(0) \equiv \frac{H}{N(0)}$, $H > 0$, units of a paper asset. Since all people are alike there are $N(t) = (1 + n(t))\,N(t-1)$ young people at each period $t \geq 1$.

Before we discuss specific market structures, we characterize the set of feasible and optimal allocations of this economy.

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[4]We ignore the discrete nature of fertility. In addition, this condition is crucial in establishing the existence of an interior solution to the planner problem discussed below. In particular, we will show that ε must be greater than $1 - \delta$ to avoid the existence problem noted by Deardorff (1976).

Definition

An allocation $\{C_1(t),\ C_2(t-1),\ n(t+1),\ K(t+1)\}$ for all $t \geq 1$ is feasible if it satisfies

$$(2.2) \quad C_1(t) + \frac{C_2(t-1)}{1+n(t)} + (1 + n(t+1))e = f(k(t)) + (1-\delta)k(t) - (1+n(t+1))k(t+1)$$

where k is per-worker capital. A stationary allocation is defined such that the variables in (2.2) are independent of the time index t.

Definition

A feasible allocation $\{\overline{C}_1(t),\ \overline{C}_2(t-1),\ \overline{n}(t+1),\ \overline{k}(t+1)\}$ for all $t \geq 1$ is optimal if and only if there does not exist another feasible allocation $\{C_1(t),\ C_2(t-1),\ n(t+1),\ k(t+1)\}$ for all $t \geq 1$ such that

(a) $V(C_1(t),\ C_2(t),\ 1+n(t+1)) \geq V(\overline{C}_1(t),\ \overline{C}_2(t),\ 1+\overline{n}(t+1))$ for all $t \geq 1$

and

(b) $C_2(0) \geq \overline{C}_2(0)$

with at least one strict inequality.

This optimality definition maximizes the welfare of the representative consumer of each generation and ignores the size of the population in the welfare criterion. If $n(t) > \overline{n}(t)$ for some t, the economy with a larger population size would have an inefficient allocation.

Hence, the optimal allocation, " – ", has less people, and some agents are, trivially, worse-off since they don't exist. Usually the two allocations will be Pareto non-comparable. We view this implication as an important drawback to the above definition of optimality when population size is subject to choice. However, this definition follows the conventional approach (see Koopmans, 1965), and we adopt it even though it permits a particular allocation to be inefficient only because there are too many children.

In some of our analysis we focus on stationary allocations and thus neglect all generations prior to the steady state. For the steady state, an optimality definition would consist of condition (a) above as a strict inequality for generations that are at the stationary allocation. This optimality definition is identical to a social planner problem that we formulate and analyze in the next section.

The Decentralized Economy

In this section we describe the equilibrium of the competitive market when population is subject to choice. Each young of generation t saves $K(t+1)$ units of the good for his production at time $t+1$. He holds $h(t)$ units of the paper asset whose price in terms of the consumption good at time t is $P(t)$. At time t the individual supplies exactly one unit of labor and receives a wage of $W(t)$ in terms of consumption goods at time t. At time $t+1$ each old person hires $L(t+1)$ units of labor for production using his accumulated capital $K(t+1)$, and consumes his capital rent and the non-depreciated quantity of capital.

Formally the problem of a young person of generation t who is born at time $t-1$ is to maximize, for all $t \geq 1$,

$$(2.3) \qquad V(C_1(t),\ C_2(t),\ 1 + n(t+1))$$

subject to

$$(2.4) \qquad C_1(t) = W(t) - K(t+1) - P(t)\,h(t) - e(1 + n(t+1))$$

$$(2.5) \qquad C_2(t) = F(K(t+1),\ L(t+1)) - W(t+1)\,L(t+1) + (1 - \delta)\,K(t+1) + P(t+1)\,h(t)$$

by choice of $C_1(t)$, $C_2(t)$, $K(t+1)$, $h(t)$, $n(t+1)$ and $L(t+1)$.

The first order necessary conditions for a maximum are

$$(2.6) \quad -V_1 + V_2[F_1(K(t+1),\ L(t+1)) + (1 - \delta)] \leqq 0 \quad \text{with} = \text{ if } K(t+1) > 0$$

$$(2.7) \quad -V_1 P(t) + V_2\,P(t+1) \leqq 0 \quad \text{with} = \text{ if } h(t) > 0$$

$$(2.8) \quad -V_1\,e + V_3 \leqq 0 \quad \text{with} = \text{ if } n(t+1) > -1$$

$$(2.9) \quad V_2[F_2\,(K(t+1),\ L(t+1)) - W(t+1))] \leqq 0 \quad \text{with} = \text{ if } L(t+1) > 0$$

Observe that (2.9) implies that the real wage is equal to the marginal product of labor, and that, if $K(t+1)$, $h(t)$ and $P(t)$ are positive, the net rate of return on capital is equal to that on the paper asset.

Definition

A perfect-foresight competitive equilibrium consists of non-negative values of $W(t)$, $h(t+1)$, $k(t+1)$, $P(t)$, $h(t)$ and $L(t+1)$ for all $t \geq 1$, that are consistent with the necessary conditions for the maximum problem of the young (2.3) and

(i) $N(t+1) = (1+n(t+1))\, N(t)$ is the law of motion for population

(ii) $L(t)\, N(t-1) = N(t)$ is the equilibrium demand and supply of labor

(iii) $N(t)\, h(t) = H$ is the equilibrium demand and supply of the paper asset.

