ECONOMIC GROWTH CENTER

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CENTER DISCUSSION PAPER NO. 598

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March 1990

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Revised: March 1990

Abstract: Though the envelope theorem is a widely used tool of applied economic analysis, the standard version of the theorem can only be used if all of the choice variables are assumed to be continuous. This limitation is significant because the natural description of many economic choice variables is as integers (e.g., the number of projects or children, as well as variables that represent various yes-no choices and choices that entail fixed costs). A continuous representation of such variables is not only unsatisfactory and a source of potential error, but it can also make certain kinds of economic analyses intractable or unproductive.

This paper shows that modified but intuitive versions of the envelope theorem can be used with integer or discrete choice variables, provided the optimization problem satisfies the usual conditions. Thus, the results presented here make it possible to use the envelope theorem in a variety of economic problems.

*We thank Alex Ascencios, James Mirrlees, Hal Varian and Martin Weitzman for their valuable suggestions.

The envelope theorem is a genuine workhorse of applied economic analysis. It is used in numerous contexts such as the indirect utility function, the cost function and the profit function.[1] A limitation of the standard envelope theorem is that choice variables in the underlying optimization are assumed to be continuous. This limitation is significant because the natural description of many economic choice variables is as integers (or, more generally, as discrete variables); for example, the number of projects, plants or children. This feature is actually more widespread since it arises in all those situations in which some of the choices are indivisible (e.g., where there are fixed costs associated with non-zero levels of choices) or where there are binary (yes-no) choices such as labor-force participation, purchase of a house, and migration.

Because of the seeming difficulties in working with integer choice variables, economists typically represent these variables as continuous. Such representations are not only unsatisfactory but also a source of error. For example, a continuous representation is potentially more erroneous for a binary choice variable or for a variable with a small magnitude (such as the number of children produced by a couple) than for a variable with a large magnitude (such as the number of widgets produced in a factory). Another important consideration is as follows. The conventional view is that economic analysis becomes more tractable if a continuous representation is employed for intrinsically discrete variables. This is not valid for certain kinds of economic analyses in which a continuous representation leads to a substantial loss of useful information. In such cases, the analysis may become intractable or unproductive if a continuous representation is employed.[2]

This paper presents results that ameliorate this limitation. It shows that modified versions of the standard envelope theorem can always be employed with integer or discrete choice variables under the usual conditions. As we shall see, the final, usable results presented in this paper are quite intuitive, but they are not obvious, nor, to our knowledge, have these results been reported in the earlier literature.

The paper is organized as follows. Section I presents an informal description of some of the results. These and other results are formally presented in Section II. Some of the possible generalizations are then discussed in Section III.

Note that this paper is concerned only with the envelope theorem for discrete optimization; it does not

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[1] See Diewert (1982) and Varian (1984) for example.

[2] See Sah (1989) for example. This paper establishes analytically what has been observed by numerous empirical studies, namely, that the number of children produced by a couple decreases if the mortality rate declines. It turns out that an important reason why it has been difficult to predict this pattern in even the simplest theoretical models is that the number of children (born or surviving) has typically been represented by a continuous variable. Thus, realism and tractability go hand in hand in this context.

deal with other issues involving discreteness.[3] The results presented here are useful primarily for comparative analysis of the type economists routinely employ for qualitative theoretical predictions.

