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CENTER DISCUSSION PAPER NO. 621

LECTURE NOTES ON ECONOMIC GROWTH:

INTRODUCTION TO THE LITERATURE AND NEOCLASSICAL MODELS

VOLUME I

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December 1990

Notes: Center Discussion Papers are preliminary materials circulated to stimulate discussion and critical comments.

These notes were developed while I was assisting Robert J. Barro to teach the first year graduate macroeconomics class at Harvard in 1988-89 and 1989-90. The two vintages of first year students suffered the first draft of these notes and made all kinds of useful comments/amendments. Thank you kids. I am especially grateful to Susan Guthrie for carefully reading them and helping me think about section 1. I also received comments from Serge Marquié and (Kid) Casey Mulligan from Harvard, and Julie Lee from Yale.

This paper is the revised version of the first draft, November 1988.

See Discussion Paper No. 622 for Volume II of Lecture Notes on Economic Growth.

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VOLUME I

Abstract

This is a survey of the literature on Economic Growth. In the introduction we analyze the main differences between exogenous and endogenous growth models using fixed savings rate analysis. We argue that in order to have endogenous growth there must be constant returns to the factors that can be accumulated. A graphical tool is then developed to show that changes in the savings rate have different effects on long run growth in the two kinds of models; we show that only endogenous growth models are affected by shifts in the savings rate. We then explore two versions of the Ramsey-Cass-Koopmans neoclassical model where savings are determined optimally; one with exogenous productivity growth and one without.

KEY WORDS: Economic Growth, Increasing Returns, Externalities, Endogenous Growth

"The consequences for human welfare involved in questions like these are simply staggering: once one starts to think about them, it is hard to think about anything else". Lucas (1988), p. 5.

(1) INTRODUCTION TO GROWTH MODELS.

(a) Exogenous versus Endogenous Growth models: An Introduction

Most of the recent economic growth literature deals with "optimizing growth models" where consumers choose a consumption path by maximizing some kind of utility function subject to some intertemporal budget constraint¹. The complicated mechanics of dynamic optimization, however, obscures some of the important points and issues. Hence, before studying such models it will be convenient to start with the assumption that the savings rate is an exogenous constant: people save a constant fraction "s" of their income. This is what Solow (1957) and others, following the Keynesian multiplier hypothesis, do. Within an intertemporally optimizing framework, there is a configuration of parameters that will yield a constant savings rate². Hence, economists that do not believe in Keynesian

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Early economists used to confine the intertemporal optimization analysis to normative issues. The celebrated Ramsey 1928 paper (which deals with intertemporal optimizing economies) starts with the sentence "*The first problem I propose to tackle is this: how much of its income should a nation save?*" (p.543). Contemporaneous economists use intertemporal optimizing models for descriptive or positive analysis as well. Following Barro (1974), the representative agent is assumed to be a family or group of individuals linked to each other through bequests.

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Kurtz (1968) showed that if the production function is Cobb Douglas, necessary and sufficient conditions for constant transitional optimal savings rates are

(1) the utility function be Constant Elasticity of Intertemporal Substitution of the form $c^{1-(1/s)}/(1-(1/s))$, where s is the savings rate,

multipliers may want to think of an economy described by such configuration. Suppose also that the only asset in this (closed) economy is something we call K_t . You may want to think of K as being physical CAPITAL but it may also include other inputs that can be accumulated, such as knowledge or skills. Now imagine that the production function is Cobb-Douglas and that there are two aggregate inputs. One of them, K_t , can be purposely accumulated and the other L_t , cannot be accumulated, or it grows at a rate which is independent of individual choices (think of L as labor but it may also include other unreproducible resources such as land or energy).

$$(1.1) Y = K_t^\beta L_t^\alpha$$

The increase in K over time, which we will call³ $\dot{K} = dK/dt$ is aggregate net INVESTMENT. In a closed economy net investment must equal to SAVINGS minus DEPRECIATION. Using (1.1) and the fixed savings rate assumption:

$$(1.2) \dot{K}_t = sK_t^\beta L_t^\alpha - \delta K_t$$

Where δ is the (constant) depreciation rate. Population is assumed to be equal to employment (so we abstract from unemployment and labor force participation issues) and is assumed to grow at an exogenous constant rate, $\dot{L}/L = n$. Let us define lower case k as the capital-labor ratio (or capital per worker) K/L . By taking derivatives of k_t with respect to

(2) the discount rate be related to the parameters of the model through $\rho = \beta - s$, where ρ is the discount rate, and β is the share of capital in the production function.

See also Barro and Sala-i-Martin (1990) chapter 1 for an extension of this result.

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Throughout these notes we will denote time derivatives by "dots" on top of variables.

time we can rewrite (1.2) in per capita terms as⁴

$$(1.2)' \quad \dot{k}_t = s k_t^{\beta} L_t^{\alpha+\beta-1} - (\delta+n)k_t$$

Let us divide both sides of (1.2)' by k_t and define the growth rate of capital per worker $\dot{k}_t/k_t = \gamma_k$. We will call STEADY STATE the state where all variables grow at a constant (possibly zero) rate. Thus, in steady state γ_k is constant. Take logarithms and derivatives of both sides and get

$$(1.3) \quad 0 = (\beta-1)\gamma_k + n(\alpha+\beta-1)$$

This KEY equality deserves further attention. In the original Neoclassical growth model (Solow (1956) and Swan (1956)) the production function is assumed to exhibit Constant Returns to Scale in capital and labor (ie, $\alpha+\beta=1$) but Decreasing Returns to Capital alone ($\beta<1$). Notice that by virtue of the CRS assumption ($\alpha+\beta=1$), the second term in the right hand side of (1.3) is zero so we are left with

$$(1.3)' \quad 0 = (\beta-1)\gamma_k$$

but due to the Decreasing Returns to Capital assumption ($\beta<1$), equality (1.3)' says that the only sustainable steady state growth rate is $\gamma_k=0$. That is, in the CRS neoclassical model, the only possible steady state growth rate is zero. If the only possible growth rate is zero, how did the neoclassical theorists of the 50's and 60's explain long run growth?. They basically assumed that the economy gets (exogenously) more productive over time. In other words, they extended the technology in (1.1) to a more

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Notice that the difference between expressing the accumulation equation in levels or in per capita terms is the term nk added to δk . We can in fact think of nk as some extra depreciation since it represents the loss of capital per person due to the fact that, when population grows, we have to share capital with an increasing number of people.

general

$$(1.1)' Y_t = A(t)K_t^\beta L_t^\alpha$$

where $A(t)$ reflects the level of the technology which is assumed to be growing at the constant rate g , so $A(t)=A(0)e^{gt}$. The parameter " g " is often referred as the "exogenous productivity growth rate"⁵. In section 1.3 we will present an optimizing version of this model. We will see that income per capita, capital per capita, and investment per capita will end up growing at this exogenously given rate. We will also expand on the term $A(t)$ and on different ways to model productivity growth.

A second (and possibly more interesting) way to read equation (1.3) is the following: "In a CONSTANTS RETURNS TO SCALE model ($\alpha+\beta=1$) in order to have positive steady state growth ($\gamma_k>0$), the production function must exhibit CONSTANT RETURNS TO THE INPUTS THAT CAN BE ACCUMULATED, $\beta=1$."⁶ This simple fact underlies the CONSTANT RETURNS ENDOGENOUS GROWTH models developed in the late 80's. The implied production function is the following:

$$(1.1)'' Y_t = AK_t$$

The simplest growth model using this type of production function (Rebelo (1990)) is outlined in section 1.4. Notice that this type of production function does not give any role to exhaustible or non reproducible resources such as raw labor or land. One could argue, however, that what matters for production is not raw labor but, rather, quality

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It is called exogenous because it is unaffected by any of the parameters of the model such as the capital share or the savings rate.

⁶ Notice that we are saying CR to K and not CR to Scale. The distinction is important: the production function $Y=K^\beta L^{1-\beta}$ with $0<\beta<1$, exhibits constant returns to scale (if we multiply all inputs by $\lambda>1$ we get λ times as much output) but Decreasing Returns to Capital (since if we multiply capital by λ we get less than λ times as much as output).

adjusted labor. The quality of the labor force (often called Human Capital) is accumulated as each generation is more knowledgeable than the one before. When one combines physical and human capital into some broad measure of capital, the aggregate production function will look like the linear AK function postulated above. This is the approach taken by Lucas (1988) and Uzawa (1956). A version of these models is presented in section 1.7.

Barro (1990) and its extensions outlined in Barro and Sala-i-Martin (1990) assume that the two inputs of production are private physical capital and publicly provided inputs such as roads, infrastructure or law enforcement. Output exhibits constant returns to both inputs. In section 1.5 we will show that this setup ends up being similar to postulating an AK production function where K must again be interpreted as a broad measure of capital.

Notice that (as can be seen from equation (1.3)', the steady state growth rate γ_k derived from these models is positive without assuming exogenous productivity growth. As we will see in the next subsection, the parameters of the model (in particular the savings rate) will determine this growth rate. Because the growth rate is determined within the model, (in other words, it depends on the other parameters of the model) these are often called "ENDOGENOUS GROWTH MODELS".

Finally, equation (1.3) allows for one more possibility. If the population growth rate is zero ($n=0$), we can have nonreproducible inputs ($\alpha>0$) together with ENDOGENOUS GROWTH ($\gamma_k>0$) if there are CONSTANT RETURNS TO THE INPUTS THAT CAN BE ACCUMULATED ($\beta=1$). But notice that this implies OVERALL INCREASING RETURNS TO SCALE ($\alpha+\beta>1$). This possibility gives rise to the so called "INCREASING RETURNS ENDOGENOUS GROWTH MODELS"⁷.

Of course, if we plainly postulate an Increasing Returns to Scale (IRS) production function we run into trouble since we cannot find a set of

⁷ As can be seen from equation (1.3), when the population growth rate is positive, the increasing returns to scale models ($\alpha+\beta>1$) run into trouble since there is no γ_k that satisfies the key equality. What happens in this circumstances is that the growth rate is never constant but, rather, it increases over time.

prices to support a general competitive equilibrium. There are at least two ways to get around this problem.

(a) The first one (due to Alfred Marshall) is to introduce IRS at the aggregate level but CRS at the firm level. This can be formulated through production externalities or spillovers: each firm's decision affects all other firms output, but none of the firms takes this into account. Hence, all the firms face a "concave" problem which has a competitive solution. The economy as a whole, however, faces an IRS production function which, under some conditions that we will mention in a second, generates endogenous growth. The Cobb Douglas version of this production function is

$$(1.1.)'' Y_t = AK_t^\beta L_t^{1-\beta} \kappa_t^\psi$$

where K_t is private capital and κ_t is the aggregate capital in the economy. Individual firms do not think they can affect κ so they take it as given. Notice that under these circumstances firms face a perfectly defined concave problem so the Kuhn-Tucker theorems apply. In the aggregate, however, total capital will equal the sum of individual capitals and therefore $\kappa=K$. Thus the aggregate production function will be

$$(1.1)'''' Y_t = AK_t^{\beta+\psi} L_t^{1-\beta}$$

Notice that if the size of the externality is "correct" (that is if $\beta+\psi=1$) we will have CONSTANT RETURNS TO CAPITAL in an INCREASING RETURNS TO SCALE world. Thus, by modeling IRS through externalities we get around the problem of inexistence of competitive equilibrium. As it is well known, however, these competitive equilibrium models with externalities will be NON OPTIMAL. In section 1.6 we show how Romer (1986), following Arrow (1962) and Sheshinski (1967), postulates capital spillovers (externalities) in the aggregate production function and finds that the model generates steady endogenous growth when $\beta+\psi=1$.