A monetary equolibrium (ME) satisfies the above definition with $P(t) > 0$ for all $t \geq 1$. A non-monetary equilibrium (NME) satisfies the above definition with $P(t) = 0$ for all $t \geq 1$.[5] Hence, in a NME condition (iii) is not necessary.

A stationary equilibrium (SE) satisfies the above definition where we ignore the index t on all variables besides $P(t)$, $h(t)$ and $N(t)$ and we require that $P(t)\, h(t) = g$ for all $t \geq 1$. Then a stationary monetary equilibrium (SME) is a SE where $g > 0$ and a stationary non-monetary equilibrium (SNME) is a SE where $g = 0$.

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[5] If $P(T) > 0$ for some $T > 1$, then $P(t) > 0$ for any $t < T$.

Observe that in SE, C_1, C_2 and n are unchanged for each member of each generation. Then, the first order necessary conditions for SE with k, $1+n > 0$ can be written as

$$(2.10) \qquad -V_1 + V_2[f'(k) + (1 - \delta)] = 0$$

$$(2.11) \qquad -V_1 P(t) + V_2 P(t+1) \leq 0 \qquad \text{with} = \text{ if } \quad g > 0 .$$

$$(2.12) \qquad -V_1 e + V_3 = 0$$

$$(2.13) \qquad f(k) - k f'(k) - W = 0$$

Let the supercript "c" correspond to the values of the real stationary variables W, k, n, C_2 and C_1 for the competitive solution of (2.10) - (2.13). Furthermore let "$\bar{c}$" and "$\hat{c}$" correspond to the SME and the SNME, respectively. Observe that for SME we require that (2.11) holds in equality. It is evident that in SME conditions (i) and (iii) in the definition of competitive equilibrium together with $g = P(t)\,h(t)$ imply that $\frac{P(t+1)}{P(t)} = 1 + \bar{n}^c$. Hence, (2.10) and (2.11) imply that for the SME the economy is on the Golden Rule such that

$$(2.14) \qquad f'(\bar{k}^c) = \bar{n}^c + \delta$$

and

$$(2.15) \qquad \frac{\bar{V}_1^c}{\bar{V}_2^c} = 1 + \bar{n}^c$$

As such we show that similar to the case where fertility is exogenous, the competitive monetary equilibrium may converge to the Golden Rule where the net marginal product of capital per-capita is equal to the rate of population growth. In addition, a non-monetary equilibrium may converge to a stationary path where the net marginal product of capital per-capita exceeds the rate of population growth. Hence, the endogeneity of population growth does not alter these characteristics of competitive growth models.

Having described the structure of the model, two areas of analysis may be explored. One avenue is the Social Planner solution to the allocation problem in this economy and the comparison to the decentralized economy. A second area of analysis is to explore the positive features of the model. In particular, empirical regularities with respect to economic growth, population growth, and per-capita income across countries and over time should be explicable in the context of the model. We discuss these issues in turn.

3. The Social Planner and the Competitive Allocation

It is the tradition in growth models to compare the Social Planner solution to that of the competitive equilibrium. In growth models of the Cass-Koopmans type in which agents are assumed to be infinitely lived, there is a straightforward equivalence between the two allocations. In models where agents have finite horizons but the economy exists forever, there always exists the problem of welfare tradeoffs across members of different generations. Therefore, unlike the Cass-Koopmans model, here it is not clear what we should consider as the problem of the Social Planner. Furthermore, there is no equivalence between Pareto optimality in this model (see definition in section 2) and the particular Social Planner problem that we discuss in this section.[6] Here we follow Diamond (1965) and Samuelson (1975) who characterize the planner problem as a choice between stationary allocations where each member of each generation has an identical utility level.

Samuelson (1975) solved for the stationary planner allocation of the economy described in section 2 ignoring the fertility choice, i.e., maximizing (2.1) with respect to C_1, C_2 and k excluding the third argument of (2.1), subject to (2.2) at the stationary allocation with $e = 0$. Given this allocation, Samuelson then described the necessary conditions for choosing the level of fertility that maximized the utility of the representative agent on the stationary path. Samuelson denoted that stationary allocation the Golden Golden Rule of population growth and capital accumulation. Deardorff (1976)

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[6] Although the planner allocation is Pareto optimal, it does not necessarily characterize all optimal allocations.

pointed out that if $f(k)$ is unbounded, as is the case for a Cobb-Douglas technology, the necessary conditions may correspond to a minimum welfare level. The fact that there always exists a large k that dominates an interior solution is easily seen by setting $n = -\delta$ in the steady state version of (2.2) ($e = 0$), for then the consumption possibility frontier can be expanded indefinitely in the steady state with larger and larger levels of savings. Thus, the conclusion of the Solow model that population growth should be as small as possible may not be different in the Diamond model, contrary to Samuelson's conjecture.

The rationale for including fertility as part of the individual choice calculus as is done here is based on the view that optimal fertility can only be a meaningful concept where there exists a mechanism to achieve such a goal. Notice that in (2.2), setting $n(t+1) = -\delta$ ($e \neq 0$) also leads to an unbounded consumption possibilities frontier. It must therefore be the case that the marginal utility of children becomes infinite at a level of n larger than $-\delta$, which given condition (ii) in the specification of $V(\cdot)$ in section 2 implies that $\varepsilon > 1 - \delta$.