I. AN INFORMAL DESCRIPTION OF SOME RESULTS

Consider a function $f(n, \theta)$, where n is a scalar choice variable and θ is a scalar parameter. f is differentiable in θ. The choice variable n is restricted to being an integer. f is defined throughout the range of n. Also, to keep matters simple at present, assume that f is strictly concave in n. Let $n(\theta)$ denote a value of n that maximizes f. (The analysis is unchanged if, instead, f is strictly convex and one deals with a minimization problem). Assume that at least one optimal n exists for all values of θ under consideration. Denote the envelope function or the extreme value function by $e(\theta)$. That is,

(1) $$e(\theta) \equiv f(n(\theta), \theta) \equiv \max_n : f(n, \theta) .$$

Among the properties of the extreme value function that are of interest are its continuity, convexity, differentiability or one-sided differentiability, subgradients [Rockafellar (1970)] and generalized gradients [Clarke (1983)]. Since our focus is on the envelope theorem and on the comparative statics, we need to be concerned primarily with matters of differentiability and one-sided differentiability. Assume, for the moment, that e is differentiable. Let $e_\theta \equiv de/d\theta$ denote the derivative of e and let $f_\theta \equiv \partial f/\partial \theta$ denote the partial derivative of f with respect to θ.

If the choice variable n were continuous, rather than restricted to being an integer, then the optimal n would be unique, and the standard envelope theorem would be

(2) $$e_\theta(\theta) = f_\theta(n, \theta)\big|_{n=n(\theta)} .$$

That is, the derivative of the envelope function with respect to a parameter is the same as that of the optimand, provided the latter derivative is evaluated at the value of the choice variable that is currently optimal. This theorem cannot, however, be used if the choice variable is an integer, because the standard theorem is based on the first-order condition of optimality (namely, that the marginal value of the optimand, $\partial f/\partial n$, is zero at the optimum). This condition does not hold in the case of integer choice variable because the marginal value of the optimand, $f(n, \theta) - f(n-1, \theta)$, can be positive, negative or zero, at an optimum.

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[3] Among them are the properties of production sets with indivisibilities [see Frank (1969) and Scarf (1981, 1986)], the equilibrium or core of an exchange economy with indivisible commodities [see Shapley and Scarf (1974)], and the computation of excess burden and related welfare measures in particular discrete choice models [see Small and Rosen (1981)].

For this reason, as well as those noted later, a mechanical application of the standard envelope theorem is inappropriate, if not incorrect, if the choice variable is an integer.

When the choice variable n is restricted to being an integer, it is easily shown that the present optimization has two possible outcomes: (i) there is a unique optimal value of n, and (ii) there are two optimal values of n. The first case is illustrated in Figure 1 at the value θ_1 of the parameter, and the second at the value θ_2. The relevant issues then are: what are the envelope theorems for these two outcomes, and under what circumstances can these theorems be used?

First, consider the case in which there is a unique optimal value of n. We show that, in this case, the derivative $e_\theta(\theta)$ exists, and that the standard envelope theorem (2) holds.

Next, consider the case in which there are two optimal values of n, which, for a given θ, will be denoted by $\bar{n}(\theta)$ and $\underline{n}(\theta)$ respectively. In this case, the derivative $e_\theta(\theta)$ does not exist in general.[4] We show, however, that a necessary and sufficient condition for $e_\theta(\theta)$ to exist is $f_\theta(\bar{n}(\theta), \theta) = f_\theta(\underline{n}(\theta), \theta)$; that is, the partial derivative of the optimand has the same value at the two optimal values of n. If this condition is satisfied, then the envelope theorem is

$$(3) \qquad e_\theta(\theta) = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)} = f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)} .$$

Now, consider the case in which there are two optimal values of n and in which $e_\theta(\theta)$ does not exist, which, as noted above, requires that $f_\theta(\bar{n}(\theta), \theta) \neq f_\theta(\underline{n}(\theta), \theta)$. Let $e_\theta^+(\theta)$ denote the right-hand derivative of $e(\theta)$ with respect to θ, and let $e_\theta^-(\theta)$ denote the left-hand derivative. We show that these one-sided derivatives always exist under the usual conditions (Assumption 1 in the next section). Moreover, we show that the following envelope theorems hold in the case under consideration.