(b) A second way to get around the existence of the competitive equilibrium problem is to drop the assumption of competitive behavior. This is sometimes called the Chamberlinian approach to increasing returns. This

approach is interesting for a variety of reasons, one of the main ones being that under imperfect competition the rewards to all inputs of production does not exhaust total output. Hence, there are rents that can be assigned to activities not directly productive but that may contribute to the expansion of the frontiers of knowledge such as R&D. Not surprisingly, therefore, this approach has been extensively used by economists that think that R&D is an important source of economic growth. In section 8 we explore a model of R&D and growth taken from Barro and Sala-i-Martin (1990, a)⁸ where firms invest in R&D in search of new varieties of capital goods. In that model, there are NO decreasing returns to the introduction of new varieties so the incentive to perform R&D never diminishes, which keeps the economy growing⁹.

Of course one could have models with both imperfect competition and externalities. In fact there is an important line of research that combines R&D (with imperfect competition) with externalities. It emphasizes R&D as some activity exercised by firms in search for new varieties of products or higher quality products. As a side product, R&D increases the general stock of knowledge which has two effects. First, it decreases the cost of further research (so the incentive to perform R&D remains positive and Knowledge grows at a constant rate forever). And second it increases the productivity of other inputs (such as labor) in the production of a manufacturing good. Therefore, given that the stock of knowledge grows at constant rate, so does the manufacturing good. Models of this type include Aghion and Howitt (1989), Grossman (1989) and Grossman and Helpman (1989 d and e).

Before showing the mechanics of all these models, let us introduce

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This model, in turn, is an extension of Romer (1987) and Grossman and Helpman (1989, a).

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There is a third way to model increasing returns in a model of perfectly competitive firms and that is to introduce imperfect financial markets. This approach has been taken by Greenwald, Salinger and Stiglitz (1990).

a graphical device that will further clarify the basic difference between exogenous and endogenous growth models. It will also help us understand why the savings (or investment) rate does not affect long-run growth in the first one and does so in the latter.

(b) The Role of Saving and Investment: a Graphical Exposition.

We can often hear economic advisors to third world countries say that one of the necessary conditions for economic growth and development is the increase in national savings rate. Higher savings will lead to higher investment (since in a closed economy the two must be equal) and higher investment will lead to more rapid economic growth. In this section we will analyze under what conditions this policy recommendation is valid.

Let us keep assuming that people save a constant fraction of their income and that the government can influence it (for instance through distortionary income taxes). Suppose that, for whatever reason, the government manages to increase the economy's savings rate. What will the long run effects of such policy be?

In order to answer this question, let us start by assuming that the production function is constant returns to scale ($\alpha + \beta = 1$) and dividing both sides of the (per capita) capital accumulation equation (1.2)' to get

$$(1.4) \quad \dot{k}_t / k_t = s A k_t^{-(1-\beta)} - (\delta + n)$$

The left hand side of this equation is the instantaneous growth rate. Equation (1.4) says that the growth rate is the difference between $s A k_t^{-(1-\beta)}$ and $(\delta + n)$. We depict these two functions in Figure 1. The function $\delta + n$ is independent of k so it is a flat line. In the neoclassical model $\beta < 1$ applies. This implies that the function $s A k_t^{-(1-\beta)}$ is downward sloping in k , and approaching zero asymptotically. Notice that the two curves cross at a point k^* , the steady state capital labor ratio. Let us now consider an economy with an initial level of capital k_0 lower than k^* . The initial growth rate of capital will be very large (notice that, according to (1.4), the growth rate is the vertical difference between the two curves) and it will be decreasing over time. Imagine for a second that

we are in the steady state and, suddenly, the savings rate s , increases. Figure 1 suggests that the curve $sAk_t^{-(1-\beta)}$ will shift to the right and nothing will happen to the $(\delta+n)$ line. We can see that the following things are true:

(a) the growth rate will immediately increase.

(b) the growth rate will be falling over time until, eventually, it goes back to zero.

(c) the steady state capital labor ratio is higher.

Hence, an increase in the savings rate generates a short term increase in the growth rate and an increase in the steady state LEVEL of capital per worker. It does not affect, however, the long run or steady state growth rate, which is still zero. Under normal parameterizations, the speed of convergence towards the new steady state is quite fast. For instance, Barro and Sala-i-Martin (1990) suggest that the model predicts that half the distance between k_0 and k^* disappears in less than 6 years!

As it was mentioned above, Figure 1 suggests that the growth rate for an economy which starts below the steady state is high and decreasing. This, of course, implies that if economies differ ONLY on the initial capital labor ratio, we should observe poor economies grow faster than rich ones (in Figure 1, different economies would be represented by different stocks of k_0 but all of them would have the same steady state k^*). Economists call this the "convergence hypothesis". This hypothesis is certainly true, but notice that there is a big ONLY on it. That is, economies may differ NOT ONLY in the capital labor ratio but also in the level of technology (A), the savings rate (s), the depreciation rate (δ), or the population growth rate (n). If countries differ in one or more of these parameters, they will end up in different steady states.

In Figure 2 we show the behavior of two economies, one called P (poor) and one called R (rich). The poor economy has a lower initial capital stock $k_{0P} < k_{0R}$, (that is why it is called poor). We assume that the poor economy also has a smaller savings rate so it converges to a smaller steady state capital labor ratio, $k_P^* < k_R^*$. Notice that in this particular example, it happens that the poor economy grows less than the rich one so

there is no convergence in the absolute sense. Yet there is CONDITIONAL convergence in the sense that each country converges to its own steady state at diminishing growth rates. Empirically, this means that if we hold constant the steady state, poor countries will grow faster than rich ones. If we don't, however, we will not see poor economies growing faster unless they are very similar (in the sense that they converge to similar steady states).

Barro and Sala-i-Martin (1990) find that this feature of the neoclassical model can be found in the data. They find that the States of the U.S. display absolute convergence while countries in the world do not. Holding constant the steady state, however, there is convergence across countries also. This makes sense if we think that the states of the U.S. are similar in the sense of having the same tastes and technology so they converge to the same steady state. This is certainly not true for the large cross section of countries, so they display conditional convergence only. For related studies on convergence see Baumol (1986), DeLong (1988), Dowrick and NGuyen (1989), Mankiw, Romer and Weil (1990), and Sala-i-Martin (1990).

Let us now expand the neoclassical production function by introducing exogenous productivity growth. Recall that the production function now looks like

$$(1.1)' \quad y_t = A(t)k_t^\beta$$

where $A(t) = A(0)e^{gt}$. Notice that, in terms of Figure 1, this specification implies that the curve $sA_t k^{-(1-\beta)}$ keeps shifting over time at a rate g . This implies that the steady state capital labor ratio k^* keeps shifting at the same rate. This is how the neoclassical model explains long run growth.

In Figure 3 we show that the implications from changing the savings rate are very different when we consider a simple endogenous growth model. If the capital share is 1 ($\beta=1$), the $sA k^{-(1-\beta)}$ curve is a flat line at sA . If we assume that the economy is productive enough so as to have $sA > \delta + n$, then the growth rate (difference between the two lines) is constant. In other words, the economy grows at a constant rate equal to $sA - (\delta + n)$. Notice also that in this case, an exogenous increase in the savings rate

increases both the short run and the steady state growth rates. Hence, contrary to the neoclassical predictions, policies directed to increases in the savings (and investment) rates will have long run growth effects. Further, notice that if economies differ in the initial capital stock ONLY, it is not true anymore that poor ones will grow faster than rich ones. Finally, this model predicts that a temporary recession will have permanent effects. That is, if the capital stock temporarily falls for some exogenous reason (an earthquake, a natural tragedy or a war that destroys part of the capital stock), the economy will not grow temporarily faster so as to go back to the prior path of capital accumulation. The endogenous model described here predicts that after such a temporary reduction in the capital stock, the growth rate will still be the same so the loss will tend to be permanent¹⁰.

Figure 4 depicts the case where $\beta > 1$ (IRS in the inputs that can be accumulated¹¹). The curve $sAk^{-(1-\beta)}$ is upward sloping (and if $\beta > 2$ its slope is increasing!). Notice that this implies growth rates that increase over time. We will refer to this case again in section 6 (Romer (1986)).

(c) The Harrod-Domar Model.

Long before the neoclassical theory came to life in the mid 50's, the most popular model of economic growth was the so called the Harrod-Domar model (developed by Harrod (1939) and Domar (1946)). We can use the graphical tool developed in the last subsection to learn about this older

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There are unbelievable amounts of papers on the existence of a unit root in macroeconomic aggregates such as GNP. There seems to some evidence that, for the United States, GNP is non stationary, which is what this simple model would predict. See Blanchard and Fischer (1989) Ch. 1 for discussion of these issues.

11 In this case the assumption of CRS $\alpha + \beta = 1$ must be dropped since a negative labor share has little economic sense. Think of this case as one where $\alpha = 0$ (so all inputs can be accumulated) and $\beta > 1$ (so there are both IRS and IR to capital).

growth model.

Harrod and Domar tried to put together two of the key features of the Keynesian economics -- the multiplier and the accelerator -- in a model that explained long run economic growth. We have been using the multiplier assumption (savings is a fixed proportion of income) all along so let us describe the differential feature of the Harrod-Domar model: the accelerator. The increase in capital required to produce a given increase in output is assumed to be a constant number. In particular, it is independent of the capital labor ratio. That is

$$(1.6) \Delta Y_t = A \Delta K_t$$

where A is constant. Notice that one production function that satisfies this relationship is the linear AK production function used by the CRS Endogenous Growth models. Thus one could be tempted to identify the Harrod-Domar model with the new Endogenous Growth Models. Yet that would be a mistake. The reason is that Harrod and Domar were very concerned about the effects of growth on long run employment and unemployment¹² (their study could be thought to be an explanation for the then existing long run unemployment of the Great Depression). Although they never introduced a specific production function, the fact that they worried so much about employment seems to indicate that they were not talking about a function such as "AK", where there is no role for inputs such as labor.

