Breakdown point theory for implied probability bootstrap

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Abstract

This paper studies robustness of bootstrap inference methods under moment conditions. In particular, we compare the uniform weight and implied probability bootstraps by analyzing behaviors of the bootstrap quantiles when an outlier takes an arbitrarily large value, and derive the breakdown points for those bootstrap quantiles. The breakdown properties characterize the situation where the implied probability bootstrap is more robust than the uniform weight bootstrap against outliers. Simulation studies illustrate our theoretical findings.

1 Introduction

Since Hansen (1982), the generalized method of moments (GMM) has been a standard tool for empirical analysis in econometrics. The GMM provides a unified framework for statistical inference in econometric models that are specified by some moment conditions (see e.g. Hall (2005) for a review on the GMM). However, recent research indicates that there are considerable problems with the GMM particularly in its finite sample performance and that approximations based on the asymptotic theory can yield poor results (see e.g. the special issue of the *Journal of Business and Economic Statistics*, vol. 14).

To refine the approximations for the distributions of the GMM estimator and related test statistics, bootstrap methods have been developed. A key issue to apply bootstrap methods to the GMM context is that one typically needs to impose the overidentified moment conditions to the bootstrap resamples. Hall and Horowitz (1996) suggested to use the uniform weight bootstrap with recentered moment conditions, and established higher-order refinements of their bootstrap inference over the asymptotic approximations. On the other hand, Brown and Newey (2002) suggested to use a weighted bootstrap based on the implied probabilities from the moment conditions (see also Hall and Presnell, 1999). These implied probabilities can be computed based on the GMM (Back and Brown, 1993), empirical likelihood

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(Owen, 1988), or generalized empirical likelihood (Newey and Smith, 2004). This implied probability bootstrap also provides a refinement over the asymptotic approximations.[1]

The purpose of this paper is to study robustness of these uniform weight and implied probability bootstraps based on the breakdown theory in the literature of robust statistics (see, e.g., Hampel, 1971, and Donoho and Huber, 1983). The need for robust statistical procedures has been stressed by many authors and is now widely recognized; see, e.g., Huber (1981), Hampel, Ronchetti, Rousseeuw and Stahel (1986), Maronna, Martin and Yohai (2006). To be more precise, by extending the approach of Singh (1998), we analyze behaviors of bootstrap quantiles of the uniform weight bootstrap and implied probability bootstrap (using Back and Brown's (1993) weight) when an outlier takes an arbitrary large value, and compare the breakdown points for these bootstrap quantiles. Our breakdown analysis characterizes the situation where the implied probability bootstrap is more robust than the uniform weight bootstrap against outliers. Therefore, researchers can decide which bootstrap approach should be adopted for each application. In particular, when all elements of the moment functions diverge to infinity as (the norm of) outliers diverge, the implied probability bootstrap is typically more robust than the uniform weight bootstrap. The literature of robustness study in the GMM context is relatively thin and is currently under development. Ronchetti and Trojani (2001) extended the robust estimation techniques for (just-identified) estimating equations to overidentified moment condition models. Gagliardini, Trojani and Urga (2005) develop a robust GMM test for structural breaks. Kitamura, Otsu and Evdokimov (2010) and Kitamura and Otsu (2010) studied local robustness against perturbations controlled by the Hellinger distance for point estimation and hypothesis testing, respectively, in moment condition models. Our breakdown analysis studies global robustness of bootstrap methods when outliers takes arbitrarily large values.

The rest of the paper is organized as follows. Section 2 investigates a benchmark example, inference for a trimmed mean, to understand the basic idea of our breakdown analysis. Section 3 generalizes the results obtained in Section 2 to a moment condition model. Section 4 illustrates the theoretical results by simulations. Section 5 concludes.