The maximization problem for the social planner may be written as follows:

(3.1) $$\text{maximize } V(C_1, C_2, 1 + n)$$

subject to

(3.2) $$C_1 = W - T - (1 + n)k - e(1 + n)$$

(3.3) $$C_2 = (1 + n)\, f(k) - (1 + n)(W - T) + (1 + n)(1 - \delta)k$$

$$k \geq 0\ , \quad n \geq -1$$

Notice that (3.2) and (3.3) together are a particular decomposition of the feasibili condition (2.2) at the stationary allocation. W may be considered a wage payment to labor and T a lump-sum transfer between generations. We employ this decomposition in order to facilitate comparisons between the planner's solution and the decentralized economy solution as it is described in section 2. In the framework as written, for given W the transfer (T) may be either negative (a tax on the old) or positive (a subsidy to the old). Later, we will consider a modification of the planner's problem in which $T \geq 0$ is imposed for a particular W ; otherwise $W - T$ could be replaced by a single variable or we could combine (3.2) and (3.3) into a single constraint, the feasibility constraint given by (2.2) at the steady state.

Now, for any $W > 0$ a planner stationary allocation (PSA) satisfies the following first order necessary conditions of problem (3.1), where we assume interior solutions for k, T and n.

(3.4) $$\frac{\partial V}{\partial k} = (1 + n)[-V_1 + V_2(1 - \delta + f'(k))] = 0$$

(3.5) $$\frac{\partial V}{\partial T} = -V_1 + V_2(1 + n) = 0$$

(3.6) $$\frac{\partial V}{\partial n} = V_3 - V_1(e + k) + V_2(T - W + (1-\delta)k + f(k)) = 0$$

Given our assumptions about the utility function, it is easy to impose sufficient conditions to ensure that the interior solution to (3.4) - (3.6) corresponds to a maximum. Below we analyze the case of Cobb-Douglas utility and production functions with full depreciation ($\delta = 1$) and show the necessary and sufficient conditions for the existence of an interior solution.[7]

In the Diamond framework it has been established that if the stationary competitive equilibrium is on the Golden Rule (monetary equilibrium) then this equilibrium is identical to the planner allocation. To see that, recall that the Diamond model is identical to that of this paper if fertility is not a choice, $n(t) \equiv n$ and $e \equiv 0$. Then, (3.4) and (3.5) are identical to (2.14) and (2.15) where the latter corresponds to all the necessary conditions that satisfy the stationary monetary equilibrium given that fertility is exogenous. This result does not follow when fertility is endogenous. Proposition 1 below establishes that the competitive Golden Rule is not identical to the social planner Golden Rule allocation.

Proposition 1: The Stationary Monetary Equilibrium (SME) is not the Planner Stationary Allocation (PSA).

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[7] As such, the problem that Deardorff mentioned can be easily eliminated in this model by assumptions on the utility and the production functions.

Proof: To prove the above proposition we demonstrate that the SME is not the same as the PSA by comparing the first order necessary conditions at the stationary allocation for problems (2.3) and (3.1). Here the competitive economy is on the golden rule. Hence, equations (2.10) and (2.11) are identical to (3.4) and (3.5), respectively. Evaluating the PSA using the SME allocation we get $W^p = f(\bar{k}^c) - (\bar{n}^c + \delta)\bar{k}^c$. Then, $T = \bar{g}^c$ from (3.2), (3.3) and (2.4) and (2.5) at the stationary state. Divide (3.6) by $\bar{V}_1^c$ and using (2.10) - **(2.13) we get that the SME and the PSA are identical if and only if**

$$-\bar{k}^c + \frac{\bar{g}^c}{1+\bar{n}^c} + \frac{-f(\bar{k}^c)+(\bar{n}^c+\delta)\bar{k}^c + f(\bar{k}^c) + (1-\delta)\bar{k}^c}{1+\bar{n}^c} = 0 \text{ , that is } \frac{\bar{g}^c}{1+\bar{n}^c} = 0 \text{ .}$$

This contradicts $\bar{g}^c > 0$ for the SME, and at this allocation equation (3.6) is equal to

$$\text{(3.7)} \qquad \frac{\bar{V}_3^c}{\bar{V}_1^c} - e + \frac{\bar{g}^c}{1+\bar{n}^c} > 0 \text{ ,}$$

where the inequality holds since the first two terms sum to zero from (2.12). Q.E.D.

Observe that there exists a "knife edge" case where $\bar{g}^c = 0$ such that **the economy is on the Golden Rule and the stationary non-monetary equilibrium (SNME) and the PSA are the same. Furthermore, at the SME the transfer (T) of the planner is positive, and in the absence of endogenous fertility the PSA and the SME are identical.**

In our model, the SNME is characterized by $f'(\hat{k}^c) > \hat{n}^c + \delta$, $(\tilde{g}^c = 0)$, while $f'(k^P) = n^P + \delta$ characterizes the PSA. Thus, the SNME does not have the same solution as the PSA. Observe that the Diamond framework exhibits the same result. Furthermore, if we impose on the planner problem that $W^P = \hat{W}^c$ and the transfer T is not permitted to be negative, i.e., no tax on the old, (3.5) may hold as strict inequality and then $T = 0$. The relationship between the SNME and the PSA is summarized by the following proposition.