Another production function which satisfies the accelerator principle and which is closer to the spirit of what Harrod and Domar had in mind is the Leontief Fixed Coefficients function. Output is assumed to be produced by a fixed proportion of capital and labor. Given this proportion, an increase in the level of one of the inputs without a corresponding increase in the other leaves output unchanged. Thus, we should replace the

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In fact, Domar's paper is called "Capital Expansion, Rate of Growth, and Employment".

production function (1.1) by

$$(1.1) \quad Y_t = \min(AK_t, BL_t)$$

where A and B are exogenous production parameters. After rewriting this function in per capita terms $y = \min(Ak, B)$ we plotted it in Figure 5. We see that there is a capital labor ratio $k^* = B/A$ that has the following property: for capital labor ratios smaller than k^* , Ak is smaller than B so output is determined by Ak . For capital labor ratios larger than k^* , Ak is larger than B so output is determined by B . In other words, this production function can be rewritten as

$$(1.1) \quad y_t = \begin{cases} Ak_t & \text{for all } k_t < k^* = B/A \\ B & \text{for all } k_t > k^* = B/A \end{cases}$$

Notice that this technology is similar to the Ak model but only for small capital labor ratios. For large ones, however, the production function is flat so the Marginal Product of Capital is equal to zero. We can now apply the basic savings equal investment equality (1.5) to this technology to get.

$$(1.7) \quad k_{t+1} = \begin{cases} sAk_t + (1-\delta-n)k_t & \text{for } k_t < k^* = B/A \\ sB + (1-\delta-n)k_t & \text{for } k_t > k^* = B/A \end{cases}$$

As Harrod and Domar pointed out, there are three possible configurations of parameters each of which will yield different implications for growth and employment.

CASE 1: $sA < \delta + n$

When the savings rate and/or the marginal productivity of capital are very small compared to the aggregate depreciation rate (which includes population growth), there is no possible steady state. This is pictured in Figure 6. Notice that the economy converges to a point where the logarithm of the capital labor ratio is minus infinity (so the capital labor ratio

converges to zero). In this case not only there will be unemployment (because $AK < BL$) but it will grow over time. Harrod and Domar thought that this was a good description of the observed large and growing unemployment rates of the 30's.

CASE 2: $sA = \delta + n$

When, by chance, the exogenously given savings rate and marginal product of capital are such that $sA = \delta + n$, the economy will reach a steady state where all the per capita variables grow at a zero rate. In Figure 7 we show that, in this case, the initial capital labor ratio will be the steady state one.

CASE 3: $sA > \delta + n$

The third case, depicted in Figure 8, is one where the marginal product of capital or/and the savings rate are very large relative to the depreciation rate. We see in Figure 8 that, for small capital labor ratios, this case looks very much like the Rebelo model. But as the capital labor ratio grows, the labor requirement gets binding (that is we hit $k_t = B/A$ at some finite point in time). After this point, the marginal product of capital is zero and the per capita growth process stops. The steady state capital labor ratio, k^* will be one where there will be unemployed machinery.

Two out of the three configurations of parameters yield long run equilibria where there are idle resources and the only that does not, would be achieved only by chance; remember that all the relevant parameters - A , s , δ and n - were given by mother nature. The question is why in the world would mother nature be so kind as to give us exactly that configuration of parameters?. In other words, the chance of them being such that the equality above is satisfied are quite small.

At the time, the Neoclassical approach was seen as a way of solving this knife edge property of the Harrod-Domar model. That is, the neoclassical production function achieves the equality between sA and $\delta + n$ by allowing for A (the marginal product of capital) to be variable in k ¹³. We

¹³ And we know that there will be a level of capital k such that the

should just mention that there are other non neoclassical ways of achieving this equality. One of them, proposed by the old Cambridge School in England was to argue that the savings rate was endogenous. They thought that workers had a different marginal propensity to save from capitalists. Hence, so they argued, in the process of economic growth there will be changes in the distribution of income that will lead to changes in the aggregate savings rate in such a way that the equality between sA and $\delta+n$ will be guaranteed¹⁴. We will not talk about the Cambridge school of thought anymore.

(d) The "Sobelow" Production Function.

Finally, with this graphical approach we can see that the growth paths are not limited to the cases seen up to now. We could find functions that behave in some other ways, we may discover new growth models and new transitional dynamics towards steady states. Consider Figure 9: The steady state is similar to the one described by the Rebelo model but the transitional dynamics are different. One production function that exhibits such dynamics is the following:

$$(1.9) \quad Y_t = AK_t + BK_t^\beta$$

This production function was first proposed by Kurtz (1968) and Gale and Sutherland (1968) and later reintroduced in the endogenous growth

marginal product of capital is equal to $(\delta+n)/s$ since the marginal product is assumed to range from zero ($f'(\infty)=0$) to infinity ($f'(0)=\infty$) in a continuous fashion.

¹⁴ This was one of the main differences between the Cambridge (U.S.) and the Cambridge (U.K.) school of thought. The other main difference was that the British rejected the Neoclassical production function and, in particular, they rejected the notion of aggregate capital stock. They thought of capital as a number of different machines which, combined with different types of workers yielded different types of output. Such a heterogeneous set of objects, they argued, is impossible to aggregate into a single variable called Aggregate Capital stock. See Robinson (1954).

literature by Jones and Manuelli (1990). Notice that this function is half way between Solow (BK^β) and Rebelo (AK)¹⁵. It has all the nice properties required by the Kuhn Tucker theorem so we can apply straightforward optimization techniques to find solutions.

In per capita terms the Solow production function is concave and, as k tends to infinite, the marginal product of capital approaches zero. The Rebelo production function in per capita terms is linear with slope equal to A for all values of k . The Sobelow production function is also concave for all capital-labor ratios. As k goes to infinity, however, the slope of the production function does not go to zero but to A . For large levels of k , therefore, it gets arbitrarily close to the Rebelo production function. Hence, the only difference between the Solow and the Sobelow functions is the latter does not satisfy the Inada condition.

We observe in figure 9 that $sf(k)/k$ now is not going to zero asymptotically but to A . As Kurtz (1968) noted, if A is sufficiently large (in this case this means if $sA > \delta + n$), then the steady state growth rate is positive, even though there is a transition period where growth rates are decreasing monotonically. It is worth noticing that if the economy has been going on for a while, the decreasing returns part of the production function will be almost irrelevant so we might as well deal with the (simpler) linear technology described above.

(e) Poverty Traps.

Another possibility could be the one in Figure 10. Here we see the function $sf(k)/k$ crossing the horizontal line $(\delta + n)$ twice so there are two steady states. The lower crossing represents a "stable poverty trap". That is, countries whose initial "capital" (here we define capital in a broad sense that includes all inputs that can be accumulated) is very low will tend to this zero growth-low income trap. In fact all countries whose

initial capital lies to the left of k_2^* will fall into this trap. Countries that start to the right of this trap will tend to a constant growth steady state à la Rebelo.

In the next two sections we will present the optimizing versions of the Neoclassical models we have been talking about in this introduction. In sections 4 through 7 we present the "new" growth models of the 80's. It is useful to think about them in terms of being optimal saving versions of the $\beta=1$ model we just presented in this section.

(2) The Ramsey-Cass-Koopmans model

(a) The Model.

All optimizing growth models we will assume that consumers choose a path of consumption so as to maximize a utility function of the form:

$$(2.0) \quad U(0) = \int_0^{\infty} e^{-\rho t} u(c_t) L_t dt = \int_0^{\infty} e^{-\rho t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] L_t dt.$$

Where ρ is the discount rate, c_t is consumption per capita at time t and L_t is population. We can think about horizons being infinite (despite the fact that, obviously, lifetimes are not) if, following Barro (1974) we think that individuals care about their utility AND about their children's utility. In this sense, we must think of the agent as being a dynasty or family the number of individuals of which grows over time. Under this interpretation, the discount rate (which was described by Ramsey (1928) as "ethically indefensible and arises only from the weakness of the imagination"¹⁶, (p. 543) at the individual level) represents the fact that individuals care more for their own utility than the one of their children.

¹⁶ Ramsey was considering the optimal choice from a government's point of view. He thought that introducing a discount rate was ethically indefensible because that meant that the government was giving a larger weight to current than to future generations.

Since c_t is consumption per capita, $u(c_t)$ is the instantaneous per capita felicity. Hence, the instantaneous felicity for the whole family is equal to the individual times the number of people in the family.

We assume that there is only one good (cookies). We will assume that households OWN the firms (or that there is only household production¹⁷) so they can consume this good or they can nail it to the floor. The reason why anyone would do such a horrendous thing is that cookies nailed to the floor can be used to produce more cookies in the future. For lack of a better name, all cookies nailed to the floor will be called "capital" and will be represented by K_t . We assume that there is nobody else in the universe, so all the cookies produced will have to be consumed or nailed. Hence the increase in existing capital (called investment) must be equal to saving. If we let k_t be per capita capital (K_t/L_t), the following resource constraint must be satisfied:

$$(2.0)' \quad \dot{k} = f(k) - c - nk - \delta k$$

Notice that n is like a "depreciation rate" because it represents the fraction of resources that we need to give to new generations. The key Neoclassical assumption is a production function that expresses NET output in per capita terms as a function of capital per capita with the following properties: twice differentiable, with $f(0)=0$, $f'(k)>0$, $f''(k)<0$, $f'(0)=\infty$ and $f'(\infty)=0$ ¹⁸. A simple Neoclassical production function that we will be using throughout is the Cobb Douglas: $f(k)=k^\beta$ with $0<\beta<1$. Population is

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As we will show in the next section, the results will be the same we would get if we assume that households own capital and labor and sell their services to competitive firms in exchange for wages and rents.

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The last two conditions (the Inada conditions) are often swept under the rug. They are of crucial importance because, as Kurtz (1968) showed, the mathematical difference between an endogenous and exogenous growth model is the condition $\lim_{k \rightarrow \infty} f'(k)=0$. This point has been emphasized

also by Jones and Manuelli (1990).

assumed to grow at the (exogenously given) rate n so we can rewrite the program as:

$$(2.1) \text{ MAX } U(0) = \int_0^{\infty} e^{-(\rho-n)t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] dt$$

s.t. $\dot{k} = f(k) - c - nk - \delta k$
 $k(0) > 0$ given

For $U(0)$ to be bounded ($U(0) < \infty$, and the program to be meaningful at all) we need the term inside the integral to go to zero as t goes to infinity. This implies

$$\lim_{t \rightarrow \infty} e^{-(\rho-n)t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] = \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \frac{c_t^{1-\sigma}}{1-\sigma} - \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \frac{1}{1-\sigma} =$$

In steady state c_t will be constant (we will show later). Hence, if this limit has to be zero, it must be the case that

$$(2.2) \quad \rho > n.$$

To solve the model, we set up the corresponding Hamiltonian

$$(2.3) \quad H(t) = e^{-(\rho-n)t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[f(k) - c - nk - \delta k \right]$$

where ν is the dynamic Lagrange multiplier (or shadow price of investment). The first order conditions are the following:

$$(2.4) \quad H_c = 0 \Leftrightarrow e^{-(\rho-n)t} c_t^{-\sigma} - \nu = 0$$

$$(2.5) \quad H_k = -\dot{\nu} \Leftrightarrow \dot{\nu} = -\nu(f'(k) - n - \delta)$$

$$(2.6) \quad \text{TVC } \lim_{t \rightarrow \infty} (k_t \nu_t) = 0$$

Equation (2.4) says that at the margin, the value we will give to

consume one more unit will be equal to the value we will give to invest one more unit (that is, we will be indifferent between consuming and investing the unique good). Take logs of (2.4) to get $-(\rho-n)t - \sigma \log(c_t) = \log(\nu)$. Now take the derivative with respect to time to get:

$$(2.7) \quad -(\rho-n) - \sigma(\dot{c}/c) = \dot{\nu}/\nu$$

so

$$(2.7)' \quad \dot{c}/c = \sigma^{-1}(-\rho+n-\dot{\nu}/\nu)$$

We can now plug this in (2.5) to get the traditional condition for consumption growth:

$$(2.8) \quad \dot{c}/c = \sigma^{-1}(f'(k) - \rho - \delta)$$

This equation can be rewritten as $\rho + \sigma(\dot{c}/c) = f'(k) - \delta$ and interpreted as follows: The left hand side represents the return to consumption. The discount rate represents the gain in utility from consuming today since we prefer consumption for ourselves rather than for our children. The return to consumption also includes $\sigma\dot{c}/c$. If we want to smooth consumption over time ($\sigma > 0$), then we want to increase consumption today, whenever we expect consumption to be higher in the future (ie, when $\dot{c}/c > 0$). The return to saving (and investment) is the marginal product of capital minus the depreciation rate, δ . Optimizing individuals should, at the margin, be indifferent between consuming and investing. This indifference is the one represented by equality (2.8).