2 Benchmark example

Consider a random sample $\{X_i\}_{i=1}^n$ of size n from $X \in \mathbb{R}$. Suppose that we wish to approximate the distribution of the 10% trimmed mean $T(0.1)$ (5% trimming for each side) with $n \geq 20$ by a bootstrap method. Let $X_{(1)} \leq \ldots \leq X_{(n)}$ be the ordered sample. Since $n \geq 20$, $T(0.1)$ is always free from the largest observation $X_{(n)}$, which is treated as an outlier. On the other hand, consider the trimmed mean $T^{\#}(0.1)$ using the (uniform weight) bootstrap resample. Since the bootstrap resample can contain $X_{(n)}$

[1] An important feature of implied probabilities is that they provide semiparametrically efficient estimators for the distribution function and it moments under the moment conditions (Back and Brown, 1993, and Brown and Newey, 1998). Antoine, Bonnal and Renault (2007) employed implied probabilities to construct an asymptotically efficient estimator for parameters in the moment conditions.

more than once, $T^{\#}(0.1)$ is not necessarily free from $X_{(n)}$. Letting $B(n,p)$ be a binomial random variable with parameters n and p, the probability that $T^{\#}(0.1)$ is free from $X_{(n)}$ is

$$p^{\#} = P\left(B\left(n, \frac{1}{n}\right) \leq 1\right).$$

Therefore, if $X_{(n)} \to +\infty$, then $100\left(1 - p^{\#}\right)\%$ of resamples of $T^{\#}(0.1)$ will diverge to $+\infty$. In other words, the bootstrap quantile $Q_t^{\#}$ of $T^{\#}(0.1)$ will diverge to $+\infty$ for all $t > p^{\#}$.

Consider the situation where we have auxiliary information

$$E\left[g\left(X_i\right)\right] = 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a scalar-valued function. Let $\bar{g} = \frac{1}{n}\sum_{i=1}^{n} g\left(X_i\right)$. Back and Brown (1993) showed that under this auxiliary information, the distribution function of X can be efficiently estimated by using the implied probabilities:

$$\pi_i = \frac{1}{n} - \frac{1}{n}\frac{\left(g\left(X_i\right) - \bar{g}\right)\bar{g}}{\frac{1}{n}\sum_{i=1}^{n} g\left(X_i\right)^2},$$

for $i = 1, \ldots, n$.[2] The second term in π_i can be interpreted as a penalty term for the deviation from auxiliary information: if $\left|g\left(X_i\right)\right|$ becomes larger, then $\left(g\left(X_i\right) - \bar{g}\right)\bar{g}$ tends to be positive and the weight π_i tends to be smaller. Let $T^{*}(0.1)$ be the trimmed mean using a bootstrap sample based on the implied probabilities $\{\pi_i\}_{i=1}^{n}$. Then the probability that $T^{*}(0.1)$ is free from the largest observation $X_{(n)}$ is written as

$$P\left(B\left(n, \pi_{(n)}\right) \leq 1\right).$$

Thus, in terms of the bootstrap quantiles, the implied probability bootstrap becomes more robust than the uniform weight bootstrap when $P\left(B\left(n, \pi_{(n)}\right) \leq 1\right) > p^{\#}$ (or $\pi_{(n)} \leq \frac{1}{n}$).

To adapt the conventional breakdown theory to our setup, we need to analyze the limiting behavior of $g\left(X_{(n)}\right)$ as $X_{(n)} \to +\infty$. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be the extended real line. Suppose that

$$g\left(X_{(n)}\right) \to g_* \in \bar{\mathbb{R}} \quad \text{as } X_{(n)} \to +\infty.$$

Then the limit of $\pi_{(n)}$ is obtained as

$$\pi_* = \begin{cases} \frac{1}{n} - \frac{1}{n}\frac{\left(1 - \frac{1}{n} - \frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*} + \frac{1}{n}\right)}{\frac{\bar{v}_-}{g_*^2} + \frac{1}{n}} & \text{if } g_* \in \mathbb{R} \setminus \{0\} \\ \frac{1}{n} + \frac{1}{n}\frac{\bar{g}_-^2}{\bar{v}_-} & \text{if } g_* = 0 \\ \frac{1}{n^2} & \text{if } \left|g_*\right| = +\infty \end{cases},$$