Proposition 2: If a SNME exists, it satisfies the PSA subject to a non-negativity constraint on T and $W^P = \hat{W}^c$.

Proof: At the SNME $\hat{g}^c = 0$. Set $W^P = f(\hat{k}^c) - f'(\hat{k}^c)\hat{k}^c$ at the PSA and observe that $T = 0$, since the constraints (3.2) and (3.3) are satisfied at the SNME. From (2.10) and (2.11) we get that

$$\frac{\hat{V}_1^c}{\hat{V}_2^c} = 1 - \delta + f'(\hat{k}^c) \geq 1 + \hat{n}^c$$

Then conditions (3.4) and (3.5) are satisfied.[8] Divide (3.6) by $\hat{V}_1^c$ we get

$$\frac{\hat{V}_3^c}{\hat{V}_1^c} - e - \hat{k}^c + \frac{1}{1-\delta + f'(\hat{k}^c)} [-f(\hat{k}^c) + f'(\hat{k}^c)\hat{k}^c + (1-\delta)\hat{k}^c + f(\hat{k}^c)] = \frac{\hat{V}_3^c}{\hat{V}_1^c} - e = 0 .$$

Hence, the SNME satisfies (3.4) - (3.6) and is equivalent to the PSA. Q.E.D.

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[8] Condition (3.5) is statisfied as an inequality since $T = 0$.

Notice that the constraint that the transfer T be positive does not affect Proposition 1. Moreover, we have not demonstrated that the planner's solution with the restriction that $T \geq 0$ implies satisfaction of the optimality criterion that would take into account the welfare of generations that are not on the stationary path. Thus, Proposition 2 does not imply that the SNME is optimal.

Propositions 1 and 2 imply that under certain restrictions on the social planner, that allocation and the stationary competitive equilibrium differ only when the Golden Rule is satisfied. The monetary equilibrium is characterized by positive transfers across generations while in the non-monetary economy there is <u>no</u> direct trade between different generations besides that which arises from the existence of the labor market. It is interesting to note that the labor market which also connects the different generations does not imply any divergence between the planner and the competitive markets. Next we suggest an intervention policy in the asset market which supports a decentralized equilibrium that is identical to the planner at the steady state. Additionally, we show that fertility in the competitive economy (SME) is higher than the social planner's optimum fertility level.

An Intervention Policy

The potential non-optimality of competitive markets in overlapping generations models, the over-accumulation of capital, serves as a justification for a social security program (Samuelson, 1958), as an explanation for valued fiat money issued by the government (Wallace, 1980) and as a justification for a welfare improving role of national debt (Diamond, 1965). All of these suggested programs remove the inefficiency of the competitive allocation by introducing a costless feasible mechanism that transfers goods across generations with a rate of return that is equal to the market rate, i.e., the population growth rate. Here, we have abstracted from the standard type of inefficiency by introducing a paper asset that is identical to fiat money. Furthermore, when fertility is subject to choice, as we have already shown, the equivalence of planner and competitive allocations cannot be achieved by introducing a fixed stock of fiat money or a pay-as-you-go social security program. In fact, the program must tie the return on the saving-transfer decision to the chosen fertility rate, n.

Consider a voluntary self-financing social security program administered by the government that promises a return on savings during the working period that is identical to the individual specific fertility rate, n. It is straightforward to verify that this program will give rise to a SE that is identical to the PSA. To see this, rewrite (2.4) and (2.5) as

$$(3.8)\quad C_1(t) = W(t) - K(t+1) - g(t) - e(1 + n(t+1))$$

$$(3.9)\quad C_2(t) = F(K(t+1), L(t+1)) - W(t+1)L(t+1) + (1-\delta)K(t+1) + (1+n(t+1))\,g(t)$$

where $g(t)$ now represents the voluntary contribution to the social security program by each of the young of generation t. A young person of generation t maximizes (2.3) subject to (3.8) and (3.9). The first-order necessary conditions for SE are

$$(3.10)\quad -V_1 + V_2[f'(k) + 1 - \delta] = 0$$

$$(3.11)\quad -V_1 + V_2[1 + n] \leq 0 \quad \text{with} = \text{if } g > 0$$

$$(3.12)\quad -V_1 e + V_2 g + V_3 = 0$$

$$(3.13)\quad f(k) - kf'(k) - W = 0$$

Note that the SME now differs from the PSA if and only if (3.12) differs from (3.6) given that (3.13) is satisfied. However, substituting the solution to (3.10)- (3.13) into (3.6) with $T = g$, we get

$$V_3 - V_1(e + k) + V_2(g + k(1 - \delta) + f'(k))$$

$$= V_3 - V_1 e + V_2 g + V_2 k(1 - \delta + f'(k) - V_1 k$$

$$= V_3 - V_1 e + V_2 g$$

which is identical to (3.12). Thus, the social security program induces an equivalence between the SME and the PSA.

An alternative policy would be to place a tax or subsidy on children financed by a lump-sum transfer between generations. Thus e in (3.8) would be replaced by $e + \tau$ where τ is the per child tax or subsidy and the right hand side of (3.9) would be augmented by $\tau(1 + n)^2$. Such a program would be identical in impact to the social security program previously discussed as can be seen by setting $g(t) = \tau(1 + n)$. An important difference, however, is that the optimal τ would need to be known by the planner since such a program would not be voluntary.