$$(4a) \qquad e_\theta^+(\theta) = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)} \text{ and } e_\theta^-(\theta) = f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)}, \text{ if } f_\theta(\bar{n}(\theta), \theta) > f_\theta(\underline{n}(\theta), \theta) .$$

$$(4b) \qquad e_\theta^+(\theta) = f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)} \text{ and } e_\theta^-(\theta) = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)}, \text{ if } f_\theta(\bar{n}(\theta), \theta) < f_\theta(\underline{n}(\theta), \theta) .$$

Finally, note that our envelope theorems and the related results deal with the effects of a small change in θ. It is possible, in addition, to make some simple but useful observations even if the change in θ is not small. For example, suppose the initial value of the parameter is θ and it is changed to θ'. Then definition (1) yields

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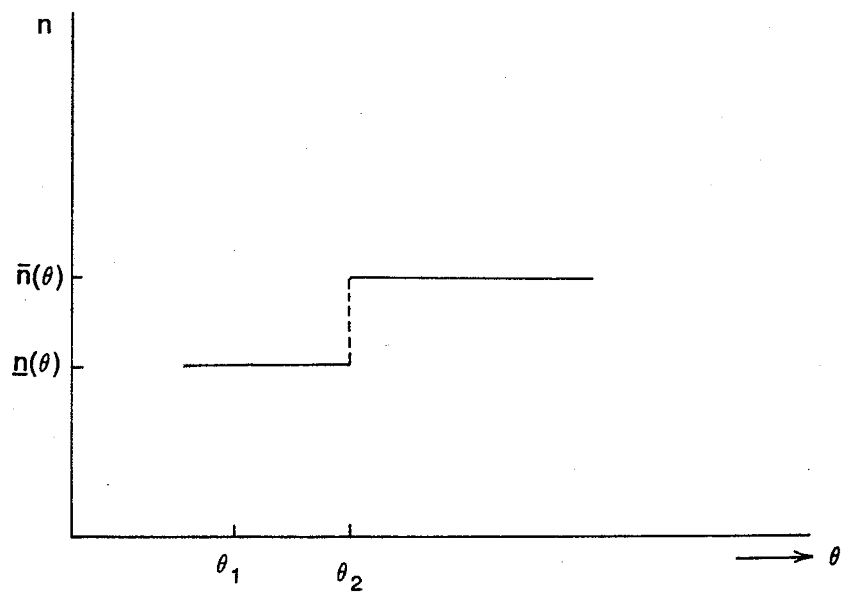
[4]Consider a simple example. Let $f(n, \theta) = -(n - \theta)^2$, where $2 > \theta > 1$. Let $N(\theta)$ denote the set of optimal values of n. Then $N(\theta) = \{1\}$ for $\theta < 3/2$, $N(\theta) = \{1, 2\}$ for $\theta = 3/2$, and $N(\theta) = \{2\}$ for $\theta > 3/2$. Also, $e(\theta) = -(1 - \theta)^2$ for $\theta \leq 3/2$, and $-(2 - \theta)^2$ for $\theta > 3/2$. Note that e_θ exists if the optimal n is unique; that is, if $\theta \neq 3/2$. But e_θ does not exist at $\theta = 3/2$.

FIGURE 1



The map from θ to the optimal values of n .

(5a) $e(\theta') - e(\theta) = f(n(\theta'), \theta') - f(n(\theta), \theta) \geq f(n(\theta), \theta') - f(n(\theta), \theta)$.

Expression (5a) has a clear interpretation. *The change in the envelope function due to a change in a parameter (large or small) is not smaller than the corresponding change in the optimand when the value of the choice variable is kept unchanged.* Moreover, it is easily shown that the former is strictly larger than the latter if the change in the parameter leads to a non-trivial change in the optimal value of the choice variable. Here, by a non-trivial change we mean that a value of n that is optimal at θ is not optimal at θ' ; that is, $n(\theta) \notin N(\theta')$, where $N(\theta')$ denotes the set of values of n which are optimal at θ' . In this case,

(5b) $e(\theta') - e(\theta) > f(n(\theta), \theta') - f(n(\theta), \theta)$.