where $\bar{g}_- = \frac{1}{n}\sum_{i=1}^{n-1} g\left(X_{(i)}\right)$ and $\bar{v}_- = \frac{1}{n}\sum_{i=1}^{n-1} g\left(X_{(i)}\right)^2$. The limit of the probability $P\left(B\left(n, \pi_{(n)}\right) \leq 1\right)$ is

$$p^* = P\left(B\left(n, \pi_*\right) \leq 1\right).$$

[2]For the breakdown analysis, we focus on Back and Brown's (1993) implied probability because of its simplicity. It is interesting to extend the analysis to other implied probabilities such as the generalized empirical likelihood-based implied probabilities discussed by Brown and Newey (2002).

If $g_* \in \mathbb{R}\backslash\{0\}$, then the sign of $\left(1-\frac{1}{n}-\frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*}+\frac{1}{n}\right)$ determines robustness of the implied probability bootstrap. If $\left(1-\frac{1}{n}-\frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*}+\frac{1}{n}\right)$ is positive (or negative), then $p^* > p^\#$ (or $p^* < p^\#$) and the implied probability bootstrap is more (or less) robust than the uniform weight bootstrap. If $g_* = 0$, then $p^\# > p^*$ is always satisfied and the uniform weight bootstrap is more robust than the implied probability bootstrap. On the other hand, if $|g_*| = +\infty$, then $p^* > p^\#$ is always satisfied and the implied probability bootstrap becomes more robust. These findings are summarized as follows.

Proposition 1. *Consider the setup of this section. If $X_{(n)} \to +\infty$, the followings hold true.*

(i) *The uniform weight bootstrap quantile $Q_t^\#$ of $T^\#(0.1)$ will diverge to $+\infty$ for all $t > p^\#$.*

(ii) *The implied probability bootstrap quantile Q_t^* of $T^*(0.1)$ will diverge to $+\infty$ for all $t > p^*$.*

(iii) *If $g_* = 0$, then $p^\# > p^*$ is always satisfied. If $|g_*| = +\infty$, then $p^\# < p^*$ is always satisfied.*

For the divergent case, $|g_*| = +\infty$, we can numerically compare $p^\# = P\left(B\left(n, \frac{1}{n}\right) \leq 1\right)$ and $p^* = P\left(B\left(n, \frac{1}{n^2}\right) \leq 1\right)$. For example, when the sample size is $n = 20$, we have $p^\# = 0.736$ and $p^* = 0.999$. This means that for the uniform weight bootstrap, a single outlier implies the divergence of more than 26% of resamples of $T^\#(0.1)$, while for the implied probability bootstrap, a single outlier implies the divergence of less than 1% of resamples of $T^*(0.1)$. Section 4.1 illustrates this proposition by simulations.

Although the results obtained in this section is insightful, there are several limitations: (i) the statistic of interest is a trimmed mean, (ii) X is scalar, and (iii) $g(\cdot)$ is a scalar-valued function and does not contain parameters. The next section discusses how to generalize the insights obtained in this section.

3 Breakdown theory

We now introduce our setup. Let $\{X_i\}_{i=1}^n$ be a random sample of size n from $X \in \mathbb{R}^d$. Consider the situation where we have the following overidentified moment conditions:

$$E\left[g\left(X_i, \theta_0\right)\right] = E\left[\begin{array}{c} g_1\left(X_i, \theta_0\right) \\ g_2\left(X_i, \theta_0\right) \end{array}\right] = 0,$$

where g_1 and g_2 are scalar-valued functions and $\theta_0 \in \mathbb{R}$ is a scalar parameter. In this case, Back and Brown's (1993) implied probabilities are defined as

$$\pi_i = \frac{1}{n} - \frac{1}{n}\left\{g\left(X_i, \hat{\theta}\right) - \bar{g}\left(\hat{\theta}\right)\right\}'\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right) g\left(X_i, \hat{\theta}\right)'\right]^{-1} \bar{g}\left(\hat{\theta}\right),$$