We have demonstrated that the SME is not identical to the planner solution and we have proposed a social security program that induces these allocations to be the same. The following proposition asserts that the SME is characterized by too few children and too much savings. Hence, our suggested program will increase fertility and reduce capital accumulation.

Proposition 3: If the utility function is such that second order conditions for the PSA hold globally, then $n^P > \overline{n}^c$, i.e., the SME has a lower stationary rate of population growth than the PSA.

Proof: See Appendix A.

The fact that the PSA has a higher stationary stock of capital per-capita, i.e., $k^P < \overline{k}^c$, is immediate from the golden rule condition that holds for both allocations. This result is due to the fact that the planner takes into consideration the direct effect of the fertility rate on the rate of interest while each young takes the rate of return on savings parametrically. As such the planner perceives a benefit from population growth that has no counterpart in the competitive economy.

In the following section we consider the popular example of a log additive utility function and a Cobb-Douglas production function in order to show analytically the results of this section.

4. An Example

In this section we analyze the monetary and non-monetary equilibria and the planner problem for the case of a log additive utility function and Cobb-Douglas production function. In this example the stationary equilibria of monetary and non-monetary economies arise immediately, that is, after only one period. We provide analytical solutions for the stationary competitive and planner allocations which can be used to demonstrate the validity of the propositions of the previous section.

The utility function is given by

$$V = \beta_1 \ell n\, C_1(t) + \beta_2 \ell n\, C_2(t) + \beta_3 \ell n\, (1 + n(t+1)) \tag{4.1}$$

and the production function by

$$f(k) = Ak^{\alpha} \quad . \tag{4.2}$$

Proceeding as in Section 2, assuming for convenience full depreciation of capital ($\delta = 1$), the first order necessary conditions for the maximization problem of the decentralized economy are

$$-\frac{\beta_1}{C_1(t)} + \frac{\beta_2}{C_2(t)} A\alpha\, k(t+1)^{\alpha-1} = 0 \tag{4.3}$$

$$-\frac{\beta_1}{C_1(t)} P(t) + \frac{\beta_2}{C_2(t)} P(t+1) \leq 0 \tag{4.4}$$

$$-\frac{\beta_1 e}{C_1(t)} + \frac{\beta_3}{1 + n(t+1)} = 0 \tag{4.5}$$

For a monetary equilibrium (ME) (4.4) is a strict equality while for a NME it is a strict inequality. Using (2.4) - (2.5) and (4.3) - (4.5), it can be shown that the equilibrium paths for capital per capita and the price of the paper asset must satisfy

$$(4.6) \qquad k(t+1)\left(\frac{A\beta_3 s(1-\alpha)k(t)^{\alpha}}{\beta_2}\right) + P(t)\frac{H}{N(t)} = s\,A(1-\alpha)k(t)$$

The right hand side of (4.6) is merely the savings of generation t with $s = \frac{\beta_2}{\beta_1+\beta_2+\beta_3}$ the fixed marginal propensity to save from income (W_t) which is equal to $A(1-\alpha)k(t)^{\alpha}$. The left hand side of (4.6) shows the division of total savings between capital and the paper asset.

Setting $P(t) = 0$ in (4.6) yields the equilibrium level of capital per capita for the NME for $t \geq 1$, since $k(1)$ is given. Thus,

$$(4.7) \qquad \hat{k}^c(t+1) = \frac{\beta_2}{\beta_3}\,e$$

Solving for the equilibrium levels of fertility, and first and second period consumption yields for $t \geq 1$

$$(4.8) \qquad 1 + \hat{n}^c(t+1) = \frac{A(1-\alpha)\beta_3 s}{\beta_2}\,k(t)^{\alpha}$$

$$(4.9) \qquad \hat{c}_1^c(t) = \frac{A(1-\alpha)\beta_1 s}{\beta_2}\,k(t)^{\alpha}$$

$$\hat{C}_2^c(t) = \frac{\beta_2}{\beta_3} A\, k(t+1)^{\alpha-1}\, \hat{C}_1^c(t) \tag{4.10}$$

It is thus obvious that the SNME is reached immediately since both $k(t+1)$ and $n(t+1)$ are independent of t, for all $t \geq 1$.

For a monetary economy P_t is positive. From inspection of (4.6) it is readily seen that if $P(t)$ takes the form $\gamma k(t)^{\alpha} \frac{N(t)}{H}$, a stationary value of k is obtained. It is easy to show that there exists a γ, which is only a function of the underlying parameters of the model, such that the stationary equilibrium is obtained for $t \geq 1$.[9] Without presenting the algebra, the solutions for $t \geq 1$ are:

$$\bar{k}^c(t+1) = \frac{\beta_2 \alpha e}{s\beta_3(1-\alpha)} \tag{4.11}$$

$$1 + \bar{n}^c(t+1) = A\alpha\, \bar{k}^c(t)^{\alpha-1} \tag{4.12}$$

$$\bar{C}_1^c(t) = A\alpha\, e\, \frac{\beta_1}{\beta_3}\, \bar{k}^c(t)^{\alpha-1} \tag{4.13}$$

---

[9] $\gamma = \dfrac{A(\beta_2 - \alpha(\beta_1 + 2\beta_2 + \beta_3)}{\beta_1 + \beta_2 + \beta_3}$. Note also that for a monetary equilibrium to exist it is necessary that $\beta_2 - \alpha(\beta_1 + 2\beta_2 + \beta_3) > 0$.

$$(4.14) \qquad \bar{C}_2^c\,(t) = A\alpha \frac{\beta_2}{\beta_1} k(t+1)^{\alpha-1}\, \bar{C}_1^c(t) \; .$$

where k(1) and n(1) are given as initial conditions.