II. RESULTS

In this section, we present the results in a bare-bones mathematical setting, working with Assumption 1 stated below. Different ways to relax this assumption are discussed in the next section.

ASSUMPTION 1: The optimand $f : \mathbf{R}^2 \to \mathbf{R}$ satisfies:

(i) $N(\theta) \equiv \{m \,|\, m$ is an integer, and $f(m, \theta) \geq f(n, \theta)$ for all integers $n\}$ is non-empty for all θ under consideration.

(ii) f is strictly concave in n . Thus,

(6) $$g(n, \theta) > g(n+1, \theta)\ , \quad \text{where } g(n, \theta) \equiv f(n, \theta) - f(n-1, \theta)\ .$$

(iii) f is differentiable in the scalar parameter θ .

Note that $N(\theta)$ is the set of optimal values of n for a given θ . Let N denote the map that takes θ to $N(\theta)$. We now establish

LEMMA 1.

(a) $N(\theta)$ **contains at most two values.**

(b) **The map** N **is upper semi-continuous.**

(c) e **is continuous in** θ .

PROOF. An optimal solution $n(\theta)$ satisfies

(7) $g(n(\theta), \theta) \geq 0 \geq g(n(\theta) + 1, \theta)$.

From (6), one of the two inequalities in (7) must be strict. (6) and (7) also imply that $g(n, \theta) > 0$ for $n < n(\theta)$, and $g(n, \theta) < 0$ for $n > n(\theta) + 1$. Thus, $N(\theta)$ can contain at most two values. This proves part (a) of the lemma. Also, note that if $N(\theta)$ contains two values, then these two values are neighboring integers.

To prove part (b), first consider the case in which the optimum is unique. Let $n^*(\theta)$ denote this unique optimum. Then (7) becomes

(8) $$g(n^*(\theta), \theta) > 0 > g(n^*(\theta) + 1, \theta).$$

Note from the definition in (6) that g is continuous in θ because f is continuous in θ. Thus, there exists a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $g(n^*(\theta), \theta') > 0 > g(n^*(\theta) + 1, \theta')$. That is, $n^*(\theta)$ remains the unique optimum for these values of θ'.

Next, consider the case in which there are two optimal values of n. Denote the larger of these two values by $\bar{n}(\theta)$ and the smaller by $\underline{n}(\theta)$. In this case (7) becomes

(9) $$g(\underline{n}(\theta), \theta) > g(\bar{n}(\theta), \theta) = 0 > g(\bar{n}(\theta) + 1, \theta).$$

Since g is continuous in θ, there exists a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $g(\underline{n}(\theta), \theta') > 0$ and $g(\bar{n}(\theta) + 1, \theta') < 0$. Thus, $N(\theta')$ contains either just $\bar{n}(\theta)$ or just $\underline{n}(\theta)$, or both. From this and the previous paragraph it follows that:

(10) There exists a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $N(\theta') \subseteq N(\theta)$.

Thus the map N is upper semi-continuous in θ. This proves part (b) of the lemma.

To prove part (c), consider the case in which $N(\theta)$ contains two values. Then, $e(\theta) = f(\bar{n}(\theta), \theta) = f(\underline{n}(\theta), \theta)$. Also, from (10), there exists a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $e(\theta')$ is either $f(\bar{n}(\theta), \theta')$ or $f(\underline{n}(\theta), \theta')$, or both. Consequently, for these values of θ', $|e(\theta') - e(\theta)| \leq |f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)| + |f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)|$. This and the continuity of f in θ yield the continuity of e in θ. The continuity of e in the case where $N(\theta)$ contains a single value is apparent. This completes the proof of Lemma 1.[5] □

[5]Note that this lemma requires only the continuity of f in θ but not its differentiability. Thus, part (iii) of Assumption 1 can be weakened to continuity. Also, note the following implication of part (b) of this lemma. Sometimes, one is interested in studying the aggregate value of individuals' choices in an economy in which a parameter can potentially affect various individual choices. Let $n(\theta, h)$ denote an optimal choice of an individual of type h. Let $m(\theta) \equiv \Sigma_h n(\theta, h)z(h)$ denote an aggregate value of optimal choices, where $z(h)$ denotes the number of individuals of type h. Let $M(\theta)$ denote the set of possible values of $m(\theta)$. Then, it follows that the map that takes θ to $M(\theta)$ is upper semi-continuous.