for $i = 1, \ldots, n$, where $\hat{\theta}$ is an estimator of θ_0 and $\bar{g}\left(\hat{\theta}\right) = \frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right)$. For simplicity and technical tractability (basically to obtain an explicit formula for π_i), we focus on the case of $\dim(g) = 2$. Remark 3 below discusses some extensions for the case of $\dim(g) > 2$. Let $\left\{X_{(i)}\right\}_{i=1}^n$ be the ordered sample, where $\|X_{(1)}\| \leq \ldots \leq \|X_{(n)}\|$ and $\|\cdot\|$ is the Euclidean norm. Suppose we are interested in a real-valued object $T_n = T_n(X_1, \ldots, X_n; \theta_0)$, where $T_n \to +\infty$ as $\|X_{(n)}\| \to +\infty$.[3] We also assume that as $\|X_{(n)}\| \to +\infty$,

$$\begin{pmatrix} g_1\left(X_{(n)}, \hat{\theta}\right) \\ g_2\left(X_{(n)}, \hat{\theta}\right) \end{pmatrix} \to \begin{pmatrix} g_{1*} \\ g_{2*} \end{pmatrix} \in \bar{\mathbb{R}}^2, \quad \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)}, \hat{\theta}\right) \\ \frac{1}{n}\sum_{i=1}^{n-1} g_2\left(X_{(i)}, \hat{\theta}\right) \\ \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)}, \hat{\theta}\right)^2 \\ \frac{1}{n}\sum_{i=1}^{n-1} g_2\left(X_{(i)}, \hat{\theta}\right)^2 \\ \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)}, \hat{\theta}\right) g_2\left(X_{(i)}, \hat{\theta}\right) \end{pmatrix} \to \begin{pmatrix} \bar{g}_{1-} \\ \bar{g}_{2-} \\ v_{11} \\ v_{22} \\ v_{12} \end{pmatrix} \in \mathbb{R}^5.$$

Note that the second condition requires that the limits are finite and restricts the form of g and/or the limit of $\hat{\theta}$. In this case, the limit of $\pi_{(n)}$ is obtained as

$$\pi_* = \begin{cases} \frac{1}{n} - c_* & \text{if } g_{1*} \in \mathbb{R} \text{ and } g_{2*} \in \mathbb{R} \\ \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{2-}^2}{v_{22}} & \text{if } |g_{1*}| = +\infty \text{ and } g_{2*} \in \mathbb{R} \\ \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{1-}^2}{v_{11}} & \text{if } g_{1*} \in \mathbb{R} \text{ and } |g_{2*}| = +\infty \\ \frac{1}{n^2} + \frac{1}{n}\frac{(\bar{g}_{1-}-\bar{g}_{2-})^2}{v_{11}+v_{22}-2v_{12}} & \text{if } |g_{1*}| = +\infty \text{ and } |g_{2*}| = +\infty \end{cases},$$

where

$$\begin{aligned} c_* &= \frac{1}{n}\left\{\left(v_{11} + \frac{1}{n}g_{1*}^2\right)\left(v_{22} + \frac{1}{n}g_{2*}^2\right) - \left(v_{12} + \frac{1}{n}g_{1*}g_{2*}\right)^2\right\}^{-1} \\ &\times \left\{\begin{array}{c} \left(v_{22} + \frac{1}{n}g_{2*}^2\right)\left\{\left(1-\frac{1}{n}\right)g_{1*} - \bar{g}_{1-}\right\}\left(\bar{g}_{1-} + \frac{1}{n}g_{1*}\right) \\ -\left(v_{12} + \frac{1}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{1}{n}\right)g_{2*} - \bar{g}_{2-}\right\}\left(\bar{g}_{1-} + \frac{1}{n}g_{1*}\right) \\ -\left(v_{12} + \frac{1}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{1}{n}\right)g_{1*} - \bar{g}_{1-}\right\}\left(\bar{g}_{2-} + \frac{1}{n}g_{2*}\right) \\ +\left(v_{11} + \frac{1}{n}g_{1*}^2\right)\left\{\left(1-\frac{1}{n}\right)g_{2*} - \bar{g}_{2-}\right\}\left(\bar{g}_{2-} + \frac{1}{n}g_{2*}\right) \end{array}\right\}. \end{aligned}$$

Therefore, we obtain the following result.

