Appendix to A Multivariate Model of Strategic Asset Allocation

John Y. Campbell, Yeung Lewis Chan, and Luis M. Viceira[1]

September 2000

[1]Campbell: Department of Economics, Littauer Center 213, Harvard University, Cambridge MA 02138, USA, and NBER. Email john_campbell@harvard.edu. Website http://www.economics.harvard.edu/faculty/jcampbell/campbell.html. Chan: Department of Finance, School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email ylchan@ust.hk. Viceira: Graduate School of Business Administration, Morgan Hall 367, Harvard University, Boston MA 02163, USA, CEPR, and NBER. Email lviceira@hbs.edu. Website http://www.people.hbs.edu/lviceira/. Campbell acknowledges the financial support of the National Science Foundation. We are grateful for helpful comments and suggestions by Anthony Lynch and seminar participants at Harvard, the 1999 Intertemporal Asset Pricing Conference hosted by the Centre Interuniversitaire de Recherche en Analyse des Organizations (CIRANO) of Montreal and the 2000 WFA Meetings.

A Appendix A: Derivation of Main Equations in Text

We first summarize three results on matrix algebra that will be convenient in deriving the expressions given in the text.

Result 1.

$$\begin{aligned} & \mathbf{z}_{t+1}\mathbf{z}'_{t+1} - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}'_{t+1}\right) \\ = & \left(\Phi_0 + \Phi_1\mathbf{z}_t + \mathbf{v}_{t+1}\right)\left(\Phi_0 + \Phi_1 z_t + \mathbf{v}_{t+1}\right)' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}'_{t+1}\right) \\ = & \Phi_0\Phi'_0 + \Phi_1\mathbf{z}_t\Phi'_0 + \mathbf{v}_{t+1}\Phi'_0 + \Phi_0\mathbf{z}'_t\Phi'_1 + \Phi_1\mathbf{z}_t\mathbf{z}'_t\Phi'_1 \\ & +\mathbf{v}_{t+1}\mathbf{z}'_t\Phi'_1 + \Phi_0\mathbf{v}'_{t+1} + \Phi_1\mathbf{z}_t\mathbf{v}'_{t+1} + \mathbf{v}_{t+1}\mathbf{v}'_{t+1} - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}'_{t+1}\right) \\ = & \mathbf{v}_{t+1}\Phi'_0 + \mathbf{v}_{t+1}\mathbf{z}'_t\Phi'_1 + \Phi_0\mathbf{v}'_{t+1} + \Phi_1\mathbf{z}_t\mathbf{v}'_{t+1} + \mathbf{v}_{t+1}\mathbf{v}'_{t+1} - \Sigma_v. \end{aligned}$$

■

Result 2.

$$\begin{aligned} & r_{i,t+1} - \mathrm{E}_t\left(r_{i,t+1}\right) \\ = & \mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1} - \mathrm{E}_t\left(\mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1}\right) \\ = & \mathbf{v}_{t+1}^{(i)} + \mathbf{v}_{t+1}^{(1)} \end{aligned}$$

where $\mathbf{x}_{t+1}^{(i-1)}$ denotes the $(i-1)$th element of the excess return vector $\mathbf{x}_{t+1}$ and likewise with $\mathbf{v}_{t+1}$.■

Result 3. (Muirhead, 1982, pp.518)

$$\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}'_{t+1}\right)\right) = \left(\mathbf{I}_{m^2} + \sum_{i,j}^{m}\left(\mathbf{Q}_{ij}\otimes\mathbf{Q}'_{ij}\right)\right)\left(\Sigma_v\otimes\Sigma_v\right),$$

where $\mathbf{Q}_{ij}$ is a $m\times m$ zero matrix except for the (i,j)th element which is equal to 1.■

Derivation of Equation (10)

The log return on the portfolio $r_{p,t+1}$ is a discrete-time approximation to its continuous-time counterpart. We begin by specifying the return processes for the short-term instrument B_t and other risky assets $\mathbf{P}_t$ in continuous time:

$$\frac{dB_t}{B_t} = \mu_{b,t}dt + \boldsymbol{\sigma}_b d\mathbf{W}_t, \tag{27}$$

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{W}_t, \tag{28}$$

where $\mu_{b,t}$ and $\boldsymbol{\mu}_t$ are the drifts, $\boldsymbol{\sigma}_b$ and $\boldsymbol{\sigma}$ are the diffusion, and $\mathbf{W}_t$ is a m-dimensional standard Brownian motion.[12] We allow the drifts to depend on other state variables, but for notational

[12]The dimensions of $\mu_b, \boldsymbol{\mu}, \boldsymbol{\sigma}_b, \boldsymbol{\sigma}$ are $1\times 1, (n-1)\times 1, 1\times m, (n-1)\times m$, respectively.

simplicity, we suppress this dependency and simply use the time subscript. Moreover, note that the same $\mathbf{W}_t$ appears in the two equations.

We can obtain the log return on each asset using Ito's Lemma:

$$d\log B_t = \left(\frac{dB_t}{B_t}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt, \tag{29}$$

$$d\log P_{i,t} = \left(\frac{dP_{i,t}}{P_{i,t}}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right)dt, \tag{30}$$

where $\boldsymbol{\sigma}_i$ is the ith row of the diffusion matrix $\boldsymbol{\sigma}$, and $i = 1, .., n-1$.

Let V_t be the value of the portfolio at time t. We will use $d\log V_t$ to approximate $r_{p,t+1}$. By Ito's Lemma,

$$d\log V_t = \left(\frac{dV_t}{V_t}\right) - \frac{1}{2}\left(\frac{dV_t}{V_t}\right)^2. \tag{31}$$

We will now derive these two terms in order:

$$\begin{aligned}
\frac{dV_t}{V_t} &= \boldsymbol{\alpha}_t'\left(\frac{d\mathbf{P}_t}{\mathbf{P}_t}\right) + \left(1-\boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\frac{dB_t}{B_t} \\
&= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t + \frac{1}{2}\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right]dt\right) + \left(1-\boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\left(d\log B_t + \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt\right) \\
&= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right) + d\log B_t \\
&\quad + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] - \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\cdot\boldsymbol{\iota}\right)dt + \frac{1}{2}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'dt,
\end{aligned}$$

where $\boldsymbol{\iota}$ is a $n\times 1$ vector of ones and the bracket $[\cdot]$ denotes a vector with $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'$ the ith entry. Next,

$$\begin{aligned}
\left(\frac{dV_t}{V_t}\right)^2 &= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)'\boldsymbol{\alpha}_t + \left(d\log B_t\right)^2 \\
&\quad + 2\boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log B_t\right) + o\left(dt\right),
\end{aligned}$$

where the $o(dt)$ terms vanish because they involve either $(dt)^2$ or $(dt)\left(d\mathbf{W}_t\right)$.