We now consider the differentiability of $e(\theta)$. Our goal is to find necessary and sufficient conditions under which the derivative or the one-sided derivatives of $e(\theta)$ exist, and then to identify the counterparts of the standard envelope theorem (2), given that the choice variable is an integer. Theorem 1 shows that the derivative $e_\theta(\theta)$ exists if and only if the partial derivative $f_\theta(n, \theta)$ has the same value at all optimal values of n. Theorem 2 shows that the one-sided derivatives of $e(\theta)$ always exist. Theorem 3 presents three versions of the envelope theorem. Version (a) holds if the derivative $e_\theta(\theta)$ exists. Versions (b) and (c) always hold.

For the remainder of this paper, whenever $N(\theta)$ contains two values, we shall let $\bar{n}(\theta)$ and $\underline{n}(\theta)$ denote the optimal values of n, not requiring, as in the previous proof, that $\bar{n}(\theta)$ exceed $\underline{n}(\theta)$.

THEOREM 1. The derivative $e_\theta(\theta)$ exists if and only if the partial derivative $f_\theta(n, \theta)$ has the same value at all optimal values of n. That is, $e_\theta(\theta)$ exists if and only if there exists a constant c such that $f_\theta(n, \theta) = c$ for all $n \in N(\theta)$.

PROOF. We first prove the sufficiency of the condition. There are two cases: when the optimal n is unique and when it is not.

Case 1. $N(\theta) = \{n^*(\theta)\}$. Then it follows from (10) that there exists a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $N(\theta') = N(\theta) = \{n^*(\theta)\}$. Thus,

(11) $$e_\theta(\theta) = \lim_{\theta' \to \theta} \frac{e(\theta') - e(\theta)}{\theta' - \theta} = \lim_{\theta' \to \theta} \frac{f(n^*(\theta), \theta') - f(n^*(\theta), \theta)}{\theta' - \theta} = f_\theta(n, \theta)\Big|_{n=n^*(\theta)}.$$

The last part of (11) follows since f is differentiable in θ. Thus, $e_\theta(\theta)$ exists.

Case 2. $N(\theta) = \{\underline{n}(\theta), \bar{n}(\theta)\}$. By assumption we have

(12a) $$f_\theta(\bar{n}(\theta), \theta) = f_\theta(\underline{n}(\theta), \theta), \text{ and}$$

(12b) $$e(\theta) = f(\bar{n}(\theta), \theta) = f(\underline{n}(\theta), \theta).$$

Again, by (10), there is a $\delta > 0$ such that for all $|\theta' - \theta| < \delta$, $e(\theta') = \max\{f(\bar{n}(\theta), \theta'), f(\underline{n}(\theta), \theta')\}$. Thus, using (12b), $e(\theta') - e(\theta) = \max\{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta), f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)\}$. In turn,

$$\frac{e(\theta') - e(\theta)}{\theta' - \theta} = \max\{\frac{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)}{\theta' - \theta}, \frac{f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)}{\theta' - \theta}\}, \text{ if } \theta' > \theta \text{; and}$$

$$\min\{\frac{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)}{\theta' - \theta}, \frac{f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)}{\theta' - \theta}\}, \text{ if } \theta' < \theta.$$