Proposition 2. *Consider the setup of this section. If $\|X_{(n)}\| \to +\infty$, the followings hold true.*

(i) *The uniform weight bootstrap quantile $Q_t^{\#}$ from the resamples $T_n^{\#}$ of T_n will diverge to $+\infty$ for all $t > p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$.*

(ii) *The implied probability bootstrap quantile Q_t^* from the resamples T_n^* of T_n will diverge to $+\infty$ for all $t > p^* = P\left(B\left(n, \pi_*\right) = 0\right)$.*

Note that this proposition is presented for the non-robust statistic T_n (i.e., s single outlier yields divergence of T_n). More generally, if the statistic T_n is robust to k outliers (the trimmed mean $T(0.1)$

[3]For brevity, as in Singh (1998) we consider only the case where $T_n \to +\infty$ as $\|X_{(n)}\| \to +\infty$. Nevertheless, following the definition of finite sample breakdown point in Donoho and Huber (1983), the results in our paper extend straightforward to the case $T_n \to +\infty$ as $X_i \to x_* \in \mathbb{R}$.

in Section 2 corresponds to the case of $k=1$), then this proposition holds for $p^{\#}=P\left(B\left(n,\frac{1}{n}\right)\leq k\right)$ and $p^{*}=P\left(B\left(n,\pi_{*}\right)\leq k\right)$.

If we impose more assumptions, this proposition can be presented by using the notion of the breakdown point for quantiles (Singh, 1998). Let b_T be the upper breakdown point of T_n, i.e., nb_T is the smallest number of observations whose Euclidean norm need to go to $+\infty$ in order to force T_n to go to $+\infty$. In our context, the upper breakdown point of quantiles $\{Q_t\}_{t\in(0,1)}$ can be defined as

$$UB_t=\inf\left\{b\in\left[\frac{1}{n},b_T\right]:nb\in\mathbb{N}\text{ and }Q_t=+\infty\right\},$$

where b is the fraction of observations $X_{(n)},X_{(n-1)},\ldots,X_{(nb+1)}$ such that $\left\|X_{(j)}\right\|\to+\infty$ for all $j=nb+1,\ldots,n$.[4] Consider the situation where as $\left\|X_{(j)}\right\|\to+\infty$,

$$\begin{pmatrix} g_1\left(X_{(j)},\hat{\theta}\right)\\ g_2\left(X_{(j)},\hat{\theta}\right)\end{pmatrix}\to\begin{pmatrix} g_{1*}\\ g_{2*}\end{pmatrix}\in\bar{\mathbb{R}}^2,$$

and for any $j\neq j'$, $g_1\left(X_{(j)},\hat{\theta}\right)/g_1\left(X_{(j')},\hat{\theta}\right)\to 1$ and $g_2\left(X_{(j)},\hat{\theta}\right)/g_2\left(X_{(j')},\hat{\theta}\right)\to 1$, where $n-k+1\leq j,j'\leq n$, $1\leq k\leq n$. In this case, for $j=n-k+1,\ldots,n$, the limit of $\pi_{(j)}$ is obtained as:

$$\pi_{*,k}=\begin{cases}\frac{1}{n}-c_{*,k} & \text{if } g_{1*}\in\mathbb{R}\text{ and }g_{2*}\in\mathbb{R}\\ \frac{k}{n^2}+\frac{1}{n}\frac{\bar{g}_{2-,k}^2}{v_{22,k}} & \text{if } |g_{1*}|=+\infty\text{ and }g_{2*}\in\mathbb{R}\\ \frac{k}{n^2}+\frac{1}{n}\frac{\bar{g}_{1-,k}^2}{v_{11,k}} & \text{if } g_{1*}\in\mathbb{R}\text{ and }|g_{2*}|=+\infty\\ \frac{k}{n^2}+\frac{1}{n}\frac{(\bar{g}_{1-}-\bar{g}_{2-})^2}{v_{11}+v_{22}-2v_{12}} & \text{if } |g_{1*}|=+\infty\text{ and }|g_{2*}|=+\infty\end{cases},$$

