Inverse problems in statistics

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Introduction

There exist many fields where inverse problems appear

- Astronomy (Hubble satellite).
- Econometrics (instrumental variables).
- Financial mathematics (model calibration).
- Medical image processing (X-rays).

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These are problems where we have indirect observations of an object (a function) that we want to reconstruct.

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Let A be a continuous linear operator from H into G.

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Importance of the notion of "noise" or "error".

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Projection of a white noise on any orthonormal basis $\{\psi_k\}$ gives a sequence of i.i.d . standard Gaussian random variables.

Discrete model of inverse problems

The standard discrete sample statistical model for linear inverse problems is

$$Y_i = Af(X_i) + \xi_i,\ i = 1, \ldots, n,$$

where $(X_1, Y_1), .., (X_n, Y_n)$ are observed (we may assume $X_i \in [0, 1]$), f is an unknown function in $L^2(0, 1)$, A is an operator from $L^2(0, 1)$ into $L^2(0, 1)$, and ξ_i are i.i.d. zero-mean Gaussian random variables of variance σ^2.

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Noise level is related to number of observations by

$$\varepsilon \approx 1/\sqrt{n}.$$

Singular value decomposition

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$\{\varphi_k\}$ o.n.b. on H, $\{\psi_k\}$ o.n.b. on G.

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A linear bounded compact operator between two Hilbert spaces may really be seen as an infinite matrix.

Projection on $\{\psi_k\}$

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where $\{\xi_k\}$ standard Gaussian sequence i.i.d., by projection of a white noise ξ on the o.n.b. $\{\psi_k\}$.

Sequence space model

Equivalent Sequence space model

$$y_k = b_k\theta_k + \varepsilon\xi_k, \quad k = 1, 2, \ldots,$$

where $\{\theta_k\}$ coefficients of f, $\xi_k \sim \mathcal{N}(0,1)$ i.i.d.,
$b_k \to 0$ singular values.

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$\longrightarrow$ Ill-posed problem.

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Thus, we obtain the model :

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(see Donoho (1995), Mair and Ruymgaart (1996), Johnstone (1999) and C. and Tsybakov (2002)...).

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Parameter β is called degree of ill-posedness.

Examples

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- Convolution.
- Tomography.
- Instrumental variables.

Circular convolution

The framework of deconvolution is perhaps one of the most well-known inverse problem. It is used in many applications as econometrics, physics, astronomy, medical image processing. For example, it corresponds to the problem of a blurred signal that one wants to recover from indirect data.

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Consider the following convolution operator

$$Af(t) = r * f(t) = \int_0^1 r(t-x)f(x)dx, \quad x \in [0,1],$$

where r is a known 1-periodic symetric real convolution kernel in $L^2[0,1]$. In this model, A is a linear bounded self-adjoint operator from $L^2[0,1]$ to $L^2[0,1]$.

Blurred cameraman

(a)

(b)

Convolution model

Define then the following model

$$Y(t) = r * f(t) + \varepsilon \, \xi(t), \quad x \in [0,1],$$

where Y is observed, f is an unknown periodic function in $L^2[0,1]$ and $\xi(t)$ is a white noise on $L^2[0,1]$.