Now, from equation (27)–(29) and ignoring dt terms,

$$d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota} = \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)d\mathbf{W}_t.$$

Thus,

$$\begin{aligned}
\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)' &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)', \\
\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log B_t\right) &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_1\right)\cdot\boldsymbol{\sigma}_b'.
\end{aligned}$$

Collecting these results and using our notation for excess returns: $\mathbf{x}_{t+1} = d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}$,

$r_{1,t+1} = d\log(B_t)$ and $dt = 1$,

$$
\begin{aligned}
& r_{p,t+1} \\
= \; & d\log V_t \\
= \; & \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left([\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'] - \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota} \right) \\
& -\frac{1}{2}\left[\boldsymbol{\alpha}_t' \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right) \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right)' \boldsymbol{\alpha}_t + 2\boldsymbol{\alpha}_t' \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right) \boldsymbol{\sigma}_b' \right]
\end{aligned}
$$

Using the notation in the VAR system with the Cholesky decomposition for $\Sigma_v = \mathbf{GG}'$, $\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_b$ is equal to the ith row of $\mathbf{G}, \mathbf{G}_i$. Hence,

$$
\begin{aligned}
\left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right) \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right)' &= \mathbf{G}_{2:n} \mathbf{G}_{2:n}' = \Sigma_{xx}, \\
\boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' &= \mathbf{G}_1 \mathbf{G}_1' = \sigma_1^2, \\
\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i' &= \mathbf{G}_i \mathbf{G}_i' + \boldsymbol{\sigma}_b \mathbf{G}_i' + \mathbf{G}_i \boldsymbol{\sigma}_b' + \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b', \\
[\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'] &= \sigma_x^2 + 2\Sigma_{1x} + \sigma_1^2 \boldsymbol{\iota}, \\
\left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b \right) \boldsymbol{\sigma}_b' &= \mathbf{G}_{2:n} \mathbf{G}_1' = \Sigma_{1x},
\end{aligned}
$$

where $\mathbf{G}_{2:n}$ denotes the submatrix formed by taking the 2nd to nth rows of $\mathbf{G}$.

With these terms, the return on the portfolio is

$$
\begin{aligned}
r_{p,t+1} &= \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left(\sigma_x^2 + 2\Sigma_{1x} \right) - \frac{1}{2}\boldsymbol{\alpha}_t' \Sigma_{xx} \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t' \Sigma_{1x}, \\
&= \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left(\sigma_x^2 - \Sigma_{xx} \boldsymbol{\alpha}_t \right).
\end{aligned}
$$

■

Solving for the Optimal Portfolio Rule.

Subtracting the log Euler equation (12) with $i = 1$ from (12), we obtain

$$
\begin{aligned}
& \mathrm{E}_t \left(r_{i,t+1} - r_{1,t+1} \right) + \frac{1}{2}\mathrm{Var}_t \left(r_{i,t+1} - r_{1,t+1} \right) \qquad (32) \\
= \; & \mathrm{Cov}_t \left(\frac{\theta}{\psi} \Delta c_{t+1} + (1-\theta) r_{p,t+1}, r_{i,t+1} \right) - \mathrm{Cov}_t \left(\frac{\theta}{\psi} \Delta c_{t+1} + (1-\theta) r_{p,t+1}, r_{1,t+1} \right) \\
& -\frac{1}{2} \left(\mathrm{Var}_t \left(r_{i,t+1} \right) - \mathrm{Var}_t \left(r_{1,t+1} \right) - \mathrm{Var}_t (r_{i,t+1} - r_{1,t+1}) \right).
\end{aligned}
$$

Using the budget constraint (11) and the trivial identity $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$,

$$
\begin{aligned}
& \frac{\theta}{\psi} \Delta c_{t+1} + (1-\theta) r_{p,t+1} \\
= \; & \frac{\theta}{\psi} \left(c_{t+1} - w_{t+1} \right) + \gamma r_{p,t+1} + \text{ time } t \text{ terms and constants.}
\end{aligned}
$$

Thus, equation (32) can be written as

$$\begin{aligned}
& \mathrm{E}_t\left(r_{i,t+1}-r_{1,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{i,t+1}-r_{1,t+1}\right) \\
= \; & \frac{\theta}{\psi}\left[\sigma_{i,c-w,t}-\sigma_{1,c-w,t}\right]+\gamma\left[\sigma_{i,p,t}-\sigma_{1,p,t}\right] \\
& -\frac{1}{2}\left(\mathrm{Var}_t\left(r_{i,t+1}\right)-\mathrm{Var}_t\left(r_{1,t+1}\right)-\mathrm{Var}_t(r_{i,t+1}-r_{1,t+1})\right).
\end{aligned}$$

We will derive these terms now.