where $\bar{g}_{1-,k}$, $\bar{g}_{2-,k}$, $v_{11,k}$, $v_{22,k}$, and $v_{12,k}$ are the limits of $\frac{1}{n}\sum_{i=1}^{n-k}g_1\left(X_{(i)},\hat{\theta}\right)$, $\frac{1}{n}\sum_{i=1}^{n-k}g_2\left(X_{(i)},\hat{\theta}\right)$, $\frac{1}{n}\sum_{i=1}^{n-k}g_1\left(X_{(i)},\hat{\theta}\right)^2$, $\frac{1}{n}\sum_{i=1}^{n-k}g_2\left(X_{(i)},\hat{\theta}\right)^2$, and $\frac{1}{n}\sum_{i=1}^{n-k}g_1\left(X_{(i)},\hat{\theta}\right)g_2\left(X_{(i)},\hat{\theta}\right)$, respectively, which are assumed to be finite, and

$$\begin{aligned}c_{*,k} &= \frac{1}{n}\left\{\left(v_{11,k}+\frac{k}{n}g_{1*}^2\right)\left(v_{22,k}+\frac{k}{n}g_{2*}^2\right)-\left(v_{12,k}+\frac{k}{n}g_{1*}g_{2*}\right)^2\right\}^{-1}\\ &\quad\times\left\{\begin{array}{c}\left(v_{22,k}+\frac{k}{n}g_{2*}^2\right)\left\{\left(1-\frac{k}{n}\right)g_{1*}-\bar{g}_{1-,k}\right\}\left(\bar{g}_{1-}+\frac{k}{n}g_{1*}\right)\\ -\left(v_{12,k}+\frac{k}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{k}{n}\right)g_{2*}-\bar{g}_{2-,k}\right\}\left(\bar{g}_{1-,k}+\frac{k}{n}g_{1*}\right)\\ -\left(v_{12,k}+\frac{k}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{k}{n}\right)g_{1*}-\bar{g}_{1-,k}\right\}\left(\bar{g}_{2-,k}+\frac{k}{n}g_{2*}\right)\\ +\left(v_{11,k}+\frac{k}{n}g_{1*}^2\right)\left\{\left(1-\frac{k}{n}\right)g_{2*}-\bar{g}_{2-,k}\right\}\left(\bar{g}_{2-,k}+\frac{k}{n}g_{2*}\right)\end{array}\right\}.\end{aligned}$$

Proposition 3. *Let T_n be a statistic with breakdown point $b_T\in(0,1)$. Under the setup of this section, the followings hold true.*

(i) *(Singh (1998, Theorem 1)) The upper breakdown point of the uniform weight bootstrap quantile $Q_t^{\#}$ is*

$$UB_t^{\#}=\frac{1}{n}\min\left\{k\in\{1,\ldots,n\}:P\left(B\left(n,\frac{k}{n}\right)\geq nb_T\right)\geq 1-t\right\}.$$

[4]The same argument applies to the lower breakdown point which focuses on the lower tail (i.e., $Q_t=-\infty$).

(ii) *The upper breakdown point of the implied probability bootstrap quantile* Q_t^* *is*

$$UB_t^* = \frac{1}{n} \min \left\{ k \in \{1, \ldots, n\} : P\left(B\left(n, k\pi_{*,k}\right) \geq nb_T\right) \geq 1 - t \right\}.$$

We close this section by remarks on the main result.