Note that in the Diamond framework the same example gives rise to a smooth path of infinitely many periods of convergence to the steady state. However, the endogeneity of fertility enables the economy to restore the steady state in one adjustment period.

The planner stationary allocation for this example can be obtained by solving (3.4) - (3.6) together with the feasibility constraint at the steady sta $C_1 + \frac{C_2}{1+n} + e(1 + n) = f(k) - (1 + n)k$, and is given by[10]

$$(4.15) \qquad k^P = \frac{(\beta_1+2\beta_2+\beta_3)\ \alpha e}{(\beta_2+\beta_3) - \alpha(\beta_1+2\beta_2+\beta_3)}$$

$$(4.16) \qquad 1 + n^P = A\alpha\ (k^P)^{\alpha-1}$$

$$(4.17) \qquad c_1^P = \left(\frac{\beta_1}{\beta_2+\beta_3}\right)(k^P + e)(1 + n^P)$$

$$(4.18) \qquad c_2^P = \frac{\beta_2}{\beta_1}\, c_1^P(1 + n^P)$$

---

[10] Note that for a planner's solution to exist it is necessary that

$(\beta_2 + \beta_3) - \alpha(\beta_1 + 2\beta_2 + \beta_3) > 0$.

Given these solutions the propositions of section 3 are easily demonstrated. Comparing (4.11) - (4.14) to (4.15) - (4.18), it is readily seen that the stationary monetary equilibrium is not the same as the planner allocation (Proposition 1). Further, the number of children is less in the monetary equilibrium, i.e., $1 + \overline{n}^c < 1 + n^p$, as stated in Proposition 3. To see that it is only necessary to show that $\overline{k}^c > k^p$ since the golden rule holds in both cases, i.e., (4.12) and (4.16) are the same. The difference between $\overline{k}^c$ (4.12) and k^p (4.15) is positive if

$$\frac{\beta_2 - \alpha(\beta_1 + 2\beta_2 + \beta_3)}{\beta_3(1 - \alpha)} + \frac{\alpha}{(1 - \alpha)} + s > 0$$

This is true if $\beta_2 - \alpha(\beta_1 + 2\beta_2 + \beta_3) > 0$, which as noted is the condition for a monetary equilibrium to exist. As observed in the previous section, the two allocations differ because the return on the paper asset is not perceived by economic agents in competitive equilibrium as being related to their own fertility choice. Any single individual does not take into account the impact of an additional child on the rate of return to the paper asset from which all of the old benefit.

The demonstration of Proposition 2 is also straightforward. Basically, it must be shown that the SNME satisfies (3.6), where it is recognized that $T = 0$ at the SNME and W is set at the competitive wage. Now (3.6) with V and $f(k)$ given by (4.1) and (4.2) is equal to

$$\frac{\beta_3}{1+\hat{n}^c} - \frac{\beta_1}{\hat{C}_1^c}(e + \hat{k}^c) + \frac{\beta_2}{\hat{C}_2^c} \cdot \frac{\hat{C}_2^c}{1+n}$$

which upon substituting for the SNME as given by (4.7) - (4.10) reduce to zero.

Proposition 2, which states that the SNME will be equivalent to the planner allocation if the planner is restricted to pay the competitive wage and not permitted to tax the old, is thus seen to hold if it is also recognized that under these restrictions the golden rule is not a necessary condition for the planner problem.

In order to complete the comparison between the planner and the competitive allocation it would be of interest to investigate whether the planner allocation is Pareto Superior to the competitive allocation using the optimality definition in section 2. If the planner allocation dominates the competitive allocation then the competitive equilibrium is obviously non-optimal. However, if the planner allocation does not dominate the competitive allocation i.e., if in the example of this section the initial old and/or the first generation are worse off in the PSA, then it might reasonably be conjectured that the competitive equilibrium is Pareto optimal.[11]

[11] We have neither been able to find a numerical example of planner dominance nor a proof that the allocations are non-comparable.

5. Fertility and Economic Growth

In this section we consider the extent to which the implications of the overlapping generations growth model with endogenous fertility as previously presented conforms to cross-country and times series observations on fertility and economic growth. We show, using examples, that in the steady state fertility and income per-capita may be negatively related even though children are normal goods and there is no child quality component, (Becker and Lewis, 1973). Competitive equilibrium conditions alone are sufficient to generate this result. We further show that in order to generate declining fertility with increasing per-capita income along the equilibrium path as recently observed for developed countries, additional assumptions are required. In particular, an extended model is presented in which the cost of children is made a function of the wage rate (endogenous) incorporating therefore a time cost to rearing children (Becker, 1960, Willis, 1973, Razin and Ben-Zion, 1975). Simulations of that model which can replicate the observed phenomenon are presented.