Using the Cobb-Douglas technology, $(y = k^\beta)$, equation (2.8) can be written as

$$(2.8)' \quad \gamma_t = \sigma^{-1}(\beta k^{-(1-\beta)} - \rho - \delta)$$

If we define steady state as the state where all the variables grow at a constant (and possibly zero) rate, equations (2.8) together with the capital accumulation equation (2.0)' say that there is a unique steady state k^* which ensures that capital and consumption per capita do not grow¹⁹. Hence this model says that, in the steady state, all variables in per capita terms do not grow at all. Alternatively, all "level" variables grow at the same rate as population, which is assumed to be exogenous.

(b) Competitive Solution.

Since this model is concave (concave preferences and technology) and there are no externalities of any kind, the OPTIMAL PROGRAM (command economy solution) will yield the same solution as the COMPETITIVE EQUILIBRIUM PROGRAM, provided that consumers and firms have RATIONAL EXPECTATIONS (since these models do not have uncertainty, rational expectations implies PERFECT FORESIGHT). We can show that the competitive solution is the same as the one we solved. On the consumption side, individuals maximize (2.0) subject to

$$(2.9) \quad \dot{k}_t = w_t + r_t k_t - c_t$$

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We can show that the only sustainable growth rate is zero: take the constraint $\dot{k} = k^\beta - c - nk - \delta k$ and divide it by k . Define $k/k = \gamma_k$ which in steady state will be a constant (by definition of steady state!!). Realize that $k^{(\beta-1)} = (\gamma\sigma + \rho)/\beta$. Rearrange to get $c/k = (\gamma\sigma + \rho)/\beta - \gamma_k - n - \delta = \text{constant}$. Take logs and derivatives to conclude that $\dot{c}/c = \dot{k}/k = \gamma_k = \gamma$. Now consider again the equality $k^{(\beta-1)} = (\gamma\sigma + \rho)/\beta$. The RHS is a constant. Take logs and derivatives of both sides to conclude that $(\beta-1)\gamma_k = 0$. This is another way to show what we saw in section 1: if there are DR to k ($\beta < 1$), then the steady state growth must be zero. The only way to achieve nonzero growth rates is to have CR to k ($\beta = 1$).

where w_t is the return to labor (wage) and rk_t is the return to capital (we are abstracting again from depreciation and population growth). In the other side, competitive firms will price factors at marginal costs so:

$$(2.10) \quad r_t = f'(k_t) - \delta k_t$$

$$w_t = f(k_t) - k_t f'(k_t)$$

Notice that $w + rk = f(k) - kf'(k) + kf'(k) - \delta k = f(k) - \delta k$ so substituting (2.10) into the individual budget constraint will give the original resource constraint (2.0)'.

(c) Transitional Dynamics, Golden Rule, and Dynamic Efficiency

The neoclassical model just outlined is NOT a very interesting model of steady state growth (because steady state growth is zero). It is nevertheless an interesting model of the transition towards the steady state. This transition is shown in Figure 11. The vertical line is the $c=0$ locus. The upward sloping line is the $k=0$ locus representing the resource constraint (2.0)'. Notice that the economy can converge to the steady state from below or from above. The interesting case is the one where we converge from below so we actually grow. Along this path $k/k > 0$. Per capita capital grows, but it does so at a decreasing rate (which ends up being zero in steady state). As the capital labor ratio increases, the marginal product of capital falls and, therefore so does the interest rate.

It is worth noticing that, in Figure 11, there is a level of capital called k_{gold} (for Golden Rule). This is the capital level that maximizes steady state consumption. From the budget constraint we see that when $k=0$, steady state consumption is equal to $c^* = f(k) - (\delta + n)k$. The capital labor ratio that maximizes c^* is the one that satisfies $f'(k_{gold}) = (n + \delta)$. This level capital divides the set of capital labor ratios in two. Capital levels above the Golden Rule have the property that in order to achieve higher steady state consumption, the economy needs to get rid of some

capital. In other words, in order to achieve higher consumption in the future the economy would need to dissave (which of course means higher consumption today). Therefore, if the economy were to find itself in one of such capital levels, everybody could increase consumption at all points in time. The points above k_{gold} are called the DYNAMIC INEFFICIENT REGION because some generations could be made better off without making any generation worse off. Notice that for capital levels below the Golden Rule, if the economy wants to increase the steady state consumption, it needs to accumulate or save: higher consumption tomorrow would have to be traded for lower consumption today. This region is called DYNAMIC EFFICIENT REGION.

We can integrate (2.5) forward between 0 and t and get

$$(2.5)' \quad \nu_t = \nu_0 e^{-\int_0^t (f'(k_s) - \delta - n) ds}$$

which, after substituting in the TVC yields

$$(2.6)' \quad \lim_{t \rightarrow \infty} \nu_0 e^{-\int_0^t (f'(k_s) - \delta - n) ds} k_t = 0$$

Since ν_0 is positive, it must be the case that the second term in (2.6)' is equal to zero. Notice also that this implies that in the steady state, the marginal product of capital must be larger than $\delta + n$. This condition is always satisfied in steady state if we assume that utility is bounded. Recall that this condition required $\rho > n$ and the steady state implies $f'(k) = \rho + \delta$ so this ensures that $f'(k) > n + \delta$. Notice how this inequality implies that the capital per capita in the steady state will be dynamically efficient (to the left of the golden rule)²⁰.

(d) Ruling out explosive paths.

It just remains to be shown that, given the saddle-path stability property of the model, the economy will find itself on the stable arm. To show this we must rule out all other possible paths. Suppose that we start with the capital stock k_0 in Figure 11. Let c_0 the consumption level that corresponds to the saddle path. Let us imagine first that the initial consumption level is $c'_0 > c_0$. If this is the case, the economy will follow the path depicted in Figure 11: at first both c and k will be growing. At some point the economy will hit the $\dot{k}=0$ schedule and, after that, consumption will keep growing yet capital will be falling. Hence, the economy will hit the zero capital axes in finite time. At this point, there will be a jump in c (because with zero capital there is zero output and therefore zero consumption) which will violate the first order condition (2.8). In order to show that the economy will hit the $\dot{k}=0$ axes in finite time just realize that k_t can be rewritten as

$$(2.11) \quad k_t = k_T + \int_T^t \dot{k}_s ds$$

Suppose that T is the time at which we hit the $\dot{k}=0$ schedule. After that moment, k_t evolves according to (2.11). If we show that dk/dt is negative, we will have that k is negative and falling so k is falling at increasing rates. This, of course implies that there is a time T' at which it will be zero. The derivative of k with respect to time is ((from 2.0)')

Recall that k^* is such that $f'(k^*) = \rho + \delta$ and that the bounded utility condition (2.2) implies that $\rho > n$. Therefore $f'(k^*) = \rho + \delta > n + \delta = f'(k_{gold})$. Since the production function is concave ($f'' < 0$) it follows that $k^* < k_{gold}$.

$$(2.12) \quad \dot{k}/dt = \left[f'(k) - (n+\delta) \right] k - \dot{c} < 0$$

notice that since $k_t < k^*$, we know that $f' > n+\delta$. We also know that $\dot{k} < 0$ and $\dot{c} > 0$ so overall, (2.12) is negative which implies that k is falling at increasing rates. Hence, if we are in this region in finite time (ie if we hit the $k=0$ schedule in finite time), then k_t will hit zero in finite time. Therefore, it ONLY remains to be shown that we will hit the $k=0$ schedule in finite time). We can show that this is the case because around the $k=0$ schedule, consumption increases at increasing rates so it will reach the $k=0$ schedule in finite time. Notice that the derivative of $\dot{c}$ with respect to time is

$$(2.14) \quad \dot{dc}/dt = (1/\sigma) \left[f'(k) - (\delta+\rho) \right] \dot{c} + (c/\sigma) \left[f''(k) \right] \dot{k}$$

notice that the first term is positive and, around the $k=0$ schedule it dominates the second term so overall $\dot{dc}/dt > 0$. Hence, if initial consumption is larger than the one required by the stable arm we will first hit the $k=0$ schedule in finite time and then hit the $k=0$ axes in finite time. This will imply a finite time jump in consumption which will violate the first order condition (2.8). Hence, it is not optimal to start above the stable arm.

Let us imagine next that we start below the stable arm. The dynamics in Figure 11 tell us that we will converge to k^{**} . Notice that this path will violate the transversality conditions since $k^{**} > k_{gold}$. That is

$$\lim_{t \rightarrow \infty} k^{**} e^{-\int_0^t (f'(k^{**}) - \delta - n) ds} > 0$$

which is positive since the term inside the integral is negative.

Hence, initial consumption levels below the stable arm are not optimal either. We are left, therefore, with the stable arm as the UNIQUE optimal path of this model.

(e) Convergence and Convergence Regressions.