Remark 1 (Implication for confidence interval and hypothesis testing)**.** The upper breakdown point of the bootstrap quantile describes the minimal fraction of outliers in the original sample such that the bootstrap quantile diverges to infinity. It turns out that when this occurs, inference based on the bootstrap distribution becomes meaningless. For example, if the researcher wish to construct a bootstrap confidence interval for a parameter of interest, the breakdown of the bootstrap quantiles implies noninformative confidence intervals for the parameter of interest. Thus, the quantile breakdown point can be considered as a diagnostic tool to check robustness of bootstrap-based inference by describing up to which fraction of contaminations the bootstrap distribution still provides some reliable information.

Remark 2 (Statistics by recentered moments)**.** For the uniform weight bootstrap, the bootstrap statistic $T_n^{\#}$ is typically computed by using recentered moments, i.e., $g^{\#}(X_i, \theta) = g\left(X_i^{\#}, \theta\right) - \frac{1}{n}\sum_{i=1}^n g\left(X_i^{\#}, \theta\right)$ (Hall and Horowitz, 1996). This recentering is required to satisfy the moment conditions by bootstrap resamples. On the other hand, the implied probability bootstrap does not require such recentering since the bootstrap resamples always satisfy the moment conditions at $\hat{\theta}$ by construction. Therefore, it is possible that the breakdown point of the bootstrap statistic $T_n^{\#}$ (say $b_T^{\#}$) is different from the breakdown point b_T of T_n and T_n^*. In this case, b_T in Proposition 3 (i) should be replaced by $b_T^{\#}$.

Remark 3 (Higher dimension case)**.** Our breakdown analysis can be extended to the case of $\dim(g) > 2$. However, if each element of $g\left(X_{(n)}, \hat{\theta}\right)$ takes a different limit as $\left\|X_{(n)}\right\| \to +\infty$, we need to explicitly evaluate the limit of the inverse $\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right) g\left(X_i, \hat{\theta}\right)'\right]^{-1}$ and the result becomes more complicated and less intuitive. To obtain a comprehensible result, it would be reasonable to consider the case where all elements of $g\left(X_{(n)}, \hat{\theta}\right)$ take only two limiting values. In this case, we can split $g\left(X_{(n)}, \hat{\theta}\right)$ into two sub-vectors and apply the partitioned matrix inverse formula for $\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right) g\left(X_i, \hat{\theta}\right)'\right]^{-1}$.

Remark 4 (Time series data)**.** In order to suitably capture the dependence of the data generating process in a time series framework, the bootstrap requires some modifications. Consequently, besides the conventional uniform bootstrap also the implied probability bootstrap analyzed in our study cannot be directly applied to the time series case. Nevertheless, we think that some extensions of the implied probability approach for the residual bootstrap and the block bootstrap (see, e.g., Hall, 1985, Carlstein, 1986, and Künsch, 1989) could provide a valid method for defining robust bootstrap methods for time series (see, e.g., Camponovo, Scaillet and Trojani, 2010). For example, as Kitamura (1997) did for empirical likelihood inference, we can allocate implied probability weights to (overlapping or non-overlapping) blocks of samples and modify the implied probability bootstrap inference for dependent data. We expect that the breakdown analysis of this paper can be adapted to such a modified bootstrap method.

4 Simulations

We study through Monte Carlo simulations the statistical properties of the conventional bootstrap and implied probability bootstrap in estimating the distribution of the trimmed mean.

4.1 Benchmark example

To evaluate our theoretical results in finite samples, we first conduct a simulation study for the benchmark example in Section 2. The setup is as follows. Let $\{X_1, \ldots, X_{20}\}$ be a scalar iid sample of size $n = 20$ from $X \sim N(0, 1)$. As in Section 2, we wish to estimate the distribution of the 10% trimmed mean $T(0.1) = \frac{1}{18}\sum_{i=2}^{19} X_{(i)}$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ is the ordered sample. In order to study the robustness of the bootstrap methods, we consider two data generating processes: (i) no contamination ($X_{(n)}$ is as is), and (ii) contamination ($X_{(n)}$ =3, 5, 10, 20, and 100). We compare the uniform weight bootstrap quantile $Q_t^{\#}$ and the implied probability bootstrap quantile $Q_t^{*}$to estimate the quantile Q_t of the trimmed mean $T(0.1)$. For the implied probability bootstrap, we consider three moment functions: $g_1(X) = X$, $g_2(X) = X^2 - 1$, and $g_3(X) = \frac{1}{X} - E\left[\frac{1}{X}\right]$. For g_1 and g_2, Proposition 1 says that the implied probability bootstrap is more robust than the uniform weight bootstrap. On the other hand, for g_3, the implied probability bootstrap becomes less robust.