Using the equation for log return on the portfolio and ignoring time t terms and constants,

$$\begin{aligned}
\sigma_{i,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1}+r_{1,t+1},r_{i,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\left(\Sigma_{xx}^{(i-1)}+\Sigma_{1x}\right)+\Sigma_{1x}^{(i-1)}+\sigma_1^2, \\
\sigma_{1,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1}+r_{1,t+1},r_{1,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\Sigma_{1x}+\sigma_1^2.
\end{aligned}$$

To evaluate the conditional covariances $\sigma_{i,c-w,t}$ and $\sigma_{1,c-w,t}$, we use the conjectured policy rule for the consumption-wealth ratio.

$$\begin{aligned}
& \sigma_{i,c-w,t} \\
= \; & \mathrm{Cov}_t\left(c_{t+1}-w_{t+1}-\mathrm{E}_t\left(c_{t+1}-w_{t+1}\right),r_{i,t+1}-\mathrm{E}_t(r_{i,t+1})\right) \\
= \; & \mathrm{Cov}_t\left(\mathbf{B}_1'v_{t+1}+\mathbf{B}_2'\left(\mathbf{v}_{t+1}\Phi_0'+\mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1'+\Phi_0\mathbf{v}_{t+1}'+\Phi_1\mathbf{z}_t\mathbf{v}_{t+1}'+\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\mathbf{B}_2,\,\mathbf{v}_{t+1}^{(i)}+\mathbf{v}_{t+1}^{(1)}\right) \\
= \; & \mathbf{B}_1'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)+2\mathbf{B}_2'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)\Phi_0'\mathbf{B}_2+2\mathbf{B}_2'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)\mathbf{z}_t'\Phi_1\mathbf{B}_2,
\end{aligned}$$

where the second equality follows from using Result 1 and 2, and $\Sigma_v^{(i)}$ denotes the ith column of Σ_v. Similarly,

$$\sigma_{1,c-w,t}=\mathbf{B}_1'\Sigma_v^{(1)}+2\mathbf{B}_2'\Sigma_v^{(1)}\Phi_0'\mathbf{B}_2+2\mathbf{B}_2'\Sigma_v^{(1)}\mathbf{z}_t'\Phi_1\mathbf{B}_2.$$

With these expressions,

$$\begin{aligned}
& \boldsymbol{\sigma}_{c-w,t}-\sigma_{1,c-w,t}\boldsymbol{\iota} \\
= \; & \left(\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_1+2\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_2\mathbf{B}_2'\Phi_0\right)+\left(2\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_2\mathbf{B}_2'\Phi_1\right)\mathbf{z}_t \\
= \; & \left[\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_1+2\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_2\mathbf{B}_2'\Phi_0\right]+\left[2\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right]\mathbf{z}_t \\
= \; & \Lambda_0+\Lambda_1\mathbf{z}_t,
\end{aligned}$$

as claimed in equation (17).■

Solving for the Optimal Consumption Rule

We derive first equation (24). To derive this equation, note that log consumption growth verifies the following trivial identity: $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$. Substituting the log-linearized budget constraint (11) into this equation and taking expectations we obtain

$$\begin{aligned} \mathrm{E}_t(\Delta c_{t+1}) &= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t\left(\Delta w_{t+1}\right) \\ &= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t(r_{p,t+1}) + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k. \end{aligned} \tag{33}$$

Combining the two equations (22) and (33), we obtain a difference equation in $c_t - w_t$, given in (24).

Next we show that both the expected log return on the wealth portfolio $\mathrm{E}_t r_{p,t+1}$ and the variance term $\chi_{p,t}$ in equation (22) for expected log consumption growth are quadratic functions of the vector of state variables.

Taking conditional expectations of equation (10) and substituting the portfolio policy rule $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{z}_t$,

$$\begin{aligned} & \mathrm{E}_t\left(r_{p,t+1}\right) \\ = {} & \boldsymbol{\alpha}_t' \mathrm{E}_t\left(\mathbf{x}_{t+1}\right) + \mathrm{E}_t\left(r_{1,t+1}\right) + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\sigma_x^2 - \Sigma_{xx}\boldsymbol{\alpha}_t\right) \\ = {} & \left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\mathbf{H}_x\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) + \mathbf{H}_1\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) \\ & + \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\sigma_x^2 - \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\Sigma_{xx}\left(\mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t\right) \\ = {} & \Gamma_0 + \Gamma_1\mathbf{z}_t + \Gamma_2 \mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right), \end{aligned}$$

where

$$\begin{aligned} \Gamma_0 &\equiv \mathbf{A}_0'\mathbf{H}_x\Phi_0 + \mathbf{H}_1\Phi_0 + \frac{1}{2}\mathbf{A}_0'\sigma_x^2 - \frac{1}{2}\mathbf{A}_0'\Sigma_{xx}\mathbf{A}_0, \\ \Gamma_1 &\equiv \Phi_0'\mathbf{H}_x'\mathbf{A}_1 + \mathbf{A}_0'\mathbf{H}_x\Phi_1 + \mathbf{H}_1\Phi_1 + \frac{1}{2}\sigma_x^2\mathbf{A}_1 - \mathbf{A}_0'\Sigma_{xx}\mathbf{A}_1, \\ \Gamma_2 &\equiv \mathrm{vec}\left(\mathbf{A}_1'\mathbf{H}_x\Phi_1\right)' - \frac{1}{2}\mathrm{vec}\left(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1\right)', \end{aligned}$$

and $\mathbf{H}_1$ and $\mathbf{H}_x$ are selection matrices that select the short-term real interest rate and the vector of excess returns from the full state vector.

We now evaluate the variance term

$$\chi_{p,t} = \frac{1}{2}\left(\frac{\theta}{\psi}\right)\mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right).$$

Using the trivial identity for Δc_{t+1} and the budget constraint (11), substituting the conjecture for the consumption rule

$$\begin{aligned} c_t - w_t &= b_0 + \mathbf{B}_1'\mathbf{z}_t + \mathbf{B}_2'\mathbf{z}_t\mathbf{z}_t'\mathbf{B}_2 \\ &= b_0 + \mathbf{B}_1'\mathbf{z}_t + vec\left(\mathbf{B}_2\mathbf{B}_2'\right)' vec\left(\mathbf{z}_t\mathbf{z}_t'\right) \end{aligned}$$

and $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t$, and ignoring time t terms and constants, we can write the argument of the variance as:

$$\begin{aligned}
& \Delta c_{t+1} - \psi r_{p,t+1} \\
= \; & \left[\mathbf{B}_1' + 2\mathbf{z}_t'\Phi_1\mathbf{B}_2\mathbf{B}_2' + (1-\psi)\,\boldsymbol{\alpha}_t'\mathbf{H}_x + (1-\psi)\,\mathbf{H}_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2'\right]\mathbf{v}_{t+1} \\
& + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\,\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right) \\
= \; & \left[\Pi_1 + \mathbf{z}_t'\Pi_2\right]\mathbf{v}_{t+1} + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\,\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right),
\end{aligned}$$

where

$$\begin{aligned}
\Pi_1 & \equiv \mathbf{B}_1' + (1-\psi)\,\mathbf{A}_0'\mathbf{H}_x + (1-\psi)\,\mathbf{H}_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2', \\
\Pi_2 & \equiv 2\left(\Phi_1\mathbf{B}_2\mathbf{B}_2'\right) + (1-\psi)\,\mathbf{A}_1'\mathbf{H}_x.
\end{aligned}$$