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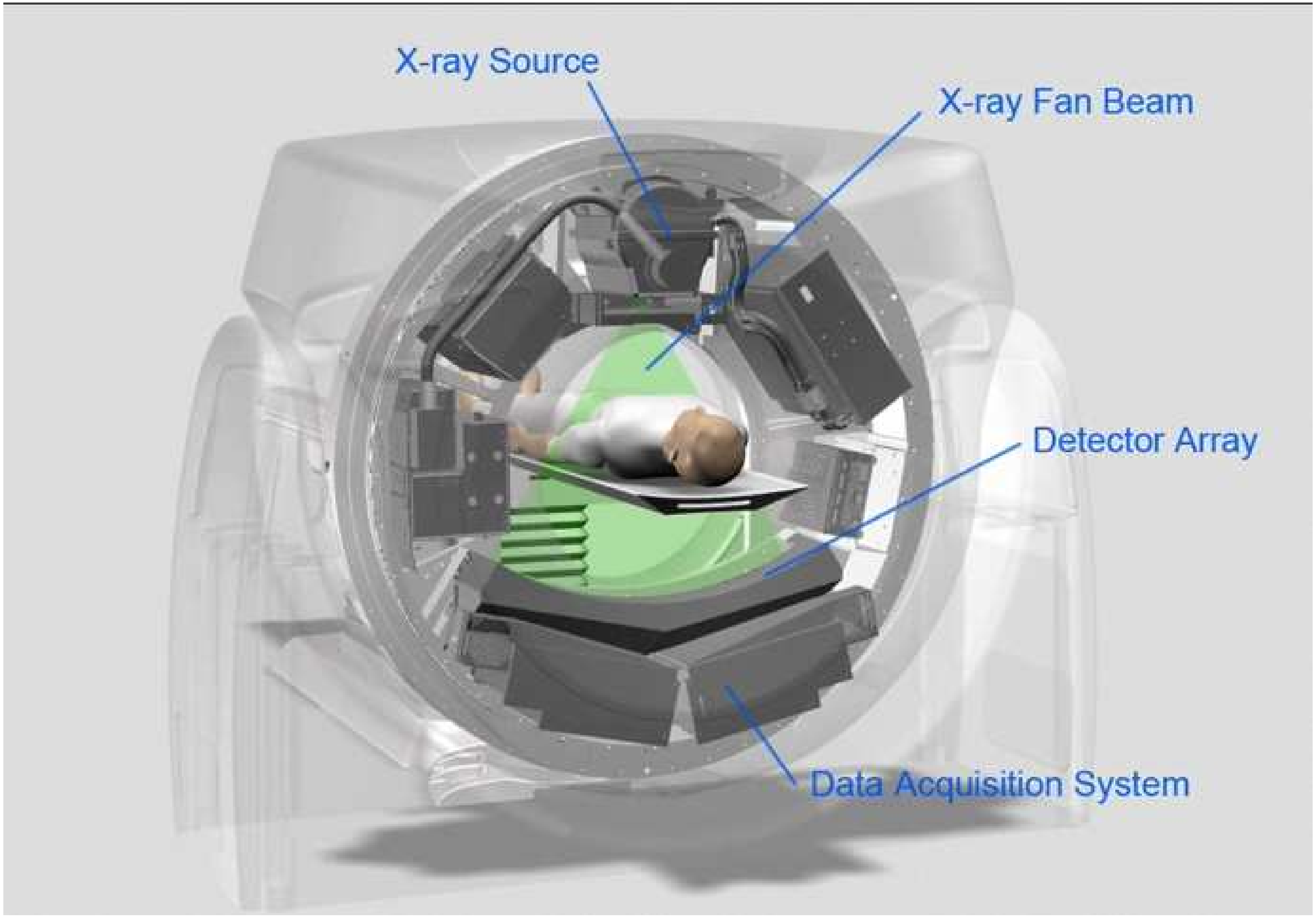
We make the projection on $\{\varphi_k(t)\}$, in the Fourier domain, and obtain

$$y_k = b_k \theta_k + \varepsilon \xi_k,$$

where $b_k = \sqrt{2}\int_0^1 r(x)\cos(2\pi k x)dx$ for even k, θ_k are the Fourier coefficients of f, and ξ_k are i.i.d. $\mathcal{N}(0,1)$.

Tomography scan



Instrumental variables

An economic relationship between a response variable Y and a vector X of explanatory variables is represented by

$$Y_i = f(X_i) + U_i, \ i = 1, \ldots, n,$$

where f has to be estimated and U_i are the errors.

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High ability tends to have high level of education, then education and ability are correlated, and thus X and U also.

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Inverse problems have been the topic of many articles in the econometrics literature, see Florens (2003), Hall and Horowitz (2005), Chen and Reiss (2009).

Inverse problem and sequence space

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Linear estimators

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The L^2-risk of a linear estimator is

$$E\|\hat{f}(\lambda) - f\|^2 = R(\theta, \lambda) = \sum_{k=1}^{\infty} (1 - \lambda_k)^2 \theta_k^2 + \varepsilon^2 \sum_{k=1}^{\infty} \sigma_k^2 \lambda_k^2.$$

Classes of linear estimators

- Projection estimators (spectral cut-off),

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$\longrightarrow$ Choice of N, γ or n?