The Neoclassical model just described has the additional implication that, if all countries share the same production and utility parameters, then poor countries tend to grow at a faster rate than rich ones. In other words, income or output levels will converge over time. Following Sala-i-Martin (1990), we can show this important implication we can linearize the two key differential equations (2.8) and the capital accumulation equation (2.0)' budget constraint around the steady state²¹. If we express all variables in logarithms the system becomes

$$\dot{\ln(c_t)} = (1/\sigma)(\alpha e^{-(1-\alpha)\ln(k_t)} - (\rho+\delta))$$

(2.15)

$$\dot{\ln(k_t)} = e^{-(1-\alpha)\ln(k_t)} - e^{(\ln(c_t) - \ln(k_t))} - (n+\delta)$$

In steady state the two equations are equal to zero so

$$e^{-(1-\alpha)\ln(k^*)} = (\rho+\delta)/\alpha$$

(2.16)

$$e^{(\ln(c^*) - \ln(k^*))} = e^{-(1-\alpha)\ln(k^*)} - (n+\delta) = h > 0$$

where c^* and k^* are the steady state values of c_t and k_t respectively and $h = (\rho+\delta(1-\alpha) - \alpha n)/\alpha$. We can now Taylor-expand the system (2.15) around (2.16)

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See King and Rebelo (1990) and Barro and Sala-i-Martin (1990) for a discussion of convergence when the economy is far away from the steady state.

and get

$$\dot{\ln(c_t)} = -\mu[\ln(k_t) - \ln(k^*)] \quad (2.17)$$

$$\dot{\ln(k_t)} = -h[\ln(c_t) - \ln(c^*)] + (\rho - n)[\ln(k_t) - \ln(k^*)]$$

where $\mu \equiv (1 - \alpha)(\rho + \delta)/\sigma > 0$. or

$$(2.18) \quad \begin{bmatrix} \dot{\ln(c_t)} \\ \dot{\ln(k_t)} \end{bmatrix} = \begin{bmatrix} 0 & -\mu \\ -h & (\rho - n) \end{bmatrix} \begin{bmatrix} \ln(c_t) - \ln(c^*) \\ \ln(k_t) - \ln(k^*) \end{bmatrix}$$

notice that the determinant of the matrix is $\det A = -h\mu < 0$ which implies that the system is saddle path stable. The eigenvalues of the system are

$$(2.19) \quad \begin{aligned} -\lambda_1 &= (1/2)(\rho - n - \left[(\rho - n)^2 + 4\mu h \right]^{(1/2)}) < 0 \\ \lambda_2 &= (1/2)(\rho - n + \left[(\rho - n)^2 + 4\mu h \right]^{(1/2)}) > 0 \end{aligned}$$

The solution for $\ln(k_t)$ has the usual form

$$(2.20) \quad \ln(k_t) - \ln(k^*) = \psi_1 e^{-\lambda_1 t} + \psi_2 e^{\lambda_2 t}$$

where ψ_1 and ψ_2 are two arbitrary constants. To determine them, we notice that since λ_2 is positive, the capital stock will violate the transversality condition unless $\psi_2 = 0$. The initial conditions help us determine the other constant since at time 0 the solution implies

$$(2.20)' \quad \ln(k_0) - \ln(k^*) = \psi_1 e^0$$

Hence the final solution for the log of the capital stock has the form

$$(2.21) \ln(k_t) - \ln(k^*) = [\ln(k_0) - \ln(k^*)]e^{-\lambda_1 t}$$

If we realize that $\ln(k_t) = \ln(y_t)/\alpha$ and we subtract $\ln(y_0)$ from both sides of equation (2.21) we will get what is known as the "convergence equation"

$$(2.22) (\ln(y_t) - \ln(y_0))/t = a - \theta \ln(y_0)$$

where $a = \ln(y^*)(1 - e^{-\lambda_1 t})/t$ and $\theta = (1 - e^{-\lambda_1 t})/t$. This equation says that if a set of economies have the same deep parameters (discount rate, coefficient of intertemporal elasticity of substitution, capital share, depreciation and population growth rates, etc) so they converge to the same steady state, the cross section regression of growth on the log of initial income should display a negative coefficient. In other words, poor countries should tend to grow faster. The reason for that is that countries with low initial capital would have high initial marginal product of capital. That would lead them to save, invest and therefore grow a lot.

If countries converge to different steady states, however, there should be no relation between growth and initial income, unless we hold constant the determinants of the steady state. Sala-i-Martin (1990) and Barro and Sala-i-Martin (1990) use a slightly more complicated²² version of (2.22) to show that the states of the U.S. (which we may think are described by similar production and utility parameters) converge to each other exactly the way equation (2.22) predicts. They also show that, once they hold constant the determinants of the steady state, large sample of countries ALSO converge to each other the way equation (2.22) predicts.

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It is a slightly more complicated version because they include exogenous productivity growth.

(3) EXOGENOUS PRODUCTIVITY AND GROWTH

(a) Classification of Technological Innovations.

As we just mentioned, the simple neoclassical model predicts that the long run growth rate is zero. In order to explain observed long run growth neoclassical economists amended the model and incorporated exogenous productivity growth. In section 1 we saw that, in the fixed saving rate models, the introduction of productivity growth lead to long run economic growth. The question is what kind of technological progress should be introduced. Some inventions "save" capital relative to labor (capital saving technological progress), some save labor relative to capital (labor saving technological progress) and some do not save either input relative to the other (Neutral or unbiased technological progress).

Notice that the definition of neutral innovations depends on what we mean by "saving". The two most popular definitions of unbiased or Neutral technological progress are due to Hicks and Harrod respectively.

Hicks says that a technological innovation is Neutral (Hicks-Neutral) with respect to capital and labor if and only if the ratio of marginal products remains unchanged for a given capital labor ratio. Consequently, a technological innovation is labor (capital) saving if the marginal product of capital (labor) increases by more than the marginal product of labor (capital) at a given capital labor ratio. Notice that Hicks neutrality amounts to renumbering the isoquants. Production functions with Hicks Neutral technological progress can be written as

$$(3.1) \quad Y_t = A(t)F(K_t, L_t).$$

where $A(t)$ is an index of the state of technology at moment t evolving according to $A_t = A_0 e^{gt}$ (ie, $\dot{A}/A = g$) and where $F()$ is still homogeneous of degree one. The second definition of technological unbiased is due to Harrod. He says that a technical innovation is neutral (Harrod Neutral) if the relative shares (KF_K/LF_L) remain unchanged for a given capital OUTPUT ratio. Robinson (1938) and Uzawa (1961) showed that this implied a production

function of the form

$$(3.1)' \quad Y_t = F(K_t, A(t)L_t)$$

where, again, $A(t)$ is an index of technology at time t , $A/A=g$ and F is homogeneous of degree one. Notice that this production function says that, with the same amount of capital, we need less and less labor to produce the same amount of output. Therefore, this function is also known as labor augmenting technological progress. By symmetry we could have thought of technological change as being "capital augmenting", ie $Y=F(B_t K_t, L_t)$. This would mean that, for a given number of hours of work (L_t), we need decreasing amounts of capital to achieve the same isoquant.

The reason why we care about what kind of technological progress we should postulate is that, as Phelps showed, a necessary and sufficient condition for the existence of a steady state in an economy with exogenous technological progress is for this technological progress to be Harrod Neutral or Labor Augmenting. Notice, however, that when we work with Cobb-Douglas utility functions the two types of progress are identical since

$$Y(K,AL) = K^\beta (AL)^{1-\beta} = K^\beta e^{g(1-\beta)t} L^{1-\beta} = e^{g(1-\beta)t} (K^\beta L^{1-\beta}) = BY(K,L)$$

(b) The Irrelevance of Embodiment.

All types of technological change we have been talking up to now are "DISEMBODIED" in the sense that, when a technological innovation occurs, ALL existing machines get more productive. An example of this would be improvements in computer software: it makes all existing computers better. There are a lot of inventions, however, that are not of this type. When one invention occurs, only the NEW machines are more productive (as it is the case with computer hardware). Economists call this, "EMBODIED TECHNOLOGICAL PROGRESS".

In the 60's, when the neoclassical model of exogenous productivity growth was being developed, there was a debate on the importance of

embodiment in economic growth. Proponents of what at the time was called "New Investment Theory" (embodied technological progress) said that investment in new machines had the usual effect of increasing the capital stock and the additional effect of modernizing the average capital stock. Proponents of the "unimportance of the embodiment question" argued that this new effect was a level effect but that it did not affect the steady state rate of growth. In a couple of important papers Solow (1969) and Phelps (1962) showed the following:

(1) The neoclassical model with embodied technological progress and perfect competition (so the marginal product of labor is equal for all workers no matter what the vintage of the machine they are using is) can be rewritten in a way that is equivalent to the neoclassical model with disembodied progress (Solow (1969)).

(2) The Steady State growth is independent of the fraction of progress that is embodied (it depends on the total rate of technical progress but not on its composition) (Phelps (1962)).

(3) The convergence or speed of adjustment to the steady state growth rate is faster the larger the fraction of embodied progress (Phelps (1962)).

Thus, the distinction between embodied and disembodied progress seems unimportant when studying long run issues but might be crucial when studying short run dynamics²³. The modeling of embodied technological progress is quite complicated because one has to keep track of all old vintages of capital and associated labor. Yet a simple way to think about it is to postulate a technology-free production function $Y=F(K,L)$ and an accumulation function of the form $\dot{K}=A(t)(Y_t-C_t)$ where $\dot{A}(t)/A(t)=g$ and $K(t)$ is a measure of aggregate capital. This function reflects the fact that a unit of saving

The importance of embodiment in modeling business cycles can be seen from the fact that an embodied shock affects the marginal product of capital but does NOT affect the marginal product of labor or current output supply. This is a key difference with respect to a disembodied shock, especially as far as the implications for the procyclicality of real wages and real interest rates is concerned.

(Y-C) in a later period generates a larger increase in capital than a unit of saving in earlier time. This is like saying that later vintages of capital are more productive.

(c) The Neoclassical Model with Technological progress.

Let us go back to the labor augmenting form as depicted in equation (3.1)'. To solve this model it is going to be useful to define the concept of "effective labor", $\hat{L}$.

$$(3.2) \quad \hat{L}_t = L_t e^{gt} \text{ and } L_t = L_0 e^{nt} \text{ ----> } \hat{L}_t = L_0 e^{(n+g)t}$$

In words, for a given size of physical population we get more effective labor as time passes by. Since, on the other hand, the number of physical bodies increases at the constant rate n , the effective labor force grows at rate $g+n$. Notice that using this definition we can rewrite the production function as follows.

$$(3.3) \quad Y = F(K_t, \hat{L}_t)$$

Let's divide both sides of (3.3) by $\hat{L}_t$, define $\hat{y} = Y/\hat{L}$ and $\hat{k} = K/\hat{L}$. The CRS assumption implies:

$$(3.4) \quad \hat{y} = f(\hat{k})$$

Again, the closed economy assumption implies that domestic savings equal gross domestic investment so $\dot{Y} = \dot{K} + C - \delta K$. Divide both sides by $\hat{L}$ and get $\dot{\hat{y}} = \dot{\hat{K}}/\hat{L} + \hat{c} - \delta \hat{k}$. By the definition of $\hat{k}$, we know that $\dot{\hat{k}} = \dot{K}/\hat{L} - (n+g+\delta)\hat{k}$, which we can plug in savings equal investment equality to yield:

$$(3.5) \quad \dot{\hat{k}} = f(\hat{k}) - (n+g+\delta)\hat{k} - \hat{c}$$

Consumers maximize a utility function of the form (2.0) subject to

(3.5). Notice that the utility function is defined in consumption per capita (per physical body) while the budget constraint is defined in terms of consumption per effective labor unit ($\hat{c}_t$). We can transform the utility function using the equality $c_t = C_t/L_t = (C_t/L_t)(\hat{L}_t/L_t) = \hat{c}_t(L_t e^{gt}/L_t) = \hat{c}_t e^{gt}$.

$$(3.6) \quad U(0) = \int_0^{\infty} e^{-(\rho-n)t} \left[\frac{(\hat{c}_t e^{gt})^{1-\sigma} - 1}{1-\sigma} \right] L_0 dt$$

We have to choose $\hat{c}_t$ so as to maximize (3.6) subject to (3.5) and subject to K_0 , L_0 and A_0 . Set up the Hamiltonian:

$$(3.7) \quad H() = e^{-(\rho-n)t} \left[\frac{(\hat{c}_t e^{gt})^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[f(\hat{k}) - \hat{c} - (n+g+\delta)\hat{k} \right]$$

The F.O.C. are the following:

$$(3.8) \quad H_{\hat{c}} = 0 \Leftrightarrow e^{-(\rho-n)t} e^{gt} \left[\hat{c} e^{gt} \right]^{-\sigma} - \nu = 0$$

$$(3.9) \quad H_{\hat{k}} = -\dot{\nu} \Leftrightarrow \dot{\nu} = -\nu(f'(\hat{k}) - n - g - \delta)$$

$$(3.10) \quad TVC \lim_{t \rightarrow \infty} (\hat{k}_t \nu_t) = 0$$

By following the same steps as in the previous section, we will find that:

$$(3.11) \quad \dot{\nu}/\nu = -(\rho-n) + g - \sigma \hat{c}/\hat{c} - \sigma g = -f'(\hat{k}) + n + g + \delta$$

by setting $\dot{\hat{c}}/\hat{c}=0$ we will get the steady state condition:

$$(3.12) \quad f'(\hat{k}) = \rho + \sigma g + \delta$$

Observe that this result is exactly parallel to the one in section two (equation (2.8)). The difference here is that the growth rate relates to consumption per unit of efficient labor. This means that, since

variables in efficiency units do not grow, variables in per capita terms grow at the constant rate g .

