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GROWTH AND CAPITAL WEALTH CONCENTRATION

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## Introduction

Recently there is a well-intentioned concern, on the part of the academicians<sup>1)</sup> and practioners alike<sup>2)</sup>, that in the development process of a contemporary less developed countries, there may be a conflict between growth and FID-equity (i.e. family income distribution equity). A "conflict thesis" is, of course, not new. For historically, when the Western countries (e.g. England) went into the "modern growth epoch"<sup>3)</sup>, radical writers of the 19th century (e.g. Karl Marx)<sup>4)</sup> had already made a similar but more vehement protest against the unequal accumulation of capital wealth to the extent that families are stratified into a capital-owning (bourgeois) class  $C^H$  and a capital-less (proletarian) class  $C^L$ . This "class orientation" stresses an extreme and stylized form of inequality of capital ownership. This paper aims to approach the "conflict thesis"-- i.e. growth with or without equity -- analytically.

Any analytical framework of a "conflict thesis" will have to incorporate "macro" as well as "micro" economic variables. For "macro" variables, the "class-orientation" necessitates the postulation of labor ( $l$ ) and capital ( $k$ ) that receives, respectively, wage income ( $w$ ) and property income ( $\pi=rk$ ) where  $r$  is the "rate of return to capital". The national income  $z$ , with two values added components, (i.e.  $z=w+rk$ ) in turn leads to savings ( $s$ ) and consumption ( $c$ ) (i.e.  $z=c+s$ ). In a dynamic context,  $s$  leads to a larger capital stock  $k'$  in the next period ( $k'=s+k$ ) the rapidity of accumulation is described by the rate of growth of capital ( $\eta_k=s/k$ ). Any respectable "conflict thesis" almost certainly will have to deal with these macro variables.

When there are  $n$  families, every one of the macro variables  $x$  introduced above (in a lower case letter), can be disaggregated into a pattern (i.e. a vector, indicated by a corresponding upper case letter)  $X = (X_1, X_2, \dots, X_n)$  in the sense that  $x = X_1 + X_2 + \dots + X_n$ . Thus  $Z = W + rK = S + C$  and  $K' = K + S$ . In particular  $K = (K_1, K_2, \dots, K_n)$  ( $K' = (K'_1, K'_2, \dots, K'_n)$ ) is the capital-ownership pattern of this (the next) period while  $S = (S_1, S_2, \dots, S_n)$  is the family saving pattern. If  $I(\cdot)$  is an index of inequality (e.g. a Gini coefficient), the inequality of family saving  $I(S)$  seems to be a root cause of the difference between  $I(K)$  and  $I(K')$  (i.e. the changes of the inequality of capital ownership through time). The conflict thesis in this paper involves the "macro" capital growth rate  $\eta_k$  as well as the inequalities of the patterns of the micro variables.

The above suggests a number of models of inequality analysis. For example, the so called "factor component approach" to FID analysis<sup>5)</sup> is based on  $Z = W + rK$  where the wage income pattern  $W = (W_1, W_2, \dots, W_n)$  and the property income pattern  $\pi = rK = (rK_1, rK_2, \dots, rK_n)$  are two factor components of the FID pattern  $Z = (Z_1, Z_2, \dots, Z_n)$ . It is well known<sup>6)</sup> that  $I(rK) = I(K)$  (i.e. the inequality of the property income  $\pi = rK$  or the capital ownership pattern  $K$ ) is more unequally distributed than  $Z$  and hence constitute a primary cause of the inequality of  $Z$  (i.e.  $I(K) > I(Z)$ ). Similarly, based on  $Z = S + C$ , we shall show that the family saving pattern is usually more unequally distributed than  $Z$  (i.e.  $I(S) > I(Z)$ ). Thus, in any year, unequal capital ownership is the "cause" of FID inequality while an unequally distributed saving pattern  $S$  is the consequence.

However, a very unequally distributed savings pattern  $S$  will in turn worsen the capital ownership pattern ( $I(K') > I(K)$ ) over time leading to an "inequitable orientation". We shall prove that this will occur

when families in the capital owning class  $C^H$  have a higher "capital-sensitive"<sup>7)</sup> average propensity to save" than families in  $C^L$ . Furthermore, the "inequitable orientation" occurs at a pace proportional to the capital growth rate ( $\eta_k$ ) for the whole economy. Thus the more rapidly the economy grows (i.e. large  $\eta_k$ ) the faster the "equality of capital ownership" deteriorates which, in essence, quantifies the "conflict thesis".

Conscientiousness of growth with wealth concentration has led to policy recommendations that includes revolution, fulfillment of basic needs,<sup>8)</sup> and progressive income tax system. Let  $Z=(Z_1, Z_2, \dots, Z_n)$  ( $X=(X_1, X_2, \dots, X_n)$ ) be the income pattern for n-families before (after) tax that are related by an income tax schedule  $t(z)$  (i.e.  $X_1=Z_1-t(Z_1)$ ). We shall show that a model of income tax has precisely the same abstract (mathematical) properties of the other models (i.e. models based on  $Z=W+rK=S+C$ ) mentioned earlier. Thus a theorem which we shall prove for the income tax model (i.e. one that characterizes the progressive income tax system) can be used for the analysis of the "conflict thesis".

The difficulty of "inequality analysis" is largely due to the complexities in the manipulations of "degrees of inequalities". In section 1, we shall build the tools of analysis by introducing the abstract notion of a "conjugate pairs of transformation functions". Properties of these transformation functions will be defined in section 2 where "abstract" theorems will be proved. These theorems will be applied to the income tax problem and the other growth related problems in sections 3 and 4. The "conflict thesis" will be presented in the final section.

## Section 1 Inequality Under Transformation

Let  $R$  be the  $n$ -dimensional real space  $R = \{Z | Z = (Z_1, Z_2, \dots, Z_n)\}$ . A point  $Z \in R$  is, abstractly, an economic pattern (e.g. a FID pattern) of  $n$  families. Let  $T(Z)$  be a mapping of  $R$  into  $R$ . Suppose a real valued function  $t(z)$  is given we can construct a special type of mapping (i.e. an "indexable" mapping), according to the following definition:

Definition: A mapping  $T(Z)$  of  $R$  into  $R$  is indexable by the transformation function  $t(z)$  if for any  $Z \in R$ ,  $X = (X_1, X_2, \dots, X_n) = T(Z) = (t(Z_1), t(Z_2), \dots, t(Z_n))$ .

Let  $\mathcal{M}$  be the set of all indexable mappings and let  $\mathcal{F}$  be the set of all real-valued functions  $t(z)$ . There is a one to one correspondence between  $\mathcal{M}$  and  $\mathcal{F}$ . We shall use a lower case letter (e.g.  $w(z)$ ) to denote a member of  $\mathcal{F}$  and an upper case letter (e.g.  $W(Z)$ ) to denote the corresponding mapping. We have the following definition:

Definition: The conjugate mapping of any  $T(Z) \in \mathcal{M}$  is  $T^*(Z)$  with a transformation function  $t^*(z) = z - t(z)$

It is obvious that the pair  $T(Z)$  and  $T^*(Z)$  are conjugate mappings of each other (i.e.  $T^{**}(Z) = T(Z)$ ) in view of the symmetry

$$(1.1) \quad t^*(z) + t(z) = z$$

Thus  $\mathcal{M}$  is partitioned into distinct conjugating pairs  $T(Z)$  and  $T^*(Z)$  with a self-conjugating member  $t(z) = t^*(z) = .5z$ .