Since $\mathbf{v}_{t+1}$ is conditionally normally distributed, all third moments are zero. Thus,

$$\begin{aligned}
& \mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right) \\
= \; & \Pi_1\Sigma_v\Pi_1' + \left[2\Pi_1\Sigma_v\Pi_2'\right]\mathbf{z}_t + \left[\mathrm{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\right]\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right) \\
& + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\,\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right),
\end{aligned}$$

and $\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)$ is given by the expression in Result 3 above. Putting these pieces together, we have

$$\chi_{p,t} = V_0 + \mathbf{V}_1\mathbf{z}_t + \mathbf{V}_2\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right),$$

where

$$\begin{aligned}
V_0 & \equiv \frac{\theta}{2\psi}\left[\Pi_1\Sigma_v\Pi_1' + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\,\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)\right], \\
\mathbf{V}_1 & \equiv \frac{\theta}{2\psi}\left[2\Pi_1\Sigma_v\Pi_2'\right], \\
\mathbf{V}_2 & \equiv \frac{\theta}{2\psi}\left[\mathrm{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\right].
\end{aligned}$$

Derivation of Equation (34)

Simple substitution of the expressions for $\mathrm{E}_t r_{p,t+1}$ and $\chi_{p,t}$, and the expression for the conditional expectation of $(c_{t+1} - w_{t+1})$ into the RHS of (24) yields

$$c_t - w_t = \Xi_0 + \Xi_1\mathbf{z}_t + \Xi_2\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right), \tag{34}$$

where

$$\begin{aligned}
\Xi_0 & \equiv \rho[-\psi\log\delta + k - V_0 + (1-\psi)\,\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 \\
& \quad + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\mathrm{vec}(\Phi_0\Phi_0') + \mathrm{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\mathrm{vec}(\Sigma_v)], \\
\Xi_1 & \equiv \rho\left[-\mathbf{V}_1 + (1-\psi)\,\Gamma_1 + \mathbf{B}_1'\Phi_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right], \\
\Xi_2 & \equiv \rho\left[-\mathbf{V}_2 + (1-\psi)\,\Gamma_2 + \mathrm{vec}\left(\Phi_1'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right)'\right].
\end{aligned}$$

Equation (34) confirms our initial conjecture on the form of the consumption-wealth ratio. Notice that Ξ_0, Ξ_1, Ξ_2 depend on $b_0, \mathbf{B}_1$ and $\mathrm{vec}(\mathbf{B}_2\mathbf{B}_2')$. Therefore, for the solution to be consistent, $\{b_0, \mathbf{B}_1,\ \mathrm{vec}(\mathbf{B}_2\mathbf{B}_2')\}$ must solve the following set of equations:

$$\begin{aligned} b_0 &= \Xi_0, \\ \mathbf{B}_1 &= \Xi_1', \\ \mathrm{vec}\,(\mathbf{B}_2\mathbf{B}_2') &= \Xi_2'. \end{aligned} \tag{35}$$

The resulting set of values for $b_0, \mathbf{B}_1$ and $\mathrm{vec}(\mathbf{B}_2\mathbf{B}_2')$ determines the optimal consumption rule.■

Verification that $\mathbf{A}_0$ and $\mathbf{A}_1$ do not depend on ψ

¿From (21), $\mathbf{A}_0$ and $\mathbf{A}_1$ can be expressed as:

$$\begin{aligned} \mathbf{A}_0 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\sigma_x^2 + (1-\gamma)\Sigma_{1x}\right) + \frac{1}{\gamma}\frac{1-\gamma}{1-\psi}\Sigma_{xx}^{-1}\Lambda_0 \\ &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\sigma_x^2 + (1-\gamma)\Sigma_{1x}\right) + \frac{1-\gamma}{\gamma}\Sigma_{xx}^{-1}\left[(\Sigma_v\mathbf{H}_x')'\,\frac{\mathbf{B}_1}{1-\psi} + 2\,(\Sigma_v\mathbf{H}_x')'\,\frac{\mathbf{B}_2\mathbf{B}_2'}{1-\psi}\Phi_0\right] \\ \mathbf{A}_1 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\,(\mathbf{H}_x\Phi_1) + \left(\frac{1}{\gamma}\Sigma_{xx}^{-1}\right)\frac{1-\gamma}{1-\psi}\Lambda_1 \\ &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\,(\mathbf{H}_x\Phi_1) + \left(\frac{1-\gamma}{\gamma}\Sigma_{xx}^{-1}\right)\left[2\,(\Sigma_v\mathbf{H}_x')'\,\frac{\mathbf{B}_2\mathbf{B}_2'}{1-\psi}\Phi_1\right]. \end{aligned}$$

Thus, showing that $\mathbf{A}_0$ and $\mathbf{A}_1$ are independent of ψ is equivalent to showing $\mathcal{B}_1 \equiv \mathbf{B}_1/(1-\psi)$ and $\mathcal{B}_2 \equiv \mathbf{B}_2\mathbf{B}_2'/(1-\psi)$ are independent of ψ.