(c) Bounded Utility Condition.

For $U(0)$ to be bounded, again, we need the expression inside the integral to tend to zero as t goes to infinity.

$$(3.14) \lim e^{-(\rho-n-g(1-\sigma))t} c_t^{1-\sigma}/(1-\sigma) - \lim e^{-(\rho-n)t}/(1-\sigma)$$

Note that if $\rho > n+g(1-\sigma) > n$, the second term goes to zero. Since c will end up growing at rate g , the first term also goes to zero if the above condition holds. Notice, finally, that this condition implies that the TVC is satisfied and that we will end up at a point to the left of the golden rule (dynamically efficient region).

Finally, let's analyze the saving rate.

$$(3.27) \begin{aligned} s/y &= (\dot{k}/y) + (nk/y) = (\dot{k}/k)(k/y) + n(k/y) = (\gamma+n)(k/y) = \\ &= (\gamma+n)/(k^{-(1-\beta)} e^{g(1-\beta)t}) = (\gamma+n)/[(\rho+\sigma\gamma)/\beta] = (g+n)\beta/(\rho+g\sigma). \end{aligned}$$

A patient society (low ρ) will save more and end up with a higher output LEVEL along the balanced path than an impatient one. She will not, however, grow at a faster rate. We have seen that the growth rate depends on g and n only. This is an important implication of the neoclassical model of economic growth.