As a diagrammatic aid, the transformation function  $t(z)$  is represented by a transformation curve in diagram 1. A point  $Z \in R$ , represented on the horizontal axis, is transformed into  $X = T(Z) = (X_1, X_2, \dots, X_n)$  on the vertical axis. The conjugate transformation function  $t^*(z)$  is represented by

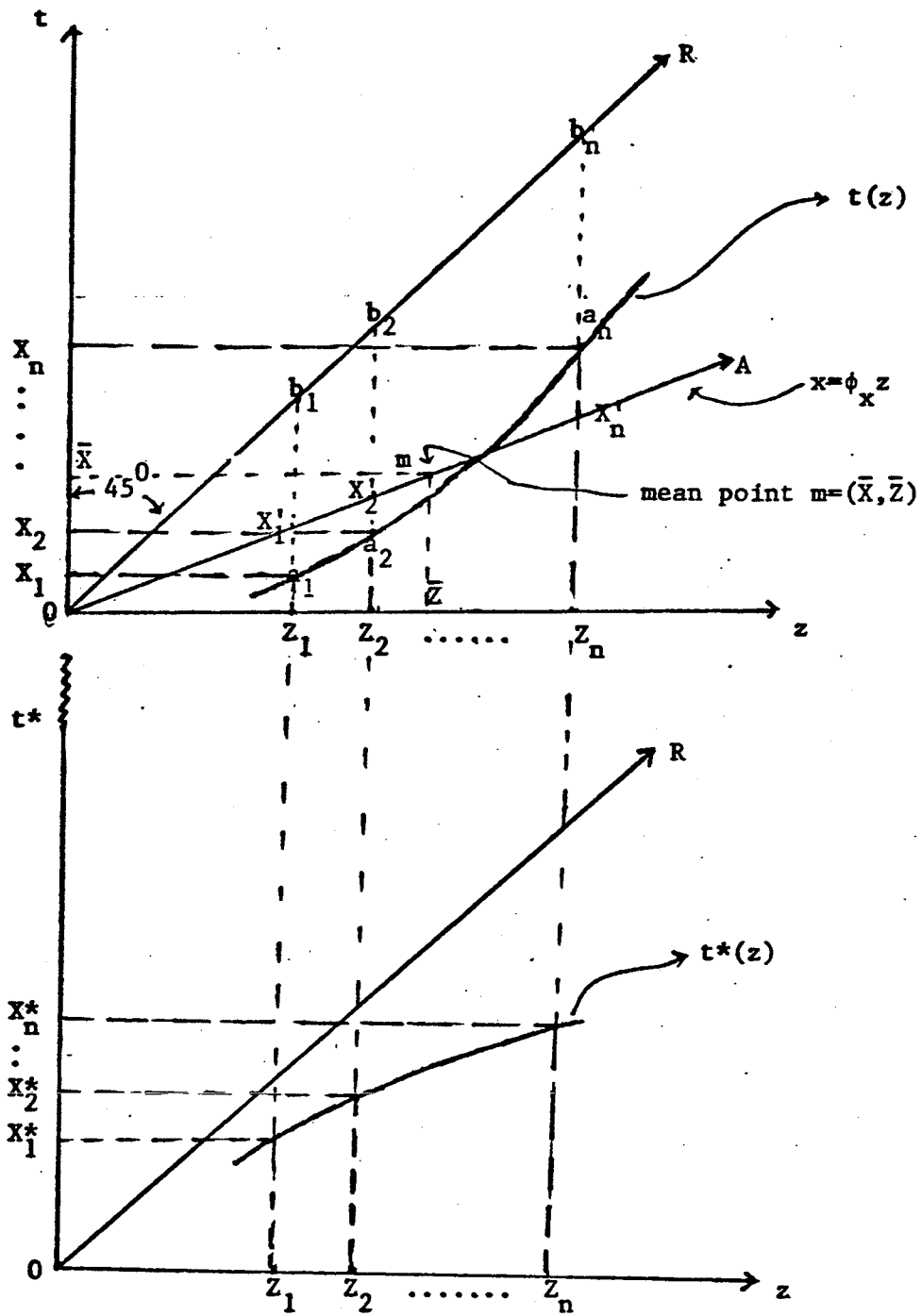


Diagram 1

the curve in the lower deck and reflects the vertical gaps (i.e.  $a_1 b_1$ ) between  $t(z)$  and the  $45^0$  line OR in the upper deck.

Let us denote the mean value of any  $W \in R$  by  $\bar{W}$ .

Definition: For any  $Z \in R$ , the average ratio of  $X=T(Z)$  is  $\phi_x = \bar{X}/\bar{Z}$

In diagram 1,  $\phi_x$  is the slope of the radial line OA passing through the mean point  $m=(\bar{X}, \bar{Z})$ . Thus when a mapping  $T(Z) \in \mathcal{M}$  is given, we have a triplet of patterns  $X=(X_1, X_2, \dots, X_n)=T(Z)$  and  $X^*=(X_1^*, X_2^*, \dots, X_n^*)=T^*(Z)$  for any  $Z \in R$  defining a pair of average ratios  $(\phi_x, \phi_{x^*})$ :

- (1.2) a)  $\phi_x = \bar{X}/\bar{Z} = \bar{T}(Z)/\bar{Z}$  and  $\phi_{x^*} = \bar{X}^*/\bar{Z} = \bar{T}^*(Z)/\bar{Z}$  satisfying  
 b)  $\phi_x + \phi_{x^*} = 1$  because  
 c)  $\bar{Z} = \bar{X} + \bar{X}^*$  by (1.1)

For any real valued function  $t(z) \in \mathcal{F}$ , a quadruplet of functions  $Q(t(z)) = ((t(z), t^*(z)), (T(Z), T^*(Z)))$  is defined. In economic applications, a triplet of variables  $(z, x, x^*)$  can form a deterministic "two-equation model"  $M = [z = x + x^*, x = t(z)]$  where  $z = x + x^*$  is an "accounting equation" and  $x = t(z)$  is a "behavior equation" when  $z$  is treated as an exogenous variable. The behavior equation  $x = t(z)$  automatically induces the quadruplets  $Q(t(z))$ . All the basic concepts of the section can be summarized in the following definition.

Definition: The behavior equation  $t(z)$  of a two-equation model

$M = [z = x + x^*, x = t(z)]$  induces the quadruplets:

(1.3)a)  $Q(t(z)) = ((t(z), t^*(z)), (T(Z), T^*(Z)))$

where  $t^*(z) (T^*(Z))$  is the conjugate function(mapping) of

$t(z) (T(Z))$ . Any  $Z = (Z_1, Z_2, \dots, Z_n) \in R$  induces a pair of

patterns:  $X = (X_1, X_2, \dots, X_n) = T(Z)$  and  $X^* = (X_1^*, X_2^*, \dots, X_n^*) = T^*(Z)$

satisfying

b)  $Z = X + X^*$

for which a triplet of mean values  $(\bar{Z}, \bar{X}, \bar{X}^*)$  determine two average ratios

c)  $\phi_x = \bar{X}/\bar{Z}$  and  $\phi_{x^*} = \bar{X}^*/\bar{Z}$

satisfying (1.2b,c). If  $I(\cdot)$  is any reasonable index of inequality, a triplet of pattern inequalities

d)  $I(Z), I(X), I(X^*)$

can be defined when  $Z \geq 0, X \geq 0$ , and  $X^* \geq 0$

As an illustration, let the triplet of economic variables be income before tax ( $z$ ), after tax ( $x^*$ ) and tax payment ( $x$ ) of a typical family that satisfies the accounting equation  $z = x + x^*$  in the two-equation model  $M = [z = x + x^*, t(z)]$  where the behavior equation  $t(z)$  is the income tax schedule. When a FID pattern  $Z$  is given exogeneously,  $X$  and  $X^*$  in 1.3b are, respectively, the tax burden pattern and the disposable income pattern of  $n$  families. The triplet  $(\bar{Z}, \bar{X}, \bar{X}^*)$  stand for average national income, tax payment and disposable income "per family" and hence  $\phi_x$  ( $\phi_{x^*}$ ) in (1.3c) stand for the average tax rate (average disposable income rate) for the whole economy.  $I(Z), I(X)$  and  $I(X^*)$  in (1.3d) stand for the degree of inequality of  $Z, X$  and  $X^*$  respectively. This model of income tax system will be studied in section 3 below where the progressiveness of the tax schedule  $t(z)$  will be defined (See introduction).