First consider $\mathcal{B}_2$. From (35), we have

$$(1-\psi)\mathrm{vec}\,(\mathcal{B}_2) = \rho\left[-\mathbf{V}_2' + (1-\psi)\Gamma_2' + (1-\psi)\mathrm{vec}\,(\Phi_1'\mathcal{B}_2\Phi_1)\right]. \tag{36}$$

Using the definition of $\mathbf{V}_2$,

$$\begin{aligned} -\mathbf{V}_2' &= \frac{1-\gamma}{2(1-\psi)}\mathrm{vec}\left[(1-\psi)^2\,(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x)\,\Sigma_v\,(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x)'\right] \\ &= \frac{1-\gamma}{2}(1-\psi)\mathrm{vec}\left[(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x)\,\Sigma_v\,(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x)'\right] \\ &\equiv (1-\psi)\overline{\mathbf{V}}_2' \end{aligned}$$

Note that $\mathbf{A}_1$, and hence $\overline{\mathbf{V}}_2'$, do not depend on ψ, given $\mathcal{B}_2$. Moreover,

$$(1-\psi)\Gamma_2' = (1-\psi)\left[\mathrm{vec}\,(\mathbf{A}_1'\mathbf{H}_x\Phi_1) - \frac{1}{2}\mathrm{vec}\,(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1)\right],$$

where the expression in brackets depends only on $\mathcal{B}_2$. Thus, (36) reduces to

$$\mathrm{vec}\,(\mathcal{B}_2) = \rho\left[\overline{\mathbf{V}}_2' + \Gamma_2' + \mathrm{vec}\,(\Phi_1'\mathcal{B}_2\Phi_1)\right].$$

This is a quadratic equation in $\mathcal{B}_2$, with coefficients independent of ψ. Consequently, the solution for $\mathcal{B}_2$ will also be independent of ψ.

Now, using the same logic, we can show that $\mathcal{B}_1$ is independent of ψ. From (35),

$$(1-\psi)\mathcal{B}_1 = \rho\left[-\mathbf{V}_1' + (1-\psi)\,\Gamma_1' + (1-\psi)\Phi_1'\mathcal{B}_1 + (1-\psi)2\left(\Phi_0'\mathcal{B}_2\Phi_1\right)'\right]. \tag{37}$$

Note that given $\mathcal{B}_1$, A_0 (given above) is independent of ψ. Now,

$$\begin{aligned}
-\mathbf{V}_1' &= \frac{1-\gamma}{(1-\psi)}\left[\left((1-\psi)\overline{\Pi}_1\right)\Sigma_v\left((1-\psi)\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)\right)'\right]' \\
&= (1-\gamma)\,(1-\psi)\left[\overline{\Pi}_1\Sigma_v\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)'\right]' \\
&\equiv (1-\psi)\overline{V}_1
\end{aligned}$$

where

$$\overline{\Pi}_1 \equiv \mathcal{B}_1' + \mathbf{A}_0'\mathbf{H}_x + \mathbf{H}_1 + 2\Phi_0'\mathcal{B}_2.$$

Also, Γ_1 is only a function of $\mathcal{B}_1$ and $\mathcal{B}_2$ via its dependence on $\mathbf{A}_0$ and $\mathbf{A}_1$, not of ψ. Therefore, (37) becomes

$$\mathcal{B}_1 = \rho\left[\overline{\mathbf{V}}_1 + \Gamma_1' + \Phi_1'\mathcal{B}_1 + 2\left(\Phi_0'\mathcal{B}_2\Phi_1\right)'\right],$$

which again implies that the solution for $\mathcal{B}_1$ does not depend on ψ. This completes our proof.■

B Appendix B: Numerical Procedure

Equations (21) and (35) show that the coefficients $\{\mathbf{A}_0, \mathbf{A}_1\}, \{b_0, \mathbf{B}_1, \text{ vec}(\mathbf{B}_2\mathbf{B}_2')\}$ in the optimal policy rules are functions of the underlying parameters. When there is one state variable as in Campbell and Viceira (1999), solving explicitly for these coefficients is manageable. However, with multiple state variables, such an exercise is practically impossible. Therefore, we employ a numerical procedure to find these coefficients instead.

We describe the numerical procedure in steps:

1. For a given set of values of $\{\gamma, \psi, \rho\}$, we start with some initial values for δ and $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$—denote these by $\delta^{(1)}$ and $\{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$. Through equation (21), this implies a set of values for $\{\mathbf{A}_0, \mathbf{A}_1\}$—denote by $\{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}$.
2. With $\rho, \delta^{(1)}, \{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}, \{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$, the coefficients $\{\Xi_0, \Xi_1, \Xi_2\}$ in the $c - w$ difference equation (34) can be calculated. By equating these coefficients with the $\{b_0, \mathbf{B}_1, \text{ vec}(\mathbf{B}_2\mathbf{B}_2')\}$ in the conjectured policy function, we have a new set of values for $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$—call them $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$. Since the initial values are arbitrary, $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$ will be different from $\{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$ in general. Thus, we recompute $\{\mathbf{A}_0, \mathbf{A}_1\}$ using $\rho, \delta^{(1)}$ and $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$ to get $\{\mathbf{A}_0^{(2)}, \mathbf{A}_1^{(2)}\}$. A new set of $\{\Xi_0, \Xi_1, \Xi_2\}$ can then be obtained. We continue until values of $\{b_0, \mathbf{B}_1,$ vec$(\mathbf{B}_2\mathbf{B}_2')\}$ and hence $\{\mathbf{A}_0, \mathbf{A}_1\}$ converge, for a given ρ and $\delta^{(1)}$. Call these new values $\{\mathbf{A}_0^{\delta^{(1)}}, \mathbf{A}_1^{\delta^{(1)}}\}$, $\{b_0^{\delta^{(1)}}, \mathbf{B}_1^{\delta^{(1)}}, \text{ vec}(\mathbf{B}_2^{\delta^{(1)}}\mathbf{B}_2^{\delta^{(1)}\prime})\}$, where the superscript emphasizes the fact that these values are based on the initial value $\delta^{(1)}$.
3. The convergence criterion for the $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$ is rather stringent. We first calculate the maximum of the squared deviations of all elements from 2 consecutive iterations. We then require for parameter convergence that the sum of 20 such consecutive maxima be less than 0.00001.
4. We now describe how to calculate the implied δ from the converged values of $\mathbf{A}_0$, $\mathbf{A}_1$, b_0, $\mathbf{B}_1$,vec$(\mathbf{B}_2\mathbf{B}_2')$. Using the fact that $\rho = 1 - \exp(\text{E}[c_t - w_t])$ and that

$$\begin{aligned} \text{E}(c_t - w_t) &= b_0 + \mathbf{B}_1' E(\mathbf{z}_t) + \text{ vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\text{E}(\mathbf{z}_t\mathbf{z}_t')) \\ &= b_0 + \mathbf{B}_1'\boldsymbol{\mu}_z + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z'), \end{aligned}$$

   we have

$$b_0 = \log(1-\rho) - \mathbf{B}_1'\boldsymbol{\mu}_z - \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z').$$