Consider first the steady state relationship between fertility and income per capita. Assume that cross-country differences are best approximated by steady state comparisons or that the paths to the steady state are sufficiently smooth so as to have the same properties. In the case of a stationary monetary equilibrium, the golden rule condition given by (2.14) must be satisfied. It is immediate that, except for production function differences (or (differences in δ), economies with high capital per capita must have low fertility, and this results solely from the golden rule competitive equilibrium condition. For example, economies with a higher cost of children (e) will have fewer children and higher income per capita even though, at the individual level savings could be independent of e. This is easily seen from (2.10) and (2.11) in which $\frac{p(t+1)}{p(t)}$ is set equal to some constant that as far as the individual is concerned is unrelated to the population growth rate. Then, $f'(k) = \frac{p(t+1)}{p(t)}$ can be solved for k in terms of the given rate of return to the paper asset, which for the individual is independent of the cost of children. There is, thus, an important distinction between partial and general equilibrium results in this case.

In the stationary non-monetary equilibrium, the golden rule no longer is satisfied, and it is, therefore, not clear how fertility will be related to per-capita income in general. For the log additive utility function, the steady state solution for capital per capita and for fertility are given by (4.7) and (4.8). It is still true that k and n are inversely related across economies differing only in the cost of children. However, in constrast to the SME, no general results emerge when preference parameters differ.

These results illustrate potential explanations for cross-sectional differences in income and fertility. Interpretation of the time series observations requires an analysis of the non-steady state features of the model. It is not necessary to solve completely for the decentralized equilibrium in order to derive some basic results. In particular, if the utility function is contemporaneously separable, it is easy to see from (2.8) that fertility and first-period consumption must move together.[12] If first period consumption is normal, then fertility will also be positively correlated with income per capita along the equilibrium path. Since on the path $k(t)$ is predetermined at $t+1$, higher levels of $k(t)$ and therefore of per-capita income at t could only produce lower levels of fertility if children were inferior. Notice that this argument is applicable both to monetary and non-monetary equilibrium paths. In the steady state, however, since $k(t) = k(t+1)$, income per capita is predetermined and this difference between the steady state and the path is crucial.[13]

A simple extension of the model is to assume that raising children requires time, and therefore, that the cost of children is a function of the endogenously determined wage rate. In order to consider a multi-period path from any initial condition to a steady state, we assume an addilog utility function of the form[14]

$$V = C_1(t)^{\beta_1} + C_2^{\beta_2} + (1 + n(t+1))^{\beta_3} \tag{5.1}$$

The cost of children e is replaced in (2.4) and (2.5) by $e_0 + e_1W(t)$, where e_0 is the fixed cost of a child as in previous sections and e_1 is the fraction of time required for each child. We consider only the non-monetary equilibrium path.

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[12] Differentiating (2.8), $-V_{11}e\, dC_1(t) + V_{33}\, d(1+n(t+1)) = 0$ which implies that $dC_1(t) = \frac{V_{33}}{eV_{11}}\, d(1+n(t+1))$.

[13] If leisure is an argument in the utility function income would no longer be predetermined.

[14] Recall that the log additive utility function yields an immediate steady state.

Table 1 reports several simulations of the model for alternative values of e_0 and e_1. It turns out that the difference between β_3 and β_2 is important to the shapes of the equilibrium paths so that β_3 is also varied in some of the simulations. The first two simulations assume only a goods cost of children ($e_1 = 0$) for two values of β_3, one larger and one smaller than β_2. In case A where β_3 is larger than β_2, n and k cycle in the same direction, while in case B, where β_3 is less than β_2, n and k move in the same direction which depends upon the initial value of k. In case C, e_0 is lowered and e_1, is raised and the propensity for n and k to move in opposite directions increases. In this case capital per worker increases throughout the equilibrium path while fertility first increases, reaches a plateau and then decreases until the steady state is reached. This example seems to be broadly consistent with the observations on fertility and economic growth as they are summarized by Kuznets.[15] In the final example, case D, there is only a time cost to raising children and in that case capital per capita rises and fertility falls throughout. These examples demonstrate that almost any set of equilibrium paths of fertility and income per-capita, including cycles, are consistent with a perfect foresight competitive equilibrium. Hence, no general propositions are feasible for this model.

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[15] Obviously some elements of the demographic transition are still missing in this model, e.g. child mortality (Schultz, 1976), but can be incorporated in straightforward fashion.

Given the time cost of children in this model, as capital accumulates and the wage rate then rises, there is a substitution away from children and towards goods consumption. At the same time, as income per-capita grows, the demand for children increases given the structure of preferences in this example. The path that actually arises depends upon the relative strengths of these two effects which are directly related to the relative magnitudes of the fixed goods cost and the time cost of raising children, (Razin and Ben-Zion, 1975).[16] Having characterized the various alternative equilibrium trends in a non-monetary economy, it does not seem that the existence of a valued paper asset would alter the general results.[17]

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[16] This can be shown for any separable utility function. Differentiating (2.8) with e replaced by $e_0 + e_1W(t)$, and using (2.9), yields

$$\frac{d(1 + n(t+1))}{dk(t)} = \frac{-k(t)f''(k(t))}{V_{33}} \left\{ e_1(1 + n(t))V_1 + (e_0 + e_1(1+n(t))W(t))V_{11} \right\}.$$

The first term inside the brackets corresponds to the substitution effect and the second to the income effect.

[17] However, the monetary path is not easily solved.