As another example, let the triplet of economic variables be income ( $z$ ), consumption ( $c$ ) and saving ( $s$ ) of families that satisfy the accounting equation  $z = s + c$  in the two-equation model:

(1.4)  $M_s = [z = s + c, s(z)]$

where the behavior equation  $s(z)$  is the "family saving function" and  $c = s^*(z) = z - s(z)$  is the family consumption function. When an FID



pattern  $Z \in R$  is given exogeneously, the triplet  $(\bar{Z}, \bar{S}, \bar{C} = \bar{S}^*)$  stand for per capita income ( $\bar{Z}$ ), saving ( $\bar{S}$ ) and consumption ( $\bar{C}$ ), and hence  $\phi_s = \bar{S}/\bar{Z}$  ( $\phi_c = \bar{C}/\bar{Z}$ ) is the average propensity to save (consume) of Keynes. The triplet of  $I(Z)$ ,  $I(C)$  and  $I(S)$  stand for the inequality of income, consumption and saving respectively. This model  $M_s$  will be used in the analysis of saving inequality in section 4 (see introduction). In this paper, any index of inequality  $I(\cdot)$  belonging to the Dalton Family<sup>9)</sup> will be referred to as a reasonable index. Using  $L(U) \geq L(V)$  to mean "U Lorenze Dominates V", it is well known:<sup>10)</sup>

$$(1.5) \quad I(U) < I(V) \text{ for any reasonable } I(\cdot) \text{ if } L(U) \geq L(V)$$

## Section 2 Reasonable Properties for $T(Z)$

In this section a number of reasonable properties for  $T(Z)$  or  $t(z)$  will be introduced by interpreting  $t(z)$  as the income tax schedule. We may wish to restrict the domain of mapping to  $\Omega = \{Z | Z \geq 0\} \subset R$ , the nonnegative orthant of  $R$ ; or to  $\Omega^0 = \{Z | Z_1 \leq Z_2 \leq \dots \leq Z_n\} \subset R$  the rank preserving subset of  $R$ . For any FID pattern  $Z$ , we can always reorder the family so that  $Z \in \Omega^0$ . The intersection of  $\Omega$  and  $\Omega^0$  will be denoted by  $\hat{\Omega} = \Omega \cap \Omega^0 \subset R$  which is the nonnegative rank preserving subset of  $\Omega^0$ .

Referring to diagram 1, the income tax schedule should be nonnegative and lie below the  $45^\circ$  line OR. This motivates the following definition:

Definition: The mapping  $T(Z)$  defined on  $\Omega$  as domain is

- A1) non-negative: if  $X = T(Z) \in \Omega$  for all  $Z \in \Omega$
- A2) non-exhaustive: if  $X^* = T^*(Z) \in \Omega$  for all  $Z \in \Omega$
- A3) regular: if it is non-negative and non-exhaustive

The regularity of  $T(Z)$  ensures that the triplet of inequalities in (1.3d)  $(I(Z), I(X), I(X^*))$  can all be defined unambiguously.

In diagram 1 the slope of the income tax schedule  $t(z)$  is the marginal tax rate  $dt/dz$  which ordinarily satisfies the following inequalities  $0 \leq dt/dz \leq 1$ , to ensure that  $t(z)$  and  $t^*(z)$  are increasing functions of  $z$ . This motivates the following definitions:

Definition: The mapping  $T(Z)$  defined on  $\Omega^0$  as domain is

- B1) non-decreasing: if  $X = T(Z) \in \Omega^0$  for all  $Z \in \Omega^0$
- B2) (marginally) inexcessive: if  $X^* = T^*(Z) \in \Omega^0$  for all  $Z \in \Omega^0$
- B3) rank preserving: if it is non-decreasing and inexcessive

Notice that when  $T(Z)$  is rank preserving, the ranks of the families in  $Z$  is preserved in both  $X$  and  $X^*$  (i.e. they are perfectly rank correlated).

Notice that the above properties (A1,A2,B1,B2) are defined in terms of a transformation function  $T(Z)$  "generated" by a real valued function  $t(z)$ . These properties can be defined, in terms of  $t(z)$ , equivalently as stated in the following lemma:

Lemma 1: In terms of  $t(z)$ , the properties A1,A2,B1,B2 can be defined equivalently as follows:

- (2.1) a) A1) non-negative  $t(z) \geq 0$  for all  $z \geq 0$   
 b) A2) non-exhaustive  $t(z) \leq z$  for all  $z \geq 0$   
 c) B1) non-decreasing  $t(z_a) \leq t(z_b)$  for all  $z_a \leq z_b$   
 d) B2) inexcessive  $t(z_a) - t(z_b) \leq z_a - z_b$  for all  $z_a \leq z_b$

Proof: To show (2.1d) is equivalent to B2, suppose B2 is valid. Let  $z_a \leq z_b$ . Construct  $U = (z_a, z_b, \dots, z_b) \in \Omega^0$ . Thus  $T^*(U) = (t^*(z_a), t^*(z_b), \dots, t^*(z_b)) \in \Omega^0$  by B2 and  $t^*(z_a) = z_a - t(z_a) \leq t^*(z_b) = z_b - t(z_b)$ . This implies  $t(z_b) - t(z_a) \leq z_b - z_a$  which proves (2.1d). Conversely suppose (2.1d) is valid,  $Z \in \Omega^0$ . Let  $i < j$  then  $Z_i \leq Z_j$ .  $t(Z_j) - t(Z_i) \leq Z_j - Z_i$  by (2.1d). Then  $X_i^* = t^*(Z_i) = Z_i - t(Z_i) \leq Z_j - t(Z_j) = t^*(Z_j) = X_j^*$  which proves  $X^* \in \Omega^0$ . Thus (2.1d) and B2 are equivalent. The proofs of the other equivalences are similar. Q.E.D.

From diagram 1 we see that the average tax rates indicated by the slopes of  $Oa_1, Oa_2, \dots, Oa_n$  form an increasing (decreasing) sequence if the income tax system is "progressive" (regressive). This motivates the following definition in which  $0/0$  is defined to be zero:

Definition: A mapping  $T(Z)$  defined on  $\Omega$  as domain is

- C1) average increasing: if  $0 \leq z_a < z_b$  implies  $t(z_a)/z_a < t(z_b)/z_b$   
 C2) average decreasing: if  $0 \leq z_a < z_b$  implies  $t(z_a)/z_a > t(z_b)/z_b$

We readily have the following theorem:

Theorem 1: The pairs (A1,A2), (B1,B2), (C1,C2) are anti-symmetrical<sup>11)</sup>

pairs of properites for the conjugating pairs of  $\mathcal{M}$ , hence

- a) T(Z) is regular if and only if T\*(Z) is regular
- b) T(Z) is rank preserving if and only if T\*(Z) is rank preserving
- c) T(Z) is average increasing (decreasing) if and only if T\*(Z) is average decreasing (increasing).