   On the other hand, from the definition of Ξ_0,

$$\begin{aligned} \Xi_0 &= \rho[-\psi \log\delta + k - V_0 + (1-\psi)\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 \\ &\quad + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Phi_0\Phi_0') + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_v)], \end{aligned}$$

   which depends on δ. Equating these two equations and solve for the new δ, we have

$$\delta^{(2)} = \exp\left[\frac{-1}{\rho\psi}(\mathcal{F}_2 - \rho\mathcal{F}_1)\right],$$

where

$$\begin{aligned}\mathcal{F}_1 &\equiv k - V_0 + (1-\psi)\,\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 + \text{vec}\,(\mathbf{B}_2\mathbf{B}_2')'\,\text{vec}(\Phi_0\Phi_0') \\ &\quad + \text{vec}\,(\mathbf{B}_2\mathbf{B}_2')'\,\text{vec}(\Sigma_v), \\ \mathcal{F}_2 &\equiv \log(1-\rho) - \mathbf{B}_1'\boldsymbol{\mu}_z - \text{vec}\,(\mathbf{B}_2\mathbf{B}_2')'\,\text{vec}\,(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z'),\end{aligned}$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are evaluated at $\{\mathbf{A}_0^{\delta^{(1)}}, \mathbf{A}_1^{\delta^{(1)}}\}$, $\{b_0^{\delta^{(1)}}, \mathbf{B}_1^{\delta^{(1)}}, \text{vec}(\mathbf{B}_2^{\delta^{(1)}}\mathbf{B}_2^{\delta^{(1)}\prime})\}$ and the VAR estimates.

5. Naturally, one might use $\delta^{(2)}$ as input again and reiterate through Step (1)–(4) until δ converges. Graphically, we attempt to find the intersection point between the implied δ curve with the 45°-line. Unfortunately, a plot of the implied δ calculated as described above against a range of input δ indicates that the implied δ curve cuts the 45°-line from above. This immediately implies that a standard iterative procedure over δ will not work in general. Therefore, we resort to a grid-search method over a range of δ.

6. The grid-search method proceeds as follows: we start with a wide range of δ and form a coarse mesh over this range. In our implementation, we choose to evaluate Step (1)–(4) over 21 points over this range. From these calculations, 21 new implied δ are obtained. We locate the inputting $\delta^{(1)}$ whose implied $\delta^{(2)}$ gives the minimum positive deviation from $\delta^{(1)}$ among these 21 pairs. Call this δ^+. Similarly, we locate the inputting $\delta^{(1)}$ whose implied $\delta^{(2)}$ gives the minimum negative deviation from $\delta^{(1)}$. Call this δ^-. Because of the downward-sloping implied δ curve, existence of this pair (δ^+, δ^-) is guaranteed as long as the initial range of δ is wide enough. Note that $\delta^+ < \delta^-$.

7. We then form a 21-point mesh again over (δ^+, δ^-) and repeat Step (6) until both the minimum positive deviation and negative deviation are less than 0.00001, or $\delta^- - \delta^+ < 0.00001$. When the first of the convergence criteria for δ is met, the converged δ is computed as the average of the $\delta^{(2)}$ with the minimum positive deviation and the $\delta^{(2)}$ with the minimum negative deviation. When the second criterion is met, a simple average of (δ^+, δ^-) is used.

C Appendix C: Construction of Hypothetical Real Bonds

Recall that the first element of our VAR system is the ex post real bill return. Therefore, the ex ante log real bill return at time $t+1$ is the first element of $\mathrm{E}_t(\mathbf{z}_{t+1}) = \Phi_0 + \Phi_1 \mathbf{z}_t$. In other words, the log real yield at time t is given by

$$\widehat{y}_{1t} = \mathbf{H}_1 \cdot \mathrm{E}_t(\mathbf{z}_{t+1}) \equiv \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+1},$$

where $\mathbf{H}_1 \equiv (1, 0, ..., 0)$ and $\widehat{\mathbf{z}}_{t,t+1} \equiv \mathrm{E}_t(\mathbf{z}_{t+1})$.

The next step is to assume that the log expectations hypothesis holds for the real term structure; that is,

$$y_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{E}_t(y_{1,t+i}),$$

where $y_{n,t}$ is the log yield on a real bond with maturity n. Note that we have implicitly assume that inflation risk premium is zero. An estimate of $y_{n,t}$ can be easily constructed as follows:

$$\widehat{y}_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} \widehat{y}_{1,t+i} = \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+i+1}.$$

To compute $\widehat{\mathbf{z}}_{t,t+i+1}$, we can iterate the VAR(1) system forward to get

$$\widehat{\mathbf{z}}_{t,t+k} = \left(\sum_{j=0}^{k-1} \Phi_1^j \right) \Phi_0 + \Phi_1^k \mathbf{z}_t.$$

Using this result, log yield can be expressed as a function of current state variables:

$$\begin{aligned}
\widehat{y}_{n,t} &= \tfrac{1}{n} \mathbf{H}_1 \sum_{i=1}^{n} \widehat{\mathbf{z}}_{t,t+i} \\
&= \frac{1}{n} \mathbf{H}_1 \sum_{i=1}^{n} \left[\left(\sum_{j=0}^{i-1} \Phi_1^j \right) \Phi_0 + \Phi_1^i \mathbf{z}_t \right] \\
&\equiv \frac{1}{n} \mathbf{H}_1 (\mathbf{Q}_c + \mathbf{Q}_n \mathbf{z}_t)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Q}_n &\equiv \sum_{i=1}^{n} \Phi_1^i = \Phi_1 (\mathbf{I}_m - \Phi_1)^{-1} (\mathbf{I}_m - \Phi_1^n), \\
\mathbf{Q}_c &\equiv (\mathbf{I}_m - \Phi_1)^{-1} (\mathbf{I}_m - \mathbf{Q}_n) \Phi_0,
\end{aligned}$$

and $\mathbf{I}_m$ is the identity matrix, $m = \dim(\mathbf{z}_t)$.