Now, evaluate the above expressions for $\theta' \to \theta$. The left-hand side is $e_\theta(\theta)$, if it exists. The right-hand side is either $f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)}$ or $f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)}$. The preceding two values are the same because of (12a). Thus $e_\theta(\theta)$ exists, and

(13) $$e_\theta(\theta) = f_\theta(n, \theta)\big|_{n \in N(\theta)} .$$

We now prove the necessity of the condition. When $N(\theta) = \{n^*(\theta)\}$, the result is trivially true. When $N(\theta) = \{\underline{n}(\theta), \bar{n}(\theta)\}$, assume, by way of contradiction, that

(14) $$f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)} > f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)} .$$

The opposite inequality leads to the same contradiction. Now, define

(15) $$F(\theta') \equiv \frac{1}{\theta' - \theta} [f(\bar{n}(\theta), \theta') - f(\underline{n}(\theta), \theta')] .$$

It follows from (14), (15) and the second equality in (12b) that

(16) $$\lim_{\theta' \to \theta} F(\theta') = \lim_{\theta' \to \theta} \frac{1}{\theta' - \theta} [\{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)\} - \{f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)\}]$$

$$= f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)} - f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)} > 0 .$$

From (16), there exists a $\delta' > 0$ such that $F(\theta') > 0$ if $0 < |\theta' - \theta| < \delta'$. In turn, definition (15) yields

(17) $$f(\bar{n}(\theta), \theta') \gtrless f(\underline{n}(\theta), \theta') \text{ if } \theta' \gtrless \theta \text{ and } 0 < |\theta' - \theta| < \delta' .$$

Now, recall from (10) that there exists a $\delta > 0$ such that $N(\theta') \subseteq \{\bar{n}(\theta), \underline{n}(\theta)\}$ if $|\theta' - \theta| < \delta$. From (17), therefore, for $0 < |\theta' - \theta| < \min(\delta, \delta')$, the following holds: $N(\theta') = \{\bar{n}(\theta)\}$ if $\theta' > \theta$, and $N(\theta') = \{\underline{n}(\theta)\}$ if $\theta' < \theta$. Thus,

$$e_\theta^+(\theta) = \lim_{\substack{\theta' \to \theta \\ \theta' > \theta}} \frac{e(\theta') - e(\theta)}{\theta' - \theta} = \lim_{\substack{\theta' \to \theta \\ \theta' > \theta}} \frac{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)}{\theta' - \theta} = f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)} .$$

Similarly $e_\theta^-(\theta) = f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)}$. Thus, by (14), $e_\theta^+(\theta) > e_\theta^-(\theta)$, which contradicts the result that $e_\theta(\theta)$ exists. □

Note that the sufficiency of the condition in Theorem 1 can also be derived from Clarke's result concerning the generalized gradient [Clarke (1983, p. 47)]. However, our proof is simpler, and it does not need to employ the set of sophisticated techniques that have been developed by Clarke for more general purposes.

Next we show that both one-sided derivatives always exist.

THEOREM 2. The one-sided derivatives of the extreme value function always exists. That is, both $e_\theta^+(\theta)$ and $e_\theta^-(\theta)$ always exist.

PROOF. If $N(\theta) = \{n^*(\theta)\}$, or if $N(\theta) = \{\bar{n}(\theta), \underline{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta), \theta) = f_\theta(\underline{n}(\theta), \theta)$, then by Theorem 1, $e_\theta(\theta)$ exists, and thus both $e_\theta^\pm(\theta)$ exist. Now consider the case in which $N(\theta) = \{\bar{n}(\theta), \underline{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta), \theta) \neq f_\theta(\underline{n}(\theta), \theta)$. It follows from the necessity half of the proof of Theorem 1 that

(18) $$e_\theta^+(\theta) = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)} \text{ and } e_\theta^-(\theta) = f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)}, \text{ if } f_\theta(\bar{n}(\theta), \theta) > f_\theta(\underline{n}(\theta), \theta).$$