Table 1 reports the Monte Carlo averages of the bootstrap quantiles Q_t, $Q_t^{\#}$, and Q_t^{*} for t =0.9, 0.95, and 0.99 over $1,000$ replications. Without contamination, both methods provide very similar results. In the presence of contamination, the results basically confirm the theoretical predictions in Proposition 1. For large values of contamination, the uniform weight bootstrap quantiles $Q_t^{\#}$ dramatically increase. In particular, for the case of $X_{(n)} = 100$, $Q_{.95}^{\#}$ becomes larger than 11 whereas $Q_{.95}$ is 0.4806. In contrast, the implied probability bootstrap quantiles using g_1 and g_2 show desirable stability. Overall the simulation for the benchmark case suggests that our breakdown analysis reasonably characterizes (lack of) robustness of the bootstrap methods in finite samples.

5 Conclusion

This paper studies robustness of the uniform weight and implied probability bootstrap inference methods for moment condition models. In particularly, we analyzed the breakdown properties of the quantiles for those bootstrap methods. Simulation studies illustrate the theoretical findings. Our breakdown analysis can be a informative guideline for applied researchers who wish to decide which bootstrap method should be applied. It is interesting to apply our breakdown analysis to more specific setups (e.g. instrumental variable regression models to evaluate the effects of outliers in dependent, endogenous, and instrumental variables). Also it is important to extend our analysis to dependent data setups, where different bootstrap methods need to be employed. These extensions are currently under investigation by the authors.

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		No Con.	Contamination				
			3	5	10	20	100
True	$Q_{.9}$	0.2862	0.3951	0.3959	0.3959	0.3959	0.3959
	$Q_{.95}$	0.3718	0.4794	0.4806	0.4806	0.4806	0.4806
	$Q_{.99}$	0.5259	0.6304	0.6318	0.6318	0.6318	0.6318
Uniform	$Q^{\#}_{.9}$	0.3082	0.4784	0.5785	0.8879	1.4683	5.9127
	$Q^{\#}_{.95}$	0.3857	0.5738	0.7183	1.1791	2.2598	11.1487
	$Q^{\#}_{.99}$	0.5305	0.7605	0.9956	1.7152	3.3194	16.6528
Implied (g_1)	$Q^{*}_{.9}$	0.3034	0.4224	0.3951	0.3381	0.3135	0.3114
	$Q^{*}_{.95}$	0.3820	0.5227	0.5186	0.4711	0.3992	0.3902
	$Q^{*}_{.99}$	0.5258	0.7600	0.8303	0.8890	0.7016	0.5495
Implied (g_2)	$Q^{*}_{.9}$	0.3087	0.3880	0.3275	0.3182	0.3118	0.3105
	$Q^{*}_{.95}$	0.3923	0.4754	0.4130	0.3958	0.3954	0.3900
	$Q^{*}_{.99}$	0.5393	0.6347	0.5767	0.5524	0.5451	0.5448
Implied (g_3)	$Q^{*}_{.9}$	0.3094	0.4859	0.5953	0.9178	1.5415	5.9869
	$Q^{*}_{.95}$	0.3928	0.5892	0.7469	1.2224	2.3186	11.2090
	$Q^{*}_{.99}$	0.5393	0.7775	1.0247	1.7610	3.3873	16.7212

Table 1: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of $T(0.1)$ based on 20,000 realizations of $T(0.1)$. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied (g_a)" report the implied probability bootstrap quantiles using the moment function g_a for a =1, 2, and 3.