The proofs are elementary and are omitted. For example the anti-symmetry of (C1,C2) follows readily from

$$(2.2) \quad t^*(z)/z + t(z)/z = 1 \quad \text{by (1.1)}$$

Notice that A1 (A2) states that T(Z) (T\*(Z)) is endomorphic in  $\Omega$  while B1 (B2) states that T(Z) (T\*(Z)) is endomorphic in  $\Omega^0$ . Since  $\hat{\Omega}$  is a subset of  $\Omega$  and  $\Omega^0$ , the above definitions are applicable when the domain of T(Z) is restricted to  $\hat{\Omega}$ . We have the following lemmas:

Lemma 2: a) T(Z) is endomorphic on  $\hat{\Omega}$  if A1 and B1 are satisfied.

- b) T\*(Z) is endomorphic on  $\hat{\Omega}$  if A2 and B2 are satisfied  
by T(Z) (Proof: omitted)

Lemma 3: a) T(Z) defined on  $\Omega$  as domain satisfies B1 if it satisfies  
A1 and C1.

- b) T(Z) defined on  $\Omega$  as domain satisfies B2 if it satisfies  
A2 and C2.

(Proof: To prove (a), if  $0 < z_a < z_b$  then  $t(z_a)/z_a < t(z_b)/z_b$  by (C1).

And hence  $t(z_b) > t(z_a) \cdot z_b/z_a > t(z_a) > 0$  by A1 and  $z_b/z_a > 1$ .

-14-

This proves (a). To prove (b) we see  $T^*(Z)$  satisfy (A1) and (C1) and hence (B1) by (a). Hence  $T(Z)$  satisfies (B2). Q.E.D.)

Notice that all the properties which we have introduced are the most "ordinary" and hence the most "useful" in the analysis of inequalities.

### Section 3 Analysis of Income Tax System

For the income tax problem of section 1, we begin by interpreting (2.1 abcd) as four "axioms" for an income tax schedule  $t(z)$ . Thus A1 rules out subsidy (negative tax). A2 is an "ability to pay" axiom. B1 implies higher income families pay no less taxes. B2 is an "axiom of incentive preservation" which implies that the disposable income will not decrease when income before tax increases. Without it, families will obviously not have the incentive to earn a higher income. It is easy to show  $\{A1, A2, B1, B2\}$  forms an "axiomatic system" (i.e. they are consistent and independent).

A popular equity oriented property of a tax schedule is its "progressiveness" according to the following definition:

Definition: A tax schedule  $t(z)$  is progressive if  $T(Z)$  is average increasing (C1)

For a progressive tax schedule, the average tax rate increases when family income increases. Given  $t(z)$  and an FID pattern  $Z \in \hat{\Omega}$ , a tax burden pattern  $X = T(Z)$  and a disposable income pattern  $X^* = T^*(Z)$  are induced (see 1.3b). We have the following theorem:

Theorem 2 A continuous tax schedule  $t(z)$  is

- a) non-negative (A1) and progressive (C1) if and only if  
 $L(T(Z)) \leq L(Z)$
- b) non-exhaustive (A2), in excess (B2) and progressive (C1)  
if and only if  $L(Z) \leq L(T^*(Z))$   
for all non-uniform  $Z \in \hat{\Omega}$

(Proof: see below)

Notice that lemma 3a implies B1 is satisfied for the tax schedule  $t(z)$  in theorem 2a and hence  $L(T(Z))$  can be unambiguously defined because  $T(Z) \in \hat{\Omega}$  by lemma 2a. Theorem 1 implies that the disposable income schedule  $t^*(z)$  for  $t(z)$  in theorem 2b is average decreasing (C2) and lemma 2b implies  $L(T^*(Z))$  can be defined unambiguously. The non-uniformity of  $Z$  (i.e.  $Z$  has at least two distinct components) is essential for otherwise the theorem is false (i.e.  $L(Z) = L(T(Z)) = L(T^*(Z))$ ). Theorem 2 will be proved later.

Since a reasonable tax schedule  $t(z)$  should satisfy all "axioms" in 2.1, we can define a progressive income tax "system" as follows:

**Definition:** An income tax system is progressive if the tax schedule  $t(z)$  is progressive (C1), non-negative (A1), non-exhaustive (A2) and in excess (B2).

Notice that a progressive tax system also satisfies B1 (by lemma 3a). We have

Corollary 1 For a progressive income tax system

- a)  $t(z)$  and  $t^*(z)$  are regular and rank preserving
- b)  $t^*(z)$  is average decreasing (C2) while  $t(z)$  is average increasing (C1)

(Proof: implied by theorem 1)

The following characterization of a progressive income tax system is a direct corollary of theorem 2.

Corollary 2 A continuous tax schedule constitutes a progressive income tax system if and only if for all non-uniform  $Z \in \hat{\Omega}$ ,

$$\underline{L(T(Z)) \leq L(Z) \leq L(T^*(Z))}$$

Thus when a legislature intends to design an "equity oriented" income tax system (in the sense of  $L(Z) \leq L(T^*(Z))$  for every  $Z \in \hat{\Omega}$ ), the tax system must be a progressive one. Corollary 2 implies the following corollary by (1.5):

Corollary 3 For a progressive income tax system,

$$I(T^*(Z)) < I(Z) < I(T(Z))$$

for any reasonable index of inequality  $I(\cdot)$ .

Thus we see, for any progressive income tax system,  $I(Z)$  is always straddled by  $I(T^*(Z))$  and  $I(T(Z))$  as the degree of inequality of the disposable income pattern  $I(T^*(Z))$  (tax burden pattern  $I(T(Z))$ ) is always lower (higher) than  $I(Z)$ .

We need the following lemma to prove theorem 2:

Lemma 4 For  $X = (X_1, X_2, \dots, X_n)$  and  $X' = (X'_1, X'_2, \dots, X'_n)$  satisfying

(3.1) a)  $X, X' \in \hat{\Omega}$

b)  $X_1 + X_2 + \dots + X_n = X'_1 + X'_2 + \dots + X'_n$

c) there exists an integer  $k$ ,  $1 \leq k \leq n$  such that

$X_i < X'_i$  for all  $i \leq k$  and

$X_i > X'_i$  for all  $i > k$

then

(3.2)  $L(X') \geq L(X)$

(Proof: See appendix)

The proof of the necessary condition of theorem 2a will be outlined with the aid of diagram 1. When a non-uniform  $Z \in \hat{\Omega}$  is given, let  $X = T(Z)$ . For the average ratio  $\phi_x = \bar{X}/\bar{Z}$ , we assert:



$$(3.3) \quad \phi_x > 0$$

(Proof:  $\bar{Z} > 0$  because  $Z$  is non-uniform,  $\bar{X} \geq 0$  because  $T(Z)$  is non-negative. If  $\bar{X}=0$  then  $X_i=0$  for  $i=1,2,\dots,n$ . This contradicts the non-uniformity of  $Z$  when  $t(z)$  is average increasing. Q.E.D.)

Let a pattern  $X'=(X'_1, X'_2, \dots, X'_n)$  be constructed in (3.4a) with properties shown in (3.4bc):

$$(3.4) \text{ a) } X' = \phi_x Z$$

$$\text{b) } X'_1 + X'_2 + \dots + X'_n = X_1 + X_2 + \dots + X_n$$

$$\text{c) } L(X') = L(Z)$$

(Proof: (b) follows from  $\sum X'_i = \phi_x \sum Z_i = n\bar{X}$  and

(c) follows from (a) Q.E.D.)