Table 1

Path Simulations : Non-Monetary Equilibrium

	Parameters				Period (t)					
	e_0	e_1	β_3		0	1	2	3	4	∞
A.	.9	0	.7	k (t)	.001	4.51	1.06	1.35	1.29	1.30
				1+n(t+1)	.022	4.86	2.02	2.33	2.28	2.28
B.	.3	0	.4	k (t)	.001	.618	.909	.930	.931	.932
				1+n(t+1)	.016	2.84	3.38	3.42	3.42	3.42
C.	.05	.15	.7	k (t)	.001	.055	.198	.401	.615	1.35
				1+n(t+1)	1.21	2.66	2.66	2.56	2.47	2.25
D.	0	.20	.7	k (t)	.001	.013	.079	.272	.647	6.24
				1+n(t+n)	3.49	2.90	2.42	2.05	1.78	1.05

$\beta_1 = \beta_2 = .5$ $\alpha = .5$ $\Lambda = 10$

6. Concluding Remarks

When population is endogenously determined within a neo-classical growth model, there exists a stationary population growth rate that is supported with stationary consumption per-capita. This steady state may or may not be on the Golden Rule; if it is on the Golden Rule, it does not correspond to the social planner stationary allocation. Moreover, the social planner Golden Rule allocation is characterized by a higher population growth rate and a lower level of capital per-capita in comparison to the competitive Golden Rule allocation.

The competitive perfect-foresight equilibrium growth model we have presented was shown to generate patterns of population and income per-capita growth that are consistent with recent time-series evidence on developed countries and with international cross-sectional evidence. An important, yet in our model omitted, ingredient to population change, particularly in less developed countries, is that of infant and child mortality. If birth rates respond to exogenous mortality of children or if their mortality is itself part of the household decision process, then the model as formulated does not capture the complete story. In particular, it has been argued that the demographic transition from high fertility and high mortality environments to low fertility and low mortality environments is due in part to (exogenous) permanent declines in mortality. A natural extension of our framework would include child mortality in the model in part to ascertain whether the observed pattern in income per-capita , fertility and mortality can be replicated.

Given our conclusion that the competitive stationary equilibrium has fewer children than the planner allocation, one might wonder whether it is possible in growth models with endogenous fertility to generate a competitive equilibrium with "too many" children. The main justification for the view that the world is, in this sense, overpopulated seems to stem from the Malthusian argument concerning fixed factors of production (land) which leads to decreasing consumption per-capita with increasing population. In fact, if one includes in our model a fixed factor, then it is easy to demonstrate that there does not exist a stationary allocation where capital per-capita is constant, except when population is constant.[18] Hence, in a stationary allocation, capital per-capita must grow to offset diminishing returns, i.e., it is required that there be land augmenting technical change, which is feasible only if there is not a high rate of depreciation. Most studies assume an exogenous rate of land augmenting technical progress which is equivalent to assuming growth without investment. In particular, if there is no capital in the model, it follows immediately that the competitive equilibrium leads to the Malthustian prediction for consumption per-capita. However, analyzing the optima and equilibria of economies with a fixed factor of production as suggested above is beyond the scope of this paper. This appears to us to be an important issue, although it is not clear that our main results would be altered.

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[18] Let the constant returns to scale aggregate production function be $F(K(t), L(t), A)$, where A is fixed.

APPENDIX A

In this appendix, we prove Proposition 3 in the text.

Proof of Proposition 3:

Although the SME is not optimal, its solution set does satisfy (3.4) and (3.5). However (3.6) evaluated at the SME is too large since it reduces to

$$V_3 - V_1 e + V_2 g > V_3 - V_1 e \quad \text{since } g > 0 .$$

Moving from the SME to the PSA must leave (3.4) and (3.5) unchanged but must reduce (3.6), where (3.4) - (3.5) are evaluated at the SME.

Thus, the following conditions, obtained by total differentiation of the necessary conditions for the PSA, must hold.

$$\text{(A.1)} \qquad \frac{\partial^2 V}{\partial k^2} dk + \frac{\partial^2 V}{\partial k \partial T} dT + \frac{\partial^2 V}{\partial k \partial n} dn = 0$$

$$\text{(A.2)} \qquad \frac{\partial^2 V}{\partial T \partial k} dk + \frac{\partial^2 V}{\partial T^2} dT + \frac{\partial^2 V}{\partial T \partial n} dn = 0$$

$$\text{(A.3)} \qquad \frac{\partial^2 V}{\partial n \partial k} dk + \frac{\partial^2 V}{\partial n \partial T} dT + \frac{\partial^2 V}{\partial n^2} dn < 0$$

where $\frac{\partial^2 V}{\partial x \partial y}$ is evaluated at the SME.

Solving (A.1) and (A.2) for dk and dT in terms of dn and substituting into (A.3) yields the following inequality expression,

$$\text{(A.4)} \qquad \frac{|A|}{|A_{22}|} \; dn \; < 0$$

where $|A| = \left|\frac{\partial V}{\partial x \partial y}\right|$ and where $|A_{22}|$ is the determinant of the matrix obtained by deleting the 2^{nd} row and 2^{nd} column. If the second-order conditions for the PSA hold at the **SME**, then $|A| < 0$ and $|A_{22}| > 0$. Thus, $dn > 0$, i.e., the **SME** is characterized by too few children. Since n and k must be inversely related, it must also be true that the **SME** is characterized by too much savings.

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