Finally, the 1-period return on a hypothetical real n-period bond is calculated as

$$\begin{aligned} r_{n,t+1} &= n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n-1,t+1} \\ &\approx n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n,t+1} \\ &= \mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_t\right) - \frac{n-1}{n}\mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_{t+1}\right) \end{aligned}$$

The next step is to construct a real consol bond from these zero-coupon bonds. Campbell, Lo and MacKinlay (1997) show how to use a loglinearization framework to construct real consol bond returns. Specifically,their equations (10.1.16) and (10.1.17) show that the log yield on a real consol $y_{c,\infty,t}$ is given by

$$y_{c,\infty,t} = (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i r_{c,\infty,t+1+i},$$

where $r_{c,\infty,t+i}$ is the one-period log return on a consol bond at time $t+i$ and $\rho_c = 1-\exp\left(\mathrm{E}[-p_{c,t}]\right)$, where $p_{c,t}$ is the log "cum-dividend" price of the consol bond including its current coupon payout.

Taking conditional expectations at time t and imposing the expectations hypothesis,

$$\begin{aligned} y_{c,\infty,t} &= (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i\mathbf{H}_1\widehat{\mathbf{z}}_{t,t+i+1} \\ &= \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j\right)\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1}\right)\mathbf{z}_t. \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} \sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j &= \frac{1}{1-\rho_c}\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}, \\ \sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1} &= \left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1. \end{aligned}$$

Thus, the log yield can be expressed as function of the VAR parameters, current state variables and the loglinearization constant ρ_c:

$$y_{c,\infty,t} = \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1\mathbf{z}_t.$$

We can now write an expression for the return on the consol bond:

$$\begin{aligned} r_{c,\infty,t+1} &\approx D_{c,\infty}y_{c,\infty,t} - (D_{c,\infty}-1)\,y_{c,\infty,t+1} \\ &= \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_0 + \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1\left(\mathbf{z}_t - \rho_c\mathbf{z}_{t+1}\right). \end{aligned}$$

D Appendix D: Tables

TABLE A

Mean Asset Demands ($\psi = 1/\gamma$ Case)

A: Annual Sample (1890 - 1995)						
	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
$\gamma = 1, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	188.44	187.46	189.20	199.43	201.97
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	127.57	146.72	155.55	155.29	231.73
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-216.01	-234.18	-244.75	-254.72	-333.70
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	98.66	100.81	101.80	132.29	132.47
	Hedging Demand	0.00	2.54	3.02	28.13	27.15
Bond	Total Demand	70.78	89.58	95.75	89.71	53.41
	Hedging Demand	0.00	9.12	12.47	6.58	-64.35
Cash	Total Demand	-69.44	-90.39	-97.55	-122.00	-85.88
	Hedging Demand	0.00	-11.66	-15.49	-34.71	37.20
$\gamma = 5, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	44.79	52.42	52.30	74.29	81.40
	Hedging Demand	0.00	7.66	7.79	27.30	34.06
Bond	Total Demand	36.71	53.74	63.75	64.28	-19.38
	Hedging Demand	0.00	13.04	23.82	24.44	-68.76
Cash	Total Demand	18.50	-6.16	-16.05	-38.57	37.99
	Hedging Demand	0.00	-20.70	-31.61	-51.74	34.69
$\gamma = 20, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	17.86	29.22	27.93	35.99	40.86
	Hedging Demand	0.00	11.22	10.54	17.59	22.53
Bond	Total Demand	19.67	35.62	48.30	46.95	15.39
	Hedging Demand	0.00	14.80	30.06	28.75	0.21
Cash	Total Demand	62.47	35.15	23.77	17.06	43.75
	Hedging Demand	0.00	-26.02	-40.60	-46.34	-22.73

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE A (ctd.)

Mean Asset Demands ($\psi = 1/\gamma$ Case)

B: Quarterly Sample (1952Q2 - 1997Q4)						
	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
$\gamma = 1, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	272.65	285.19	289.75	301.76	302.41
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	42.80	15.90	8.20	2.88	6.54
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-215.45	-201.09	-197.95	-204.64	-208.95
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	136.07	138.76	139.35	313.78	241.39
	Hedging Demand	0.00	-3.63	-5.14	163.34	90.64
Bond	Total Demand	16.28	-36.69	-68.17	-415.77	-465.79
	Hedging Demand	0.00	-39.75	-67.64	-412.62	-464.45
Cash	Total Demand	-52.36	-2.07	28.82	202.00	324.39
	Hedging Demand	0.00	43.37	72.79	249.28	373.81
$\gamma = 5, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	54.13	53.07	50.50	579.78	566.99
	Hedging Demand	0.00	-3.63	-6.84	520.15	507.23
Bond	Total Demand	0.38	-30.52	-30.40	-677.46	-1092.46
	Hedging Demand	0.00	-25.87	-24.64	-670.69	-1086.39
Cash	Total Demand	45.49	77.45	79.90	197.68	625.47
	Hedging Demand	0.00	29.51	31.48	150.55	579.16
$\gamma = 20, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	13.16	11.49	9.34	362.38	-
	Hedging Demand	0.00	-2.37	-4.42	348.14	-
Bond	Total Demand	-7.58	-17.53	7.91	-371.48	-
	Hedging Demand	0.00	-9.02	16.28	-362.91	-
Cash	Total Demand	94.42	106.03	82.75	109.11	-
	Hedging Demand	0.00	11.40	-11.86	14.77	-

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset. " - " indicates that the recursion for δ failed to converge