In diagram one the points  $(X'_i, Z_i)$   $i=1,2,\dots,n$  fall on the radial line OA with an equation

$$(3.5) \quad x = \phi_x z$$

OA has a strictly positive slope and passes through the mean point  $m=(\bar{X}, \bar{Y})$ . By (3.4c), it is sufficient to prove:

$$(3.6) \quad L(X') \geq L(X)$$

with the aid of lemma 4. Notice that (3.1a) is satisfied by (3.4a) (3.1b) is satisfied by (3.4b). Thus it is sufficient to prove (3.1c). A diagrammatical argument is OA intersects  $t(z)$  from above when  $t(z)$  is average increasing. The details will be supplied in the appendix where all other parts of theorem 2 will also be proved.

#### Section 4 Cause and Consequence of FID Inequality

The inequality of property income or capital ownership is a major cause of FID inequality while the inequality of saving is the consequence (see introduction). In this section, two separate models with the form defined in (1.3) will be used to analyze these causal relations.

The first model is  $M_s = [z = s + c, s(z)]$  defined in (1.4). It should not be a surprise that the mathematical properties (A1, A2, C1 and B2) which we have postulated for a progressive income tax system are entirely appropriate for the "saving function  $s(z)$ ". In particular, B2 implies that the consumption function  $s^*(z)$  is an increasing function of family income ( $z$ ) and C1 implies "increasing average propensity to save" reminiscent of the well known "Keynesian" property at the "family" level. All theorems in the last section are applicable. Thus corollary 3 can be rephrased as:

Corollary 3\* For a saving function satisfying (A1, A2, B2) and increasing average propensity save (C1) then

$$I(X = (S_1, S_2, \dots, S_n)) > I(Z) > I(X^* = (C_1, C_2, \dots, C_n))$$

for any reasonable index of inequality  $I(\cdot)$  and for  
any non-uniform income pattern  $Z \in \Omega$

Thus we see, while "increasing average propensity to save" has unemployment implications in the multiplier analysis of macro-economics, it leads to an equalization of current consumption welfare ( $I(Z) < I(C)$ ) and unequal distribution of capital ownership in future due to "high" saving inequality ( $I(S) > I(Z)$ ).

For the second model, let the family income ( $z$ ) be the sum of wage ( $w$ ) and property income ( $\pi$ ) (i.e. in 4.1a below) for which an empirical property income function is postulated (i.e.  $t(z)$  in 4.1b):

$$(4.1) \text{ a) } z = w + \pi$$

$$\text{b) } \pi = t(z)$$

These equations form a "two-equation model" i.e.  $M_f = [z = w + \pi, \pi = t(z)]$ . All concepts in (1.3a-d) are now applicable. The "behavior" equation  $\pi = t(z)$  has been used in empirical approaches to FID inequality via the so called "factor component approach".<sup>12)</sup> It was found that properties (A1, A2, B2, C1) are, again, satisfied. For example, A1 (A2) implies that property (wage) incomes are non-negative. Moreover, for families with higher total family income, B2 implies that family wage income is higher and C1 implies that property income share (i.e. property income as a fraction of total family income) is higher. Thus we can again rephrase corollary 3:

Corollary 3\*\* For a property income function  $\pi = t(z)$  satisfying

(A1, A2, B1, C1), then

$$I(X) = I(\pi_1, \pi_2, \dots, \pi_n) > I(Z) > I(X^*) = I(w_1, w_2, \dots, w_n)$$

for any reasonable index of inequality and for any

non-uniform  $Z \in \hat{\Omega}$

This conclusion, stating that, using an arbitrary index of inequality, property income (wage income) tends to be more (less) unequally distributed than  $Z$ , is a generalization of a known results when the Gini coefficient was used (see footnote 12).

When the Gini coefficient is used, a well known theorem in the "factor component approach" is:<sup>13)</sup>

(4.2) If  $Z, X, X^* \in \hat{\Omega}$  and  $Z = X + X^*$  then  $G(Z) = \phi_X G(X) + \phi_{X^*} G(X^*)$  where

a)  $\phi_X = \bar{X}/\bar{Z}$  (the property income share)

b)  $\phi_{X^*} = \bar{X^*}/\bar{Z}$  (the wage income share)

c)  $\phi_X + \phi_{X^*} = 1$

Since  $G(Z)$  is the weighted averaged of the factor Ginis, (4.2) implies two alternatives:

(4.3)a)  $G(X^*) < G(Z) < G(X)$  or

b)  $G(X^*) > G(Z) > G(X)$

Notice that corollary 3\*\* rules out the second alternative and implies the following theorem:

Theorem 3 For a property income function  $\pi = t(z)$  satisfying (A1, A2, B1, C1)

(4.4)a)  $G(Z) = \phi_X G(X) + \phi_{X^*} G(X^*)$  (for  $\phi_X, \phi_{X^*}$  defined in (4.2abc))

b)  $G(X^*) < G(Z) < G(X)$

for all non-uniform  $Z \in \hat{\Omega}$

(Proof: Since  $T^*(Z)$  and  $T(Z)$  are non-negative and rank preserving by corollary 1a, we have  $Z, X, X^* \in \hat{\Omega}$ . That  $Z = X + X^*$  follows from (1.3b). Thus (4.2) and corollary 3\*\* complete the proof. Q.E.D.)

This theorem, in terms of Gini coefficient, can be used to strengthen the results in the previous section.<sup>14)</sup>

In this section, we derive two major conclusions. On the one hand, corollary 3\*\* states that the inequality of "property income" is a major cause of FID inequality -- in the sense of the "factor component approach". Inferentially, the inequality of family "ownership of capital" is a root cause of FID inequality. On the other hand, corollary 3\* implies that family saving is more unequally distributed than income. That this will

in turn leads to "unequal capital accumulation" in the future, is the central theme of the next section.

## Section 5 Growth With Wealth Concentration

In a capitalistic society, families accumulate capital assets through savings. A family saving pattern  $S=(S_1, S_2, \dots, S_n)$  leads directly to the increase of privately held capital stocks  $K=(K_1, K_2, \dots, K_n)$ . Let  $K'=(K'_1, K'_2, \dots, K'_n)$  be the family ownership pattern of capital in the next period, then

$$(5.1)a) \quad K'=K+S$$

$$b) \quad K' \geq 0, \quad K \geq 0, \quad S \geq 0$$

Let the sum of all elements in  $K', K$  and  $S$  be denoted by  $H', H$  and  $J$  respectively. We have the following macro magnitudes:

$$(5.2)a) \quad H'=H+J$$

$$b) \quad \eta_k = J/H = (H'/H) - 1$$

In (5.2b),  $\eta_k$  is the capital growth rate for the whole economy. Let us assume  $K \in \hat{\Omega}$ . (i.e. let the families be ranked according to the amount of capital assets they own "this" period). Then (5.1a) shows

$$(5.3)a) \quad S \in \hat{\Omega} \quad \text{and} \quad K \in \hat{\Omega} \quad \text{implies}$$

$$b) \quad K' \in \hat{\Omega}$$

Notice that condition (5.3a) means "families that own more capital save more" and condition (5.3b) implies the family ranks of  $K$  are preserved in  $K'$ .

In case the Gini coefficient is used, (4.2) of the last section leads directly to our next theorem. In the statement of theorem 4,  $\eta_G$  in (5.4b) is the rate of change of  $G(K)$ , the Gini coefficient of the capital ownership pattern and  $\eta_k$  is the capital growth rate defined in (5.2b).