Ellipsoid of coefficients

Assume that f belongs to functional class corresponding to ellipsoids Θ in space of coefficients $\{\theta_k\}$:

$$\Theta = \Theta(a, L) = \left\{ \theta : \sum_{k=1}^{\infty} a_k^2 \theta_k^2 \leq L \right\},$$

where $a = \{a_k\}$, where $a_k > 0, a_k \to \infty$ and $L > 0$.

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Assumptions on coefficients θ_k usually related to properties (smoothness) on f.

Sobolev classes

Introduce the Sobolev classes

$$\mathcal{W}(\alpha, L) = \left\{ f = \sum_{k=1}^{\infty} \theta_k \varphi_k : \theta \in \Theta(\alpha, L) \right\}$$

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where $\Theta(\alpha, L) = \Theta(a, L)$ with $a = \{a_k\}$ polynomial such that $a_1 = 0$ and

$$a_k = \begin{cases} (k-1)^\alpha & \text{for } k \text{ odd}, \\ k^\alpha & \text{for } k \text{ even}, \end{cases} \qquad k = 2, 3, \dots,$$

where $\alpha > 0$, $L > 0$. We have also

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where $\alpha > 0$, $L > 0$. We have also

$$\mathcal{W}(\alpha, L) = \left\{ f \text{ periodic } : \int_0^1 (f^{(\alpha)}(t))^2 dt \leq \pi^{2\alpha} L \right\}.$$

Rates of convergence

Function f has Fourier coefficients in some ellipsoid, and the problem is mildly, severely ill-posed or even direct. Rates appear in the following table :

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Problem/Functions	Sobolev
Direct problem	$\varepsilon^{\frac{4\alpha}{2\alpha+1}}$
Mildly ill-posed	$\varepsilon^{\frac{4\alpha}{2\alpha+2\beta+1}}$
Severely ill-posed	$(\log \frac{1}{\varepsilon})^{-2\alpha}$

Comments

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- When β increases rates are slower.
- In direct model, standard rates for nonparametric estimation.
  $\longrightarrow$ For example, $2\alpha/(2\alpha+1)$ with Sobolev classes.

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- Even if operator A (and its degree β) is known, no real meaning to consider smoothness of f as known.
- Notion of adaptation and oracle inequalities, i.e. how to choose bandwidth N without prior assumptions on f.

Oracle

Consider now a linked, but different point of view.
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An oracle is the best in the family, but it knows the true θ.

Unbiased risk estimation

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This idea appears also in all the cross-validation techniques.

URE in inverse problems

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In this setting, the functional

$$U(X,\lambda) = \sum_{k=1}^{\infty} (1-\lambda_k)^2 (X_k^2 - \varepsilon^2 \sigma_k^2) + \varepsilon^2 \sum_{k=1}^{\infty} \sigma_k^2 \lambda_k^2$$

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is an unbiased estimator of $R(\theta,\lambda)$:

$$R(\theta,\lambda) = \mathbf{E}_\theta U(X,\lambda), \ \forall \lambda.$$

Data-driven choice

Unbiased risk estimation suggests to minimize over $\lambda \in \Lambda$ the functional $U(X, \lambda)$ in place of $R(\theta, \lambda)$.

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Define then the estimator $\theta^\star$ by

$$\theta_k^\star = \lambda_k^\star X_k.$$

Assumptions

Denote

$$S = \left(\frac{\max_{\lambda \in \Lambda} \sum_{k=1}^{\infty} \sigma_k^4 \lambda_k^2}{\min_{\lambda \in \Lambda} \sum_{k=1}^{\infty} \sigma_k^4 \lambda_k^2} \right)^{1/2}.$$

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$$0 < \sum_{k=1}^{\infty} \sigma_k^2 \lambda_k^2 < \infty, \qquad \max_{\lambda \in \Lambda} \sup_{k} |\lambda_k| \leq 1.$$

There exists a constant $C_1 > 0$ such that, uniformly in $\lambda \in \Lambda$,

$$\sum_{k=1}^{\infty} \sigma_k^4 \lambda_k^2 \leq C_1 \sum_{k=1}^{\infty} \sigma_k^4 \lambda_k^4.$$

Oracle inequality for URE

Theorem. Suppose $\sigma_k \sim k^\beta,\ \beta \geq 0$.
Assume that Λ is finite with cardinality D and belongs to the family of Projection, Tikhonov or Pinsker estimators.