Figure 2: Conditional Convergence in the Neoclassical Model

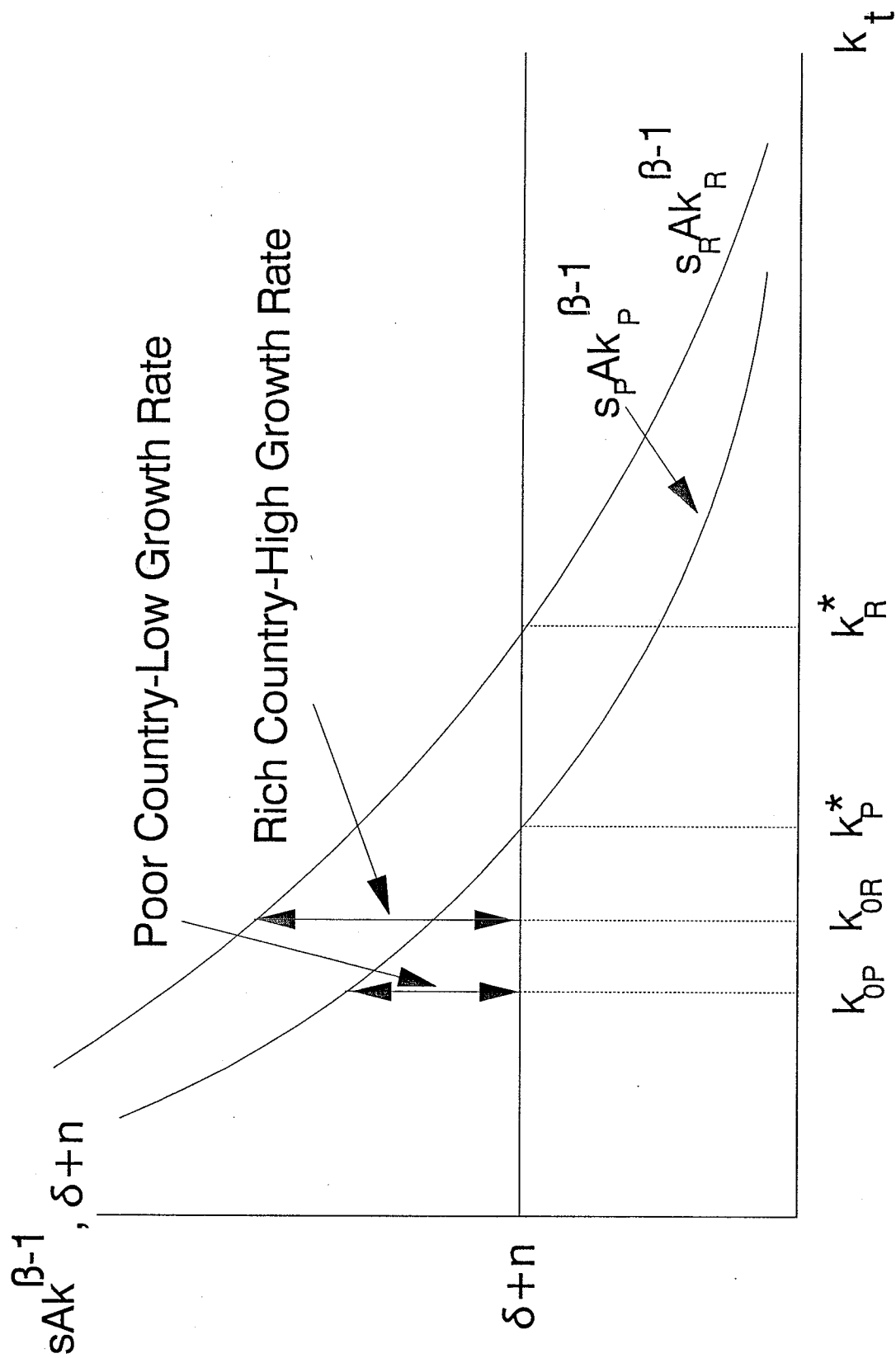


Figure 3: The Rebelo-Ak model

$$\beta = 1$$

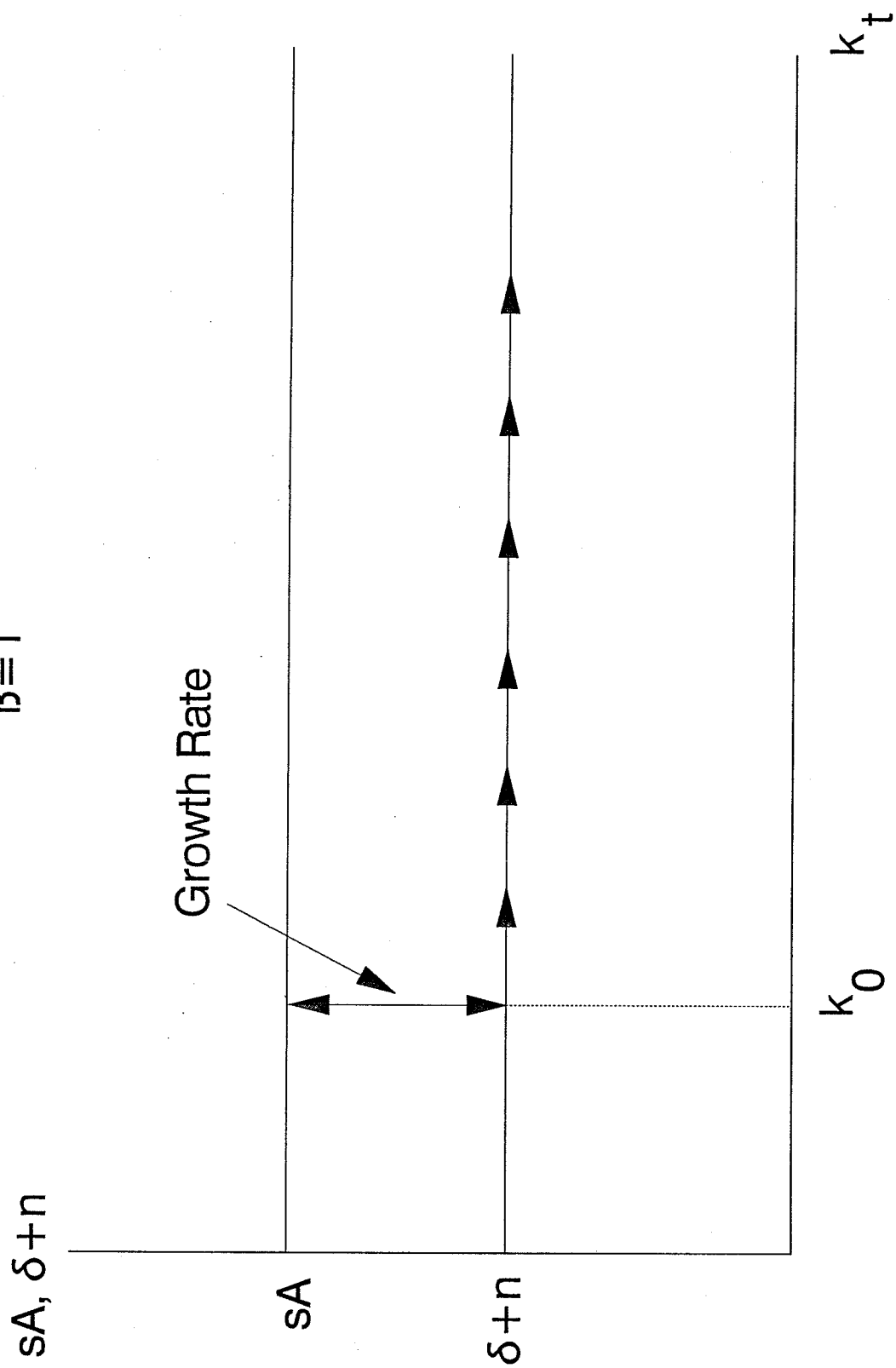


Figure 4: Increasing Returns and Increasing Growth Rates

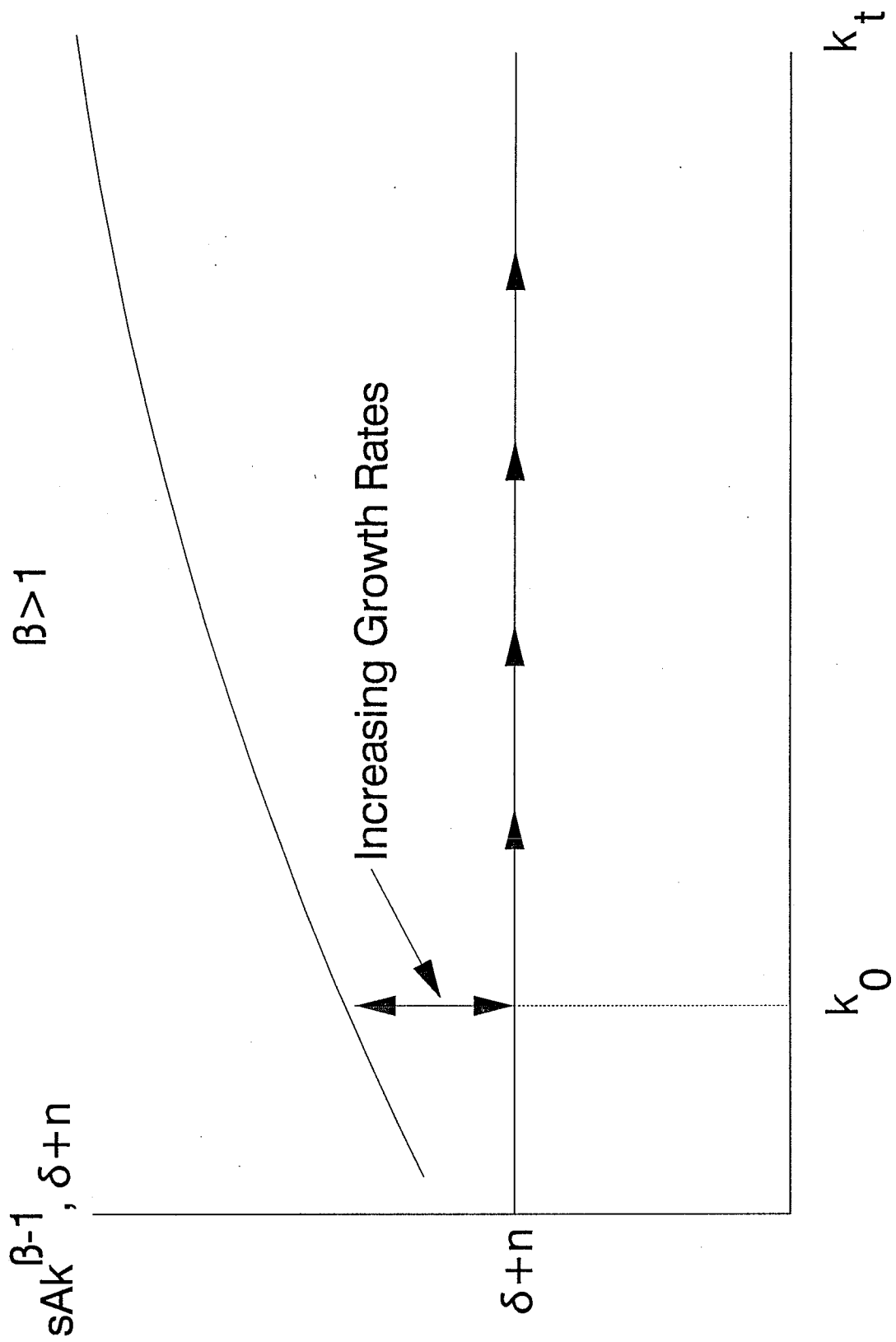


Figure 5: The Harrod-Domar Production Function

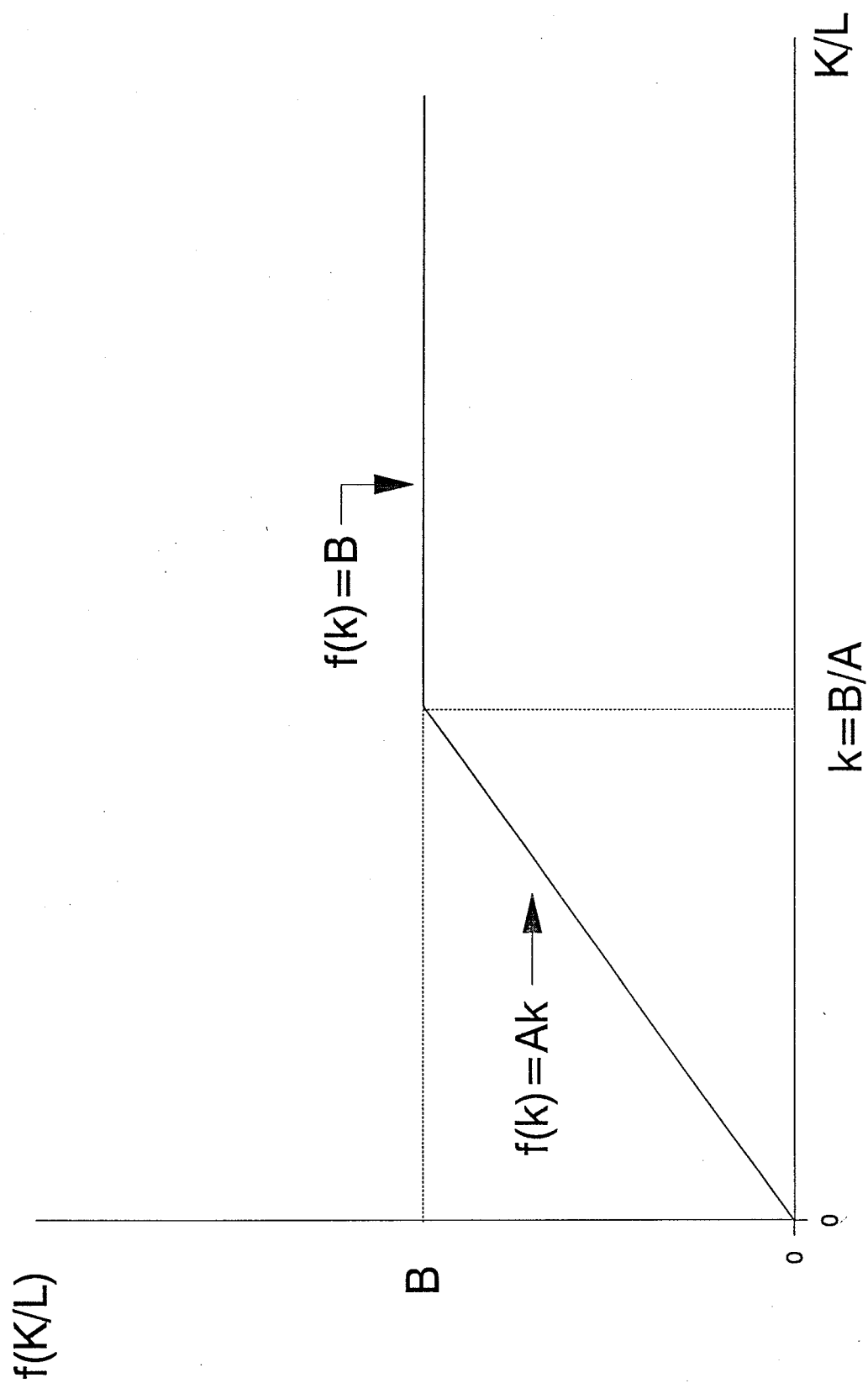


Figure 6: The Harrod-Domar Model