Analogously, it can be shown that

(19) $$e_\theta^+(\theta) = f_\theta(n, \theta)\Big|_{n=\underline{n}(\theta)} \text{ and } e_\theta^-(\theta) = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)}, \text{ if } f_\theta(\bar{n}(\theta), \theta) < f_\theta(\underline{n}(\theta), \theta).$$

Thus both $e_\theta^\pm(\theta)$ always exist. □

Define sets $N^+(\theta)$ and $N^-(\theta)$ as follows: $n \in N^+(\theta)$ (respectively, $n \in N^-(\theta)$) if and only if there are sequences $\{\theta_t\}$ and $\{n_t\}$ such that $\theta_t > \theta$ (respectively, $\theta_t < \theta$), $n_t \in N(\theta_t)$, and $\theta_t \to \theta$, $n_t \to n$, as $t \to \infty$. A consequence of Assumption 1(i) and expression (10) is that $N^+(\theta)$ and $N^-(\theta)$ are nonempty subsets of $N(\theta)$. We now present the envelope theorems.

THEOREM 3.

(a) If $e_\theta(\theta)$ exists, then $e_\theta(\theta) = f_\theta(n, \theta)\Big|_{n \in N(\theta)}$.

(b) $e_\theta^+(\theta) = f_\theta(n, \theta)\Big|_{n \in N^+(\theta)}$.

(c) $e_\theta^-(\theta) = f_\theta(n, \theta)\Big|_{n \in N^-(\theta)}$.

PROOF. Part (a) was established in the course of deriving (11) and (13).

To prove part (b), let $n \in N^+(\theta)$. Since $N^+(\theta)$ is a nonempty subset of $N(\theta)$, $n = \underline{n}(\theta)$ or $n = \bar{n}(\theta)$. Without loss of generality, assume that $n = \bar{n}(\theta)$. Therefore, there exists sequences $\{\theta_t\}$ and $\{n_t\}$ such that $\theta_t > \theta$, $n_t \in N(\theta_t)$, and $\theta_t \to \theta$, $n_t \to \bar{n}(\theta)$, as $t \to \infty$. From (10), $N(\theta_t) \subseteq N(\theta)$, for t sufficiently large. Since $n_t \in N(\theta_t) \subset \{\underline{n}(\theta), \bar{n}(\theta)\}$ and $n_t \to \bar{n}(\theta)$ as $t \to \infty$, $\bar{n}(\theta) \in N(\theta_t)$ must hold for all t sufficiently large. Hence, there exists a subsequence $\{\theta_s\}$ of $\{\theta_t\}$ such that $n_s = \bar{n}(\theta)$ for all s. In turn, by the existence of e_θ^+ and f_θ, we have

$$e_\theta^+(\theta) = \lim_{s \to \infty} \frac{e(\theta_s) - e(\theta)}{\theta_s - \theta} = \lim_{s \to \infty} \frac{f(\bar{n}(\theta), \theta_s) - f(\bar{n}(\theta), \theta)}{\theta_s - \theta} = f_\theta(n, \theta)\Big|_{n=\bar{n}(\theta)}.$$

This proves part (b).