Theorem 4 For  $K'=K+S$ , we have

(5.4)a)  $\eta_G = u(v-1)$  where

b)  $\eta_G = (G(K') - G(K)) / G(K)$

c)  $u = \eta_k / (1 + \eta_k)$

d)  $v = G(S) / G(K)$

if (5.3a) is satisfied

(Proof: (4.2) implies  $G(K') = \phi_k G(K) + \phi_s G(S)$  where

$\phi_s = 1 - \phi_k$  and  $\phi_k = H/H' = 1/(1 + \eta_k)$  by (5.2b). A routine

calculation leads to (5.4). Q.E.D.)

Since realistic empirical value of  $\eta_k$  is less than 10%, the term  $u$  has the same order of magnitude as  $\eta_k$  (e.g.  $u = .0099, .019, \dots, .056$  for  $\eta_k = 1\%, 2\%, \dots, 6\%$ ). Thus (5.4a) becomes:

(5.5)  $\eta_G \approx \eta_k(v-1), \quad v = G(S) / G(K)$

approximately. We shall refer to  $\eta_G$  as the rate (i.e. the rapidity) of "equity orientation". Since capital accumulation lies at the heart of growth,<sup>15)</sup> equation (5.5) shows that the rate of equity of orientation ( $\eta_G$ ) is proportional to the capital growth rate ( $\eta_k$ ). Hence for a fast growing economy, the "equity" of capital ownership changes rapidly. We shall refer to a positive (negative) value of  $\eta_G$  as an "inequitable" ("equitable") orientation of capital ownership because the ownership of capital stock become more unequally (equally) distributed through time. Equation (5.5) shows:

(5.6)a)  $\eta_G > 0$  if and only if  $v > 1$  or

b)  $G(S) > G(K)$

Thus growth with an inequitable orientation occurs when the saving pattern  $S$  is more unequally distributed than the pattern of

capital ownership  $K$ . Thus (5.6b) is the necessary and sufficient condition for a "conflict thesis" (i.e. growth with wealth concentration) provided condition (5.3a) is satisfied (i.e. provided families that own more capital save more). Hence the differentiated family saving habits are the root cause of growth with wealth concentration.

The fact that "families that own more capital save more" (i.e. (5.3a)), in itself, does not imply the "conflict thesis". A stronger condition on the differentiated saving habits between the high and low capital owning families is needed. Heuristically, let us postulate a "Classical" savings function (see below)  $s=s(k)$  relating the amount of family saving ( $s$ ) to the amount of capital stock ( $k$ ) that the family owns<sup>16)</sup>:

Definition: The capital sensitive saving function  $s(k)$  is "Classical" if it satisfies (A1) and (C1).

Thus a classical  $s=s(k)$  implies two conditions being satisfied. First of all, saving is non-negative. Secondly,  $s/k$  increases with  $k$  (i.e. increasing capital-sensitive average propensity to save). When  $s(k)$  is postulated we have:

Lemma 5 For  $K \in \hat{\Omega}$  we have  $S=(S_1, S_2, \dots, S_n)=(s(K_1), s(K_2), \dots, s(K_n)) \in \hat{\Omega}$   
 (5.7) if  $s(k)$  is a "classical" saving function

(Proof: Lemma 3a implies that  $s(k)$  satisfies B1. Q.E.D.)

Thus with a classical saving function (5.3ab) are satisfied. Furthermore we also have the following conclusion needed to complete the conflict thesis due to the "strongly differentiated" saving habits implied by the classical saving function.



Theorem 5 The classical saving function implies  $L(S) \leq L(K)$  for every nonuniform  $K \in \hat{\Omega}$

(Proof: (5.7) shows that  $s(k)$  corresponds to a transformation function with property B1, A1 and C1. The necessary condition of theorem 2a completes the proof.)

Thus  $I(S) > I(K)$  for any reasonable index of inequality. In particular (5.6b) is satisfied. Lemma 5 and theorem 5 imply the following corollary.

Corollary 4 Under a classical saving function

$$(5.8) \quad \eta_G \equiv \eta_k(v-1) \quad \text{where } v = G(S)/G(K) > 1$$

Thus we see that the classical saving function ensures growth has inequitable orientation with a rapidity proportional to  $\eta_k$ . Thus we know  $G(K') > G(K)$  in terms of Gini coefficient. The following theorem is a more general theorem that asserts  $I(K') > I(K)$  for any reasonable index of inequality.

Theorem 6 The "classical saving function" implies  $L(K') \leq L(K)$  for every non-uniform  $K \in \hat{\Omega}$ .

Notice theorem 6 implies the following corollary<sup>17)</sup>:

Corollary 5 The Classical saving rule is a sufficient condition for growth with an inequitable orientation in which  $I(K') > I(K)$  for any reasonable index of inequality.

(Proof: by (1.5))

To prove theorem 6, let us refer to  $\eta_i = S_i/K_i$  as the accumulation rate and  $\sigma_i = K_i/H$  ( $\sigma_i' = K_i'/H'$ ) as the capital wealth share of the  $i$ -th family in this (the next) period. When  $K \in \hat{\Omega}$ , the classical saving

rule implies:

$$(5.9)a) \quad \sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) = (K_1/H, K_2/H, \dots, K_n/H) \in \hat{\Omega}$$

$$b) \quad \eta = (\eta_1, \eta_2, \dots, \eta_n) = (S_1/K_1, S_2/K_2, \dots, S_n/K_n) \in \hat{\Omega}$$

We see

$$(5.10)a) \quad \eta_k = \sigma_1 \eta_1 + \sigma_2 \eta_2 + \dots + \sigma_n \eta_n \quad \text{where}$$

$$b) \quad \sigma_1 + \sigma_2 + \dots + \sigma_n = 1$$

Since the national capital growth rate  $\eta_k$  is the weighted average of the family accumulation rates ( $\eta_i$  in 5.10a), (5.9b) implies that the families can be partitioned into two classes (i.e. family groups), a high class  $C^H$  and a low class  $C^L$  as follows:

$$(5.11)a) \quad C^L = \{i | \eta_i \leq \eta_k\} = (1, 2, \dots, p)$$

$$b) \quad C^H = \{j | \eta_j > \eta_k\} = (p+1, p+2, \dots, n)$$

Thus the cumulation rate of any family in  $C^L$  is strictly lower than that of any family in  $C^H$  as it is lower than the national growth rate ( $\eta_k$ ). The "comparative" cumulation rate of the  $i$ -th family,  $\epsilon_i$  is defined in (5.12b) below:

$$(5.12)a) \quad \sigma'_i = \epsilon_i \sigma_i \quad i=1, 2, \dots, n \quad \text{where}$$

$$b) \quad \epsilon_i = (\eta_i + 1) / (\eta_k + 1) \quad i=1, 2, \dots, n$$

We have the following lemma:

Lemma 6 The "Classical" saving function implies that for  $C^L$  and  $C^H$  defined in (5.11),  $i \in C^L$  and  $j \in C^H$  imply

$$(5.13)a) \quad \eta_i < \eta_j$$

$$b) \quad \sigma_i \leq \sigma_j$$

$$c) \quad \sigma_i \geq \sigma'_i \quad (i=1, 2, \dots, p), \quad \sigma_j < \sigma'_j \quad (j=p+1, p+2, \dots, n)$$

(Proof: (5.13a) follows from (5.11 ). (5.13b) follows from (5.9a). (5.13c) follows from (5.12) and (5.13a), namely  $\epsilon_i < 1$  and  $\epsilon_j > 1$ . Q.E.D.)