There exist constants $\gamma, C^* > 0$ such that $\forall \theta \in \ell_2$, we have for B large enough,

$$\mathbf{E}_\theta \|\theta^\star - \theta\|^2 \leq (1 + \gamma B^{-1}) \min_{\lambda \in \Lambda} R(\theta, \lambda) + BC^* \varepsilon^2 (\log(DS))^{2\beta+1}.$$

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The data-driven choice by URE mimics the oracle.

Simulations

Discrete model : inverse problem.

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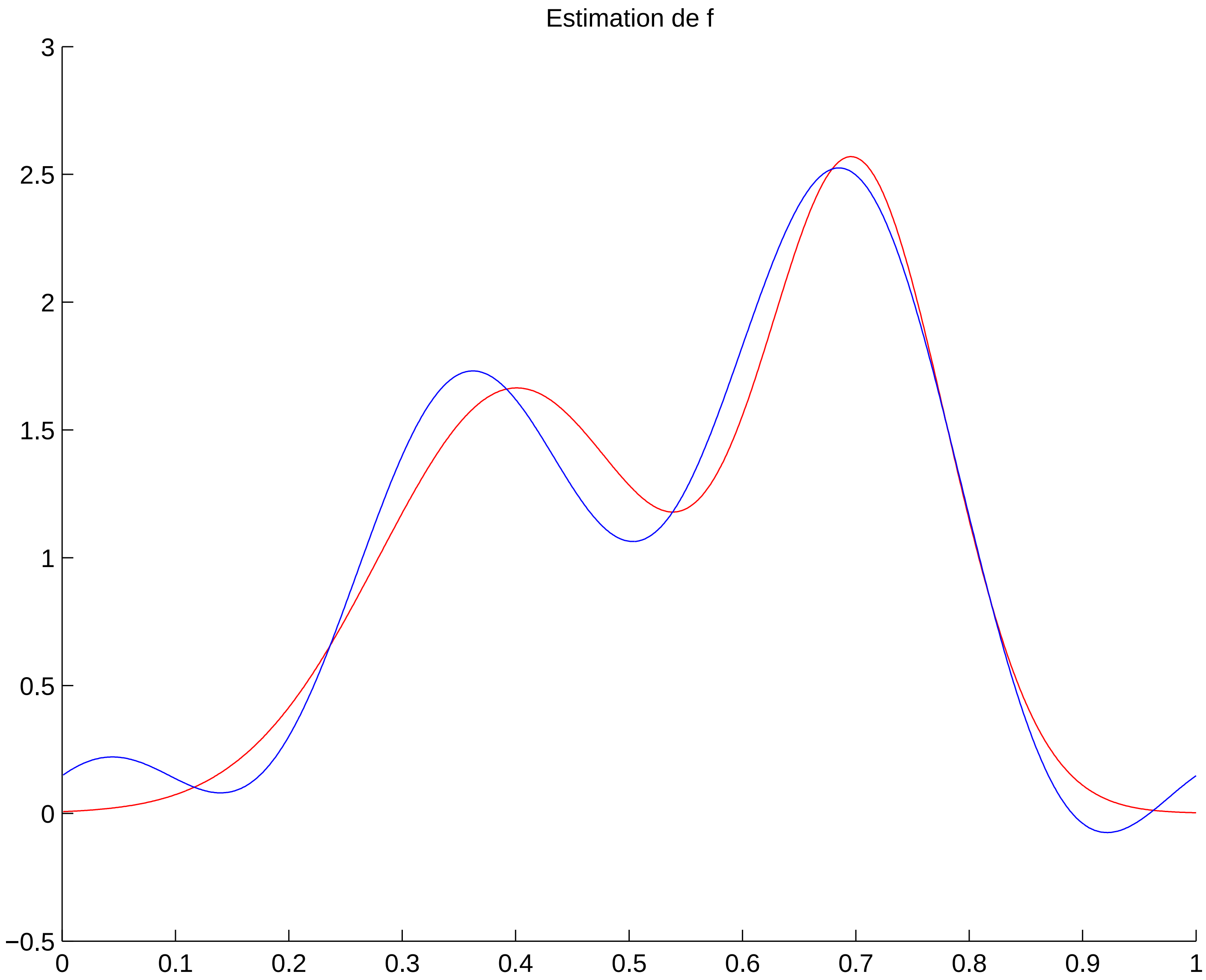
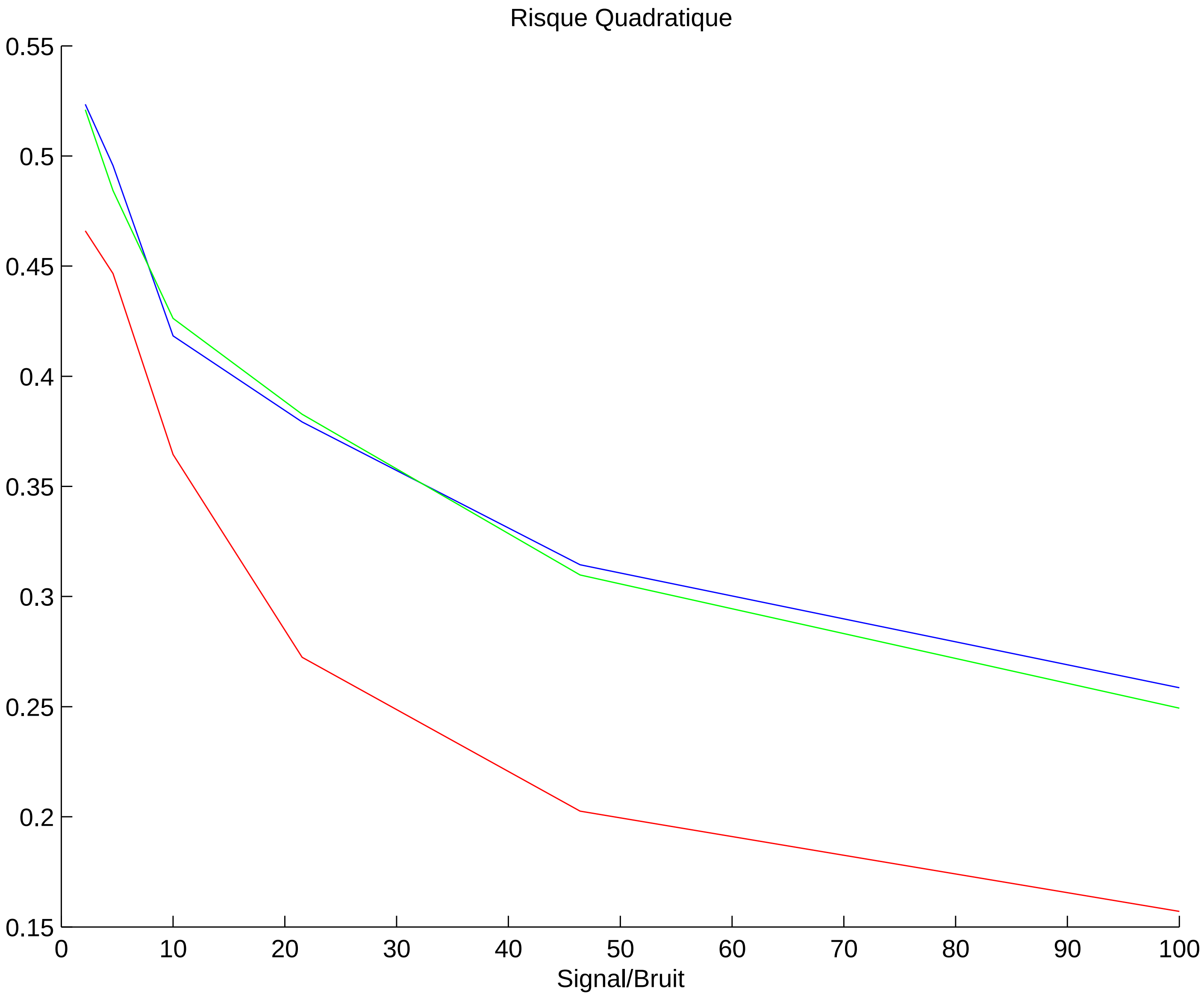
True function f. Estimator $f^{\star}$.



Oracle by projection. Estimator $f^\star$.



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- Need for stronger penalties than the URE penalty (or AIC).
- Different method called Risk Hull Method, defined in C. and Golubev (2006).