The proof of part (c) is analogous to that of part (b). □

Some of the consequences of the preceding theorems and derivations can be summarized as follows. If $N(\theta)$ is a singleton, then $N^+(\theta) = N^-(\theta) = \{n^*(\theta)\}$, and $e_\theta(\theta) = f_\theta(n, \theta)\big|_{n=n^*(\theta)}$. In the case in which $N(\theta) = \{\underline{n}(\theta), \bar{n}(\theta)\}$, we have the following. If $f_\theta(\bar{n}(\theta), \theta) = f_\theta(\underline{n}(\theta), \theta)$, then $e_\theta(\theta) = f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)} = f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)}$. If $f_\theta(\bar{n}(\theta), \theta) > f_\theta(\underline{n}(\theta), \theta)$, then $N^+(\theta) = \{\bar{n}(\theta)\}$, $N^-(\theta) = \{\underline{n}(\theta)\}$, $e_\theta^+(\theta) = f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)}$, and $e_\theta^-(\theta) = f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)}$. If $f_\theta(\bar{n}(\theta), \theta) < f_\theta(\underline{n}(\theta), \theta)$, then $N^+(\theta) = \{\underline{n}(\theta)\}$, $N^-(\theta) = \{\bar{n}(\theta)\}$, $e_\theta^+(\theta) = f_\theta(n, \theta)\big|_{n=\underline{n}(\theta)}$, and $e_\theta^-(\theta) = f_\theta(n, \theta)\big|_{n=\bar{n}(\theta)}$. An informal version of some of these results was described in Section I.

III. GENERALIZATIONS

To keep mathematical details to the necessary minimum, we have presented a bare-bones version of our results in the previous section. The analysis entailed a single choice variable and a single parameter. In this section, we briefly describe several generalizations. For expositional ease, each generalization is discussed separately.

(a) If the maximand is locally strictly concave, rather than globally strictly concave, then our envelope theorems can be rephrased as local results. However, if the optimum is not well-defined then neither our results nor the standard envelope theorem can be applied. For instance, suppose that the maximand $f(n, \theta)$ is not strictly concave and n is not restricted to being an integer. Then , the standard theorem does not apply if an optimal n does not exist. Corresponding problems arise if n is restricted to being an integer.

(b) In the earlier analysis, the choice variable could be in the entire range of integers; that is, $n = \ldots -1, 0, +1, \ldots$. The same analysis holds if the choice variable is restricted to a pre-specified set of discrete values; that is, it can not assume some of the integer values. Only one minor conclusion of the preceding analysis changes: if there are two optimal values of n, they need not be neighboring integers. This is simply because the feasible set of n does not contain all neighboring integers.

A further extension is as follows. Consider the problem in which the choice variable is restricted to a collection of compact but disjoint sets of values of n. A special case of this problem occurs when the choice variable can take only certain discrete values. It is straightforward to see that our results hold for the more general problem.

(c) Most of the results are easily generalized to many choice variables and many parameters, instead of one each as examined earlier. Suppose $\theta = (\theta_1, \ldots, \theta_I)$ is a vector of parameters, and n is a scalar.

Then our results hold for each of the parameters. Next, suppose, θ is a scalar but $n \equiv (n_1, ..., n_J)$ is a vector. (The generalization in which θ as well as n are vectors is straightforward.) Each element of n can take only restricted discrete values. Assume f is strictly concave in the vector n. Define $n(\theta)$ as an optimal vector. Define the map N, and the sets $N(\theta)$, $N^+(\theta)$ and $N^-(\theta)$ accordingly. Then, parts (b) and (c) of Lemma 1 continue to hold as stated earlier. Part (a) of Lemma 1 becomes the following: $N(\theta)$ contains at most 2^J vectors. Theorems 1, 2 and 3 hold with corresponding modifications.

Furthermore, if the elements of n can take all integer values, then the following additional result will hold: for a given θ, any element of an optimal value of the vector n will differ by at most one from the corresponding element of another optimal value of the vector n. That is, if $n^1(\theta) \in N(\theta)$ and $n^2(\theta) \in N(\theta)$, then, for $j = 1$ to J, $n_j^1(\theta) - n_j^2(\theta) = 0$ or ± 1. This will obviously not be the case if the elements of n can take only restricted discrete values.

(d) Finally, consider a constrained optimization problem in which the parameter θ affects only the objective function. Assume n is a vector, each element of which is restricted to being an integer. Assume that the objective function is strictly concave in n. Then, since the constraints determine the set of feasible n, this set is unaffected by a change in θ. As far as our results are concerned, this problem is no different from the one discussed in the previous two paragraphs in which the elements of n can take only restricted discrete values.

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