This theorem implies that the classical saving function has a natural class-oriented interpretation such that the property share and cumulation rate of every high class family is higher than those of low class family. Furthermore, (5.13c) implies that under the classical rule there is a concentration of capital wealth in the upper-class families  $C^H$  in the growth process. Notice

$$(5.14) \quad \sigma'_1 + \sigma'_2 + \dots + \sigma'_n = 1$$

and  $C^L$  and  $C^H$  are not empty (i.e.  $1 \leq p < n$ ) when  $K$  is not uniform. Condition (5.14) (5.10b) and (5.13c) and lemma 4 imply  $L(\sigma) \geq L(\sigma')$  and hence  $L(K) \geq L(K')$ . This proves theorem 6. Thus we see that the classical saving rule implies a "radical" conclusion of wealth concentration in the growth process, with a class oriented consequence.

## Conclusion

The "radical" conclusion of "growth with capital wealth concentration" was based decisively on the "classical" saving function. However, whether or not the "classical saving rule is valid is an empirical question that can be verified statistically. Suppose  $s=s(z)$  in (1.4) and  $\pi=t(z)$  in (4.1b) are estimated statistically. Then for  $\pi=rk$  we have

$$(6.1) \quad k \equiv k(s, r) \equiv t(s^{-1}(s))/r$$

which is a functional relation between  $k$  (family held capital stock) and  $s$  (family saving) taking  $r$  (the rate of return to capital) as a parameter. It is not "necessarily true" that  $s/k$  is an increasing function of  $k$ .

When the financial institutions (banks, stock markets etc.) are primitive, it may be true that the vast majority of the workers (i.e. those who receive only wage income) do not save because they can not save without becoming entrepreneurs and manage the capital assets directly. For such "under developed countries" it may be legitimate to assume:

$$(6.2)a) \quad \sigma_1 = 0 \quad \text{implies} \quad \eta_1 = 0$$

$$b) \quad \sigma_1 > 0 \quad \text{implies} \quad \eta_1 = r \quad (\text{the rate of return to capital})$$

One can readily shows that this is a "naive" classical saving rule (i.e. workers with no property income do not save (6.2a) and capital owner saves all property income (6.2b)) which is perhaps what the "class oriented" classical writers of the 19th century had in mind.<sup>17)</sup> In the contemporary world, when it is possible for the

workers to acquire titles to capital assets through the inter-mediation of financial institutions, the classical saving rule should not be an asserted one. In the age of econometrics, the statistical problem of (6.1) can be investigated with the aid of household survey data after all.

Even when (6.1) is found to be "average increasing" empirically, the pattern of capital ownership may not become more unequally distributed through time for another reason, namely, the number of families (n) will increase when population and labor force increase.<sup>18)</sup> The splitting of the property of a deceased "head of household" to more than one heir (i.e. new family starts) will obviously imply a "counter concentration" tendency and with a more equal distribution of property ownership. Thus growth with or without concentration of property ownership has other dimensions, the analysis of which is beyond the scope of this paper.

## Footnotes

- (1) See Kuznets [8], Adelman and Morris [1].
- (2) See World Bank Publications [2], and Paukert [10].
- (3) In the sense as defined by Kuznets[8].
- (4) See Schumpeter[11], pp. 439
- (5) See Fei-Ranis-Kuo [3] and [4].
- (6) See Fei-Ranis-Kuo[4], chapter 3
- (7) A Keynes saving behavior is "income-sensitive" as postulated in the well known saving function. The classical writers in the 19 century believes that saving is "capital sensitive". See discussions in section 5 (See footnote 16).
- (8) See Fields[5]
- (9) An index of the Dalton Family satisfies the axioms of transitivity, symmetry, and rank preserving equalization. It is well known that the familiar Gini coefficient, Atkinson index, Theil index and coefficient of variation are reasonable indices. See Fields and Fei [6].
- (10) See Fields and Fei [6]
- (11) A property pair  $(P_1, P_2)$  is anti-symmetrical when " $t(z)$  satisfies  $P_1$  if and only if  $t^*(z)$  satisfies the other property".
- (12) Fei-Ranis-Kuo [3] postulated a property income function (4.1b) in the linear form  $\pi = b + ay$  and found that the "correlation characteristics" is quite high. They refer to property income as "type one income" as  $b < 0$  and  $0 < a < 1$ . The readers should check that A1, A2, B2 and C1 are satisfied when the income range is properly restricted. Using this result they deduced the inequalities of corollary 3\*\* when the Gini coefficient is used.
- (13) See Fei-Ranis-Kuo [4], "monotonic model" p. 365
- (14) For example, the Gini coefficient of income before tax is a a weighted average of the Ginis of disposable income and tax burden pattern.

- (15) This is true at least for contemporary less developed countries in the early stage of development. For professor Kuznets[9] has pointed out that the epochal characteristic of modern growth is technology change embodied in new capital formation.
- (16) The relation between the income sensitive saving function  $s(z)$  in (1.4) and the capital sensitive saving function  $s(k)$  postulated here will be discussed in (6.1) below.
- (17) See Schumpeter[11], pp.641, for a discussion on the sources of capital accumulation (e.g. "capital increases by revenue's being converted into it") according to the "classic" schema of economic growth. The fact that the working and capital-owning classes have strongly differentiated saving habits was formulated as a crucial behavior hypothesis in the growth model of Kaldor[7].
- (18) Fei-Ranis-Kuo [4] found that the time series of the Gini coefficients of property income are, in fact, falling for all households and for all urban households between 1964 and 1972 in Taiwan.

## Appendix

Since lemma 4 is needed in the proof of theorem 2, we will prove lemma 4 and theorem 2 in sequence.

(1) Proof of lemma 4:

$$\text{Let } s_i = X_1 + X_2 + \dots + X_i, \quad s'_i = X'_1 + X'_2 + \dots + X'_i \quad \text{and} \quad \delta_i = s'_i - s_i.$$

We want to show that  $(\delta_1, \delta_2, \dots, \delta_n) \geq 0$ . For  $i \leq k$ , we have  $\delta_i \geq 0$  by

$$(3.1c). \quad \text{Moreover, } \delta_{i+1} - \delta_i = (s'_{i+1} - s_{i+1}) - (s'_i - s_i) = (s'_{i+1} - s'_i) - (s_{i+1} - s_i) \\ = X'_{i+1} - X_{i+1} < 0 \quad \text{for all } i > k$$

Since  $\delta_n = 0$  by (3.1b), then  $\delta_k, \delta_{k+1}, \dots, \delta_n$  strictly monotonically decreases to zero. Thus  $\delta_k > 0$  and  $\delta_i \geq 0$  for all  $i = 1, 2, \dots, n$ . Q.E.D.

(2) Proof of the necessary condition of theorem 2a:

Let  $Z \in \hat{\Omega}$ , then  $X = T(Z) \in \hat{\Omega}$  by lemma 2a and 3a. To complete the proof given in the text, let  $X'$  be constructed as in (3.4a). We want to prove (3.6) by proving (3.1c). Let the set of the first  $n$  integers be partitioned into two subsets:

$$A.1)a) \quad (1, 2, \dots, n) = \Gamma^- \cup \Gamma^+$$

$$b) \quad \Gamma^- = \{i | X_i < X'_i\}; \quad \Gamma^+ = \{i | X_i > X'_i\} \quad \text{which imply}$$

$$c) \quad \Gamma^- \cap \Gamma^+ = \phi \text{ (null set)}$$

We claim that both  $\Gamma^-$  and  $\Gamma^+$  are not empty sets. i.e.

$$A.2)a) \quad \Gamma^- \neq \phi$$

$$b) \quad \Gamma^+ \neq \phi$$

(3.1b) implies  $\Gamma^- \neq \phi$ . Suppose  $\Gamma^+ = \phi$ . (3.1b) implies  $X_i = X'_i$  and hence  $X_i/Z_i = X'_i/Z_i = \bar{X}/\bar{Z}$  for all  $i = 1, 2, \dots, n$ . This contradicts the fact that  $Z$  is nonuniform and  $t(z)$  is progressive. Thus  $\Gamma^+ \neq \phi$ .