Case 1: $sA < (\delta + n)$

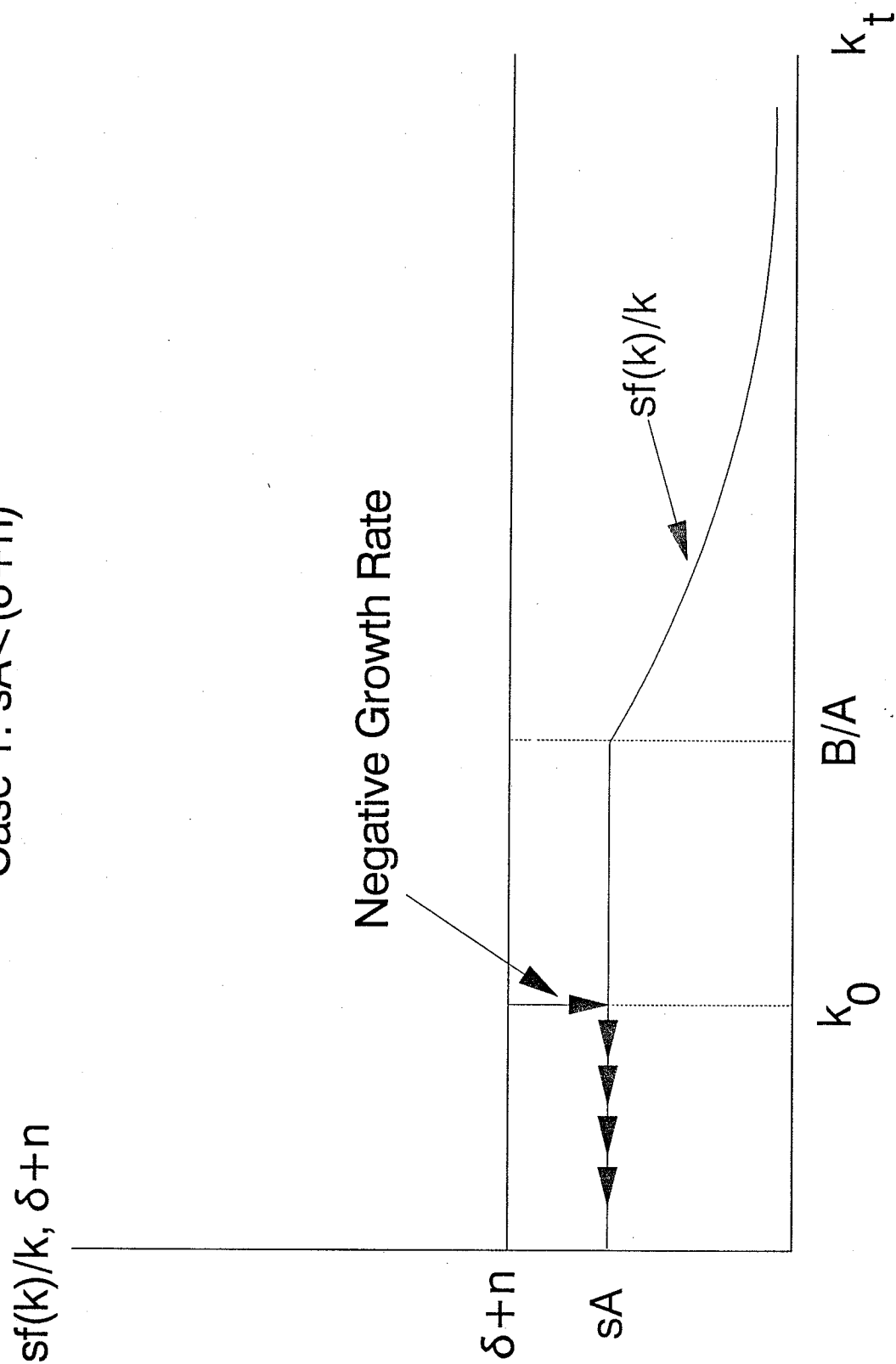


Figure 7: The Harrod-Domar Model

Case 2: $sA = (\delta + n)$

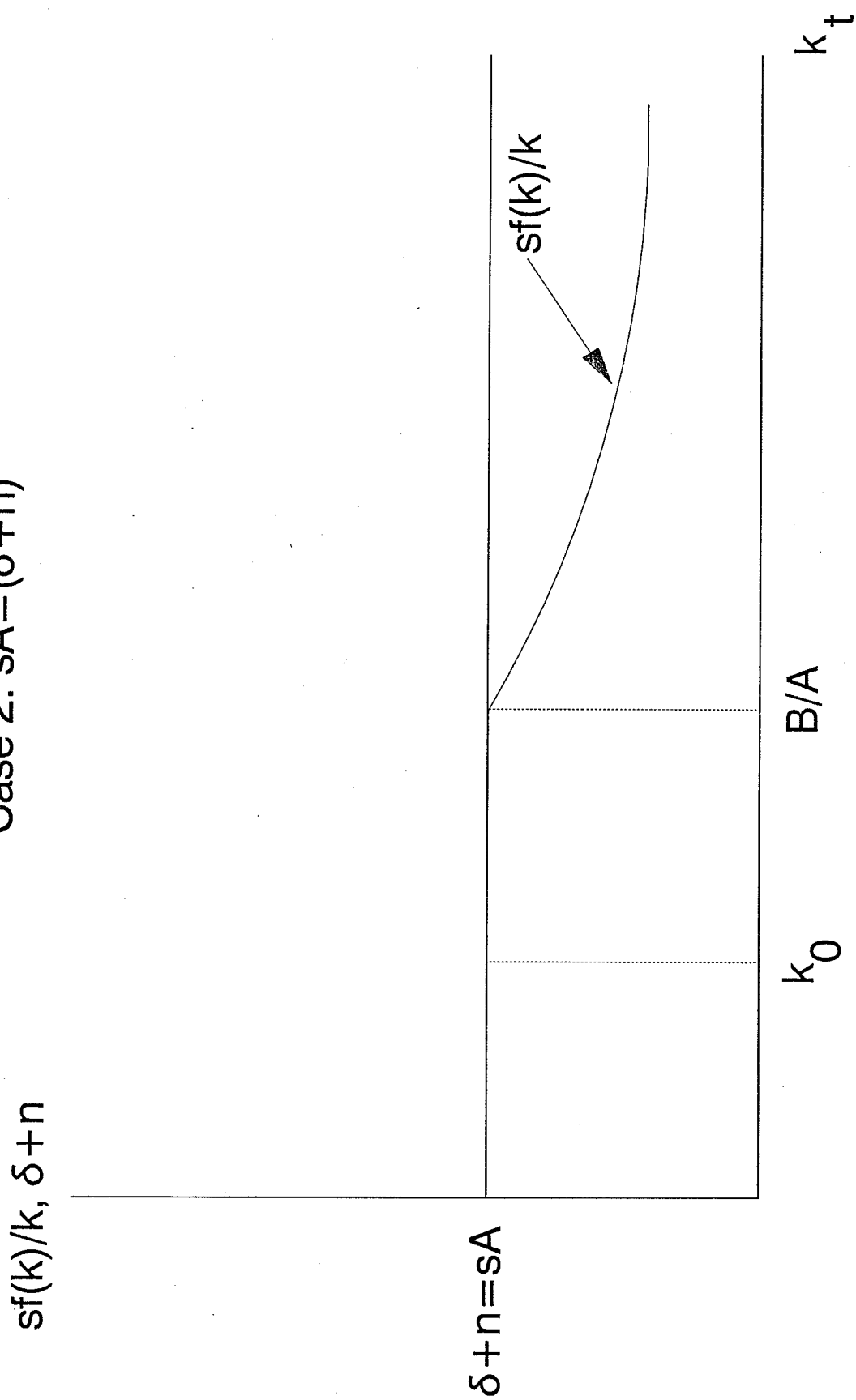


Figure 8: The Harrod-Domar Model
Case 3: $sA > (\delta + n)$

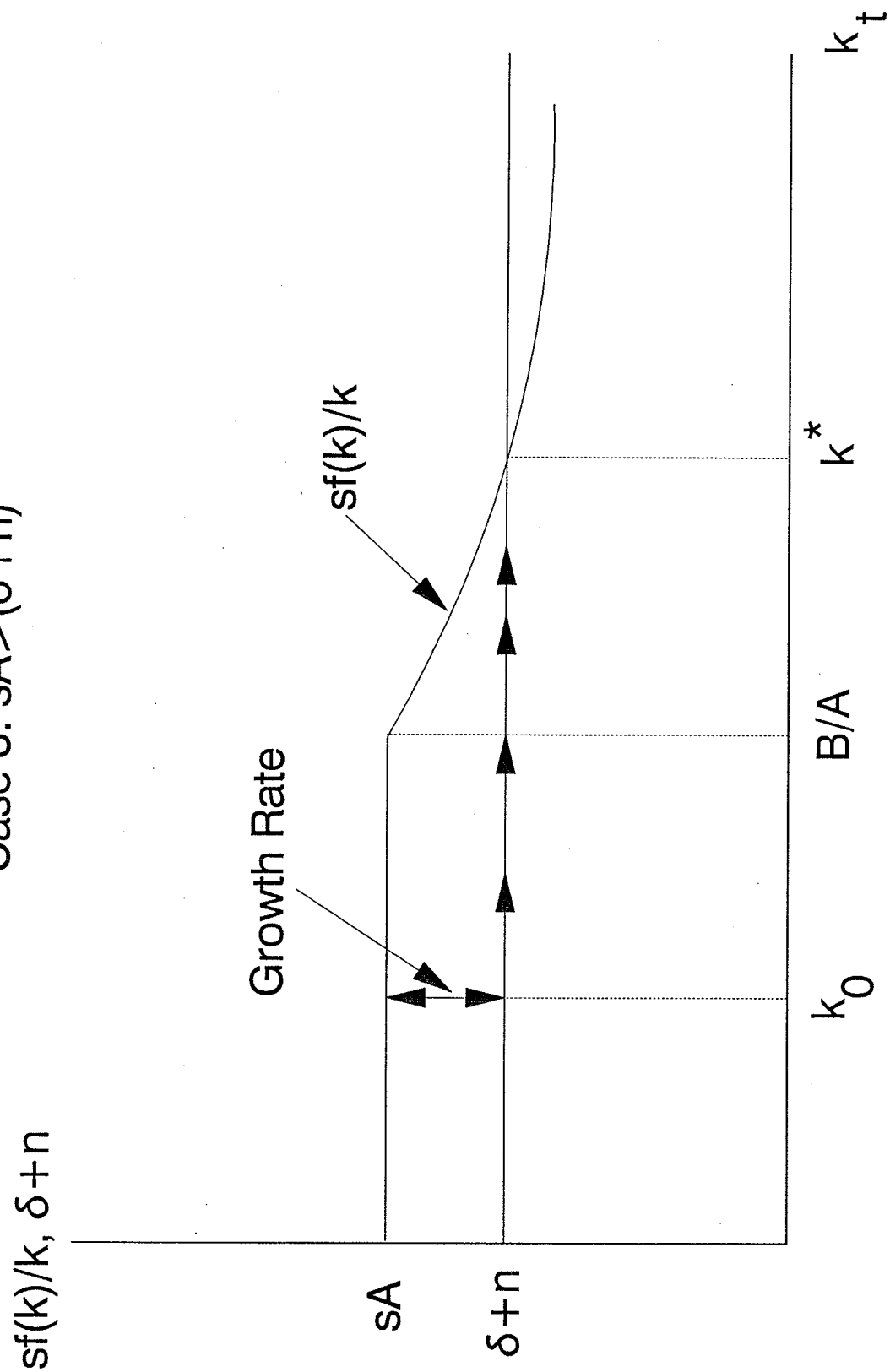


Figure 10: Stable Poverty Trap

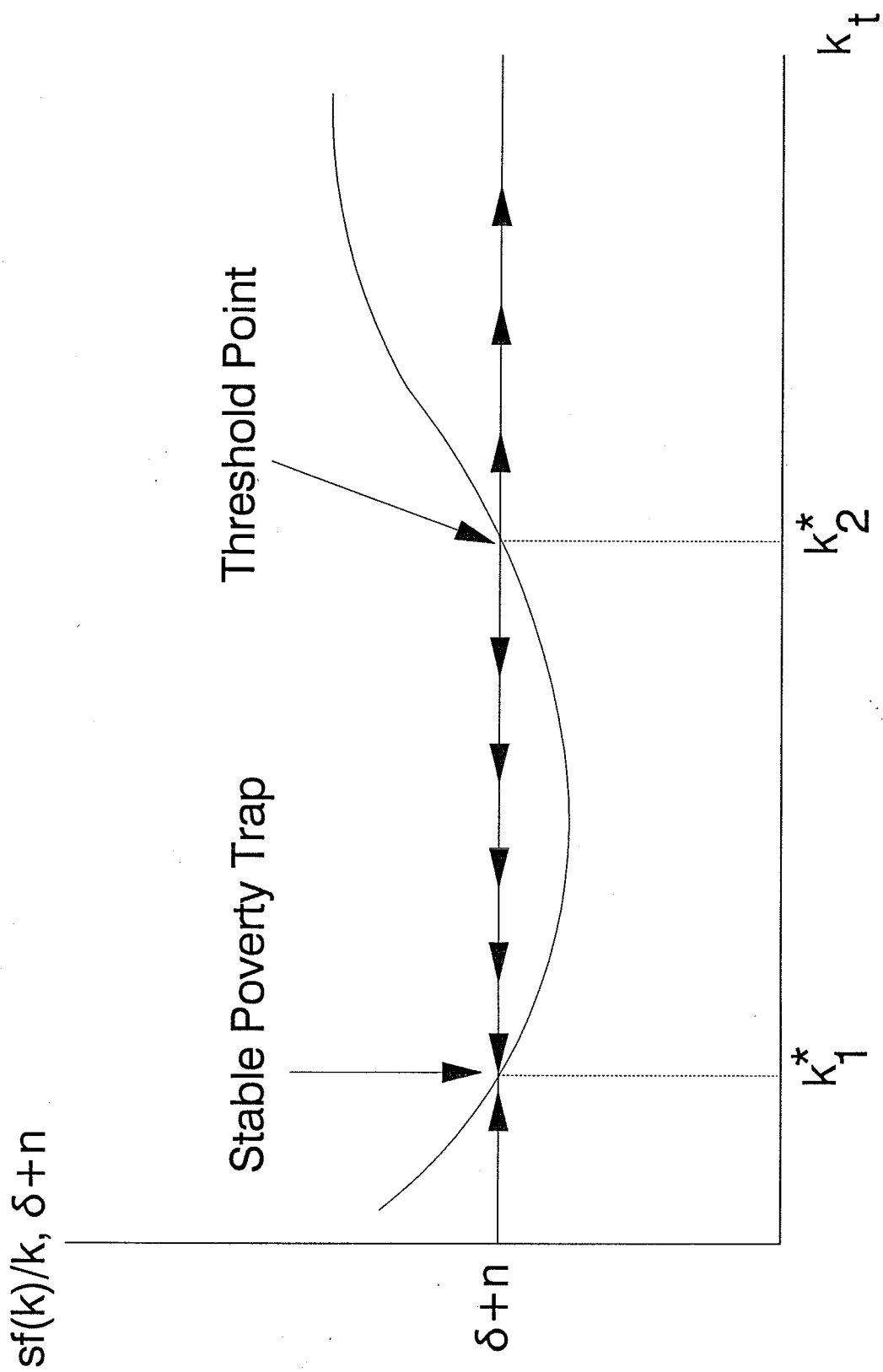


Figure 11: The Ramsey-Cass-Koopmans Phase Diagram