We claim that



A.3)  $i \in \Gamma^-$  and  $j \in \Gamma^+$  imply  $i < j$

$i \in \Gamma^-$  implies  $X_i/Z_i < X'_i/Z_i = \phi_x = X'_j/Z_i < X_j/Z_j$  (because  $j \in \Gamma^+$ )

Thus  $X_i/Z_i < X_j/Z_j$ . We have  $i < j$  because  $Z \in \hat{\Omega}$  and  $t(z)$  satisfies (C1).

Let  $k$  be the largest integer of  $\Gamma^-$ . Then  $\Gamma^- = (1, 2, \dots, k)$  and  $\Gamma^+ = (k+1, k+2, \dots, n)$   $1 \leq k < n$ . Thus (3.1c) is proved. Q.E.D.

(3) Proof of the sufficient condition of theorem 2a:

To prove the sufficient condition of theorem 2a. Suppose  $L(T(Z)) \leq L(Z)$  for all non-uniform  $Z \in \hat{\Omega}$ . Let  $Z_a, Z_b$  be any two real numbers satisfying  $0 \leq Z_a < Z_b$ . We want to prove

A.4)a)  $t(Z_a) \geq 0$  (i.e. to prove  $t(z)$  is non-negative)

b)  $\alpha \equiv t(Z_a)/Z_a < t(Z_b)/Z_b \equiv \beta$  (i.e. to prove  $t(z)$  satisfies C1)

Consider  $U = (Z_a, Z_b, \dots, Z_b)$  ;  $T(U) = (t(Z_a), t(Z_b), \dots, t(Z_b))$

The fact that  $L(T(U))$  can be unambiguously defined implies

$T(U) \geq 0$ . This proves A.4a. To prove A.4b, we know that

$t(Z_a) \neq t(Z_b)$ . For if  $t(Z_a) = t(Z_b)$ , then  $T(U)$  is a uniform

pattern and  $L(T(U)) \geq L(U)$  is impossible.  $t(z)$  is

thus either monotonically increasing or monotonically decreasing.

Case 1: for all  $Z_a, Z_b$  satisfying  $0 \leq Z_a < Z_b$  then  $t(Z_a) < t(Z_b)$

Let  $Z_a, Z_b$  be any two real number satisfying  $0 < Z_a < Z_b$

Let the sum of all elements in  $U(T(U))$  be denoted by  $s = Z_a + (n-1)Z_b > 0$

( $s_t = t(Z_a) + (n-1)t(Z_b) > 0$ ). The normalization of  $U(T(U))$  becomes

$U^* = (Z_a^*, Z_b^*, \dots, Z_b^*) = (1/s)(Z_a, Z_b, \dots, Z_b)$   $(T(U)^* = (p, q, \dots, q) = (1/s_t)(t(Z_a),$

$t(Z_b), \dots, t(Z_b)))$  for case one, with  $Z_a^* + (n-1)Z_b^* = p + (n-1)q$ .

Lemma 4 can be applied. Notice  $Z_a^* \geq p$  if and only if

$Z_b^* \leq q$ . Thus either  $L(T(U)^*) \geq L(U^*)$  or  $L(T(U)^*) \leq L(U^*)$  if  $Z_a^* \neq p$ .

Since we know  $L(U^*) = L(U) \geq L(T(U)) = L(T(U)^*)$ , so  $Z_a^* \geq Z_a/s > p \equiv t(Z_a)/s_t$ .

If  $t(Z_a)=0$  then  $t(Z_a)/Z_a < t(Z_b)/Z_b$ . Hence C1 is proved

If  $t(Z_a)>0$ , then  $Z_a/s > t(Z_a)/s_t$  can be written as

$$1/(1+(n-1)(Z_b/Z_a)) > 1/(1+(n-1)t(Z_b)/t(Z_a)) \text{ and hence}$$

$$t(Z_a)/Z_a < t(Z_b)/Z_b. \quad \text{Q.E.D.}$$

Case 2: for all  $Z_a, Z_b$  satisfying  $0 < Z_a < Z_b$  then  $t(Z_a) > t(Z_b)$

Since  $t(z)$  monotonically decreases and bounded from below by zero, we know  $\lim_{z \rightarrow \infty} t(z) = c \geq 0$ . If  $t(Z_b)=0$  then  $t(x)=0$  for all  $x > Z_b$ . The Lorenze curve for  $(Z_b, x, x, \dots, x)=0$  can not be defined.

Thus  $t(Z_b) > 0$ . For a pair of real numbers  $0 < Z_a < Z_b$ , construct

the following vector with  $n$  components:

$$V = (Z_a, Z_a, \dots, Z_a, Z_b) \in \hat{\Omega} \text{ with a sum } s = (n-1)Z_a + Z_b > 0;$$

$$T(V) = (t(Z_a), t(Z_a), \dots, t(Z_a), t(Z_b)) \geq 0 \text{ which can be reordered to}$$

$$F = (t(Z_b), t(Z_a), \dots, t(Z_a)) \in \hat{\Omega} \text{ with a sum } s_t = t(Z_b) + (n-1)t(Z_a) > 0$$

$V$  and  $F$  can be normalized to become

$$V^* = (Z_a^*, Z_a^*, \dots, Z_a^*, Z_b^*) = (1/s)(Z_a, Z_a, \dots, Z_a, Z_b)$$

$$F^* = (p, q, \dots, q) = (1/s_t)(t(Z_b), t(Z_a), \dots, t(Z_a)) \text{ with } (n-1)Z^* + Z^* = p + (n-1)q.$$

$$\text{We have } Z_b^* = Z_b/s = 1/(1+(n-1)(Z_a/Z_b))$$

$$Z_a^* = Z_a/s = 1/(n-1+(Z_b/Z_a))$$

$$p = t(Z_b)/s = 1/(1+(n-1)(t(Z_a)/t(Z_b)))$$

$$q = t(Z_a)/s_t = 1/(n-1+t(Z_b)/t(Z_a))$$

For all sufficiently large  $Z_a, Z_b$ , the ratio  $t(Z_a)/t(Z_b)$  is arbitrarily close to one (by the Cauchy property). Hence  $p$  and  $q$  can be made arbitrarily close to  $1/n$  (i.e.  $F^*$  can be made arbitrarily close to a uniform pattern  $(1/n, 1/n, \dots, 1/n)$ ).

For a fixed  $Z_a$  we can choose  $Z_b$  sufficiently large to make  $Z_a/Z_b$  arbitrarily close to zero. Thus  $Z_b^* (Z_a^*)$  is made arbitrarily

close to 1 (0), and hence  $V^*$  can be constructed to be arbitrarily close to  $(0,0,\dots,1)$ . Thus we can construct  $V^*$  and  $F^*$  such that  $L(F^*) \geq L(V^*)$ . However  $L(V) = L(V^*) \leq L(F^*) = L(F) = L(T(V))$ . This contradiction implies that case two is impossible. Q.E.D.

- (4) Proofs of the necessary and sufficient condition of theorem 2b are similar to those of theorem 2a.

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