TABLE B

VAR Estimation Results

Nominal Bills, Stocks and Real Consol Bond

A: Annual Sample (1890 - 1995)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.320	-0.057	0.012	0.593	-0.007	-0.592	0.235
	(2.367)	(-1.476)	(0.131)	(2.500)	(-0.226)	(-1.101)	(0.000)
xr_{t+1}	-0.008	0.091	-0.117	0.016	0.188	1.030	0.084
	(-0.021)	(0.745)	(-0.574)	(0.024)	(3.440)	(0.742)	(0.132)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.060	-0.011	-0.019	0.904	-0.006	0.078	0.784
	(-2.173)	(-1.555)	(-1.102)	(11.336)	(-1.204)	(0.787)	(0.000)
$(d-p)_{t+1}$	-0.467	-0.144	0.080	-0.875	0.767	-0.844	0.677
	(-1.514)	(-1.360)	(0.426)	(-1.396)	(12.730)	(-0.632)	(0.000)
spr_{t+1}	0.030	0.002	0.011	0.092	0.004	0.776	0.543
	(1.538)	(0.356)	(0.966)	(1.648)	(1.205)	(9.670)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	7.726	-0.178	-0.845	0.132	0.119	-0.161
xr	-	17.185	0.211	-0.180	-0.703	0.206
$xrcb$	-	-	12.489	-0.586	-0.077	0.627
y	-	-	-	1.220	0.234	-0.902
$(d-p)$	-	-	-	-	15.650	-0.191
spr	-	-	-	-	-	0.986

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE B (ctd.)

VAR Estimation Results

Nominal Bills, Stocks and Real Consol Bond

B: Quarterly Sample (1952Q2 - 1997Q4)							
Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.413	0.005	-0.015	0.308	-0.001	0.032	0.323
	(6.119)	(0.817)	(-0.340)	(3.918)	(-0.719)	(0.191)	(0.000)
xr_{t+1}	0.994	0.085	0.556	-2.188	0.084	1.920	0.111
	(1.004)	(1.080)	(1.214)	(-2.480)	(3.196)	(0.788)	(0.001)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.015	0.002	-0.013	0.961	0.000	0.487	0.794
	(-0.286)	(0.699)	(-0.409)	(16.967)	(0.448)	(4.394)	(0.000)
$(d-p)_{t+1}$	-1.305	-0.086	-0.479	1.358	0.932	-1.830	0.889
	(-1.298)	(-1.013)	(-0.980)	(1.489)	(34.093)	(-0.734)	(0.000)
spr_{t+1}	0.011	0.001	0.021	0.016	-0.001	0.497	0.276
	(0.294)	(0.488)	(0.954)	(0.395)	(-0.903)	(6.481)	(0.000)

Cross-Correlation of Residuals						
	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	0.601	0.229	-0.246	-0.507	-0.239	0.397
xr	-	7.594	-0.127	-0.217	-0.969	0.114
$xrcb$	-	-	1.327	-0.447	0.079	0.156
y	-	-	-	0.351	0.258	-0.899
$(d-p)$	-	-	-	-	7.949	-0.143
spr	-	-	-	-	-	0.251

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE C

VAR Estimation Results

Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

A: Annual Sample (1890 - 1995)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	xnb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates								
rtb_{t+1}	0.323	-0.054	0.020	0.153	0.704	-0.002	-0.899	0.242
	(2.419)	(-1.362)	(0.234)	(1.014)	(2.371)	(-0.060)	(-1.243)	(0.000)
xr_{t+1}	-0.012	0.085	-0.131	-0.245	-0.160	0.180	1.520	0.088
	(-0.033)	(0.693)	(-0.645)	(-0.860)	(-0.229)	(3.302)	(0.990)	(0.180)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
xnb_{t+1}	0.228	0.094	0.029	-0.086	-0.085	0.010	2.490	0.423
	(1.904)	(2.545)	(0.352)	(-0.622)	(-0.245)	(0.506)	(5.426)	(0.000)
y_{t+1}	-0.059	-0.010	-0.017	0.027	0.923	-0.005	0.025	0.786
	(-2.201)	(-1.461)	(-1.042)	(0.841)	(12.631)	(-1.077)	(0.203)	(0.000)
$(d-p)_{t+1}$	-0.457	-0.131	0.113	0.553	-0.477	0.786	-1.951	0.686
	(-1.530)	(-1.219)	(0.604)	(2.073)	(-0.736)	(13.711)	(-1.311)	(0.000)
spr_{t+1}	0.030	0.002	0.010	-0.015	0.082	0.004	0.805	0.546
	(1.543)	(0.292)	(0.895)	(-0.654)	(1.546)	(1.111)	(7.973)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	xnb	y	$(d-p)$	spr
rtb	7.690	-0.173	-0.849	-0.036	0.123	0.104	-0.156
xr	-	17.143	0.211	0.036	-0.174	-0.703	0.202
$xrcb$	-	-	12.489	0.240	-0.589	-0.078	0.629
xnb	-	-	-	4.810	-0.640	-0.126	0.269
y	-	-	-	-	1.213	0.220	-0.902
$(d-p)$	-	-	-	-	-	15.415	-0.182
spr	-	-	-	-	-	-	0.983

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE C (ctd.)
VAR Estimation Results
Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

B: Quarterly Sample (1952Q2 - 1997Q4)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	xnb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates								
rtb_{t+1}	0.537	0.010	0.085	-0.058	0.280	-0.001	0.023	0.330
	(4.940)	(1.625)	(1.097)	(-1.460)	(3.529)	(-0.606)	(0.138)	(0.000)
xr_{t+1}	0.091	0.050	-0.168	0.423	-1.987	0.083	1.984	0.113
	(0.056)	(0.562)	(-0.146)	(0.707)	(-2.128)	(3.169)	(0.819)	(0.002)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
xnb_{t+1}	0.228	-0.050	0.067	-0.080	0.329	0.002	1.004	0.043
	(0.400)	(-2.034)	(0.172)	(-0.344)	(0.805)	(0.299)	(0.942)	(0.272)
y_{t+1}	0.007	0.003	0.005	-0.010	0.956	0.001	0.486	0.794
	(0.096)	(0.914)	(0.109)	(-0.326)	(16.614)	(0.492)	(4.317)	(0.000)
$(d-p)_{t+1}$	-0.188	-0.043	0.416	-0.523	1.109	0.935	-1.910	0.889
	(-0.111)	(-0.457)	(0.348)	(-0.845)	(1.138)	(34.632)	(-0.771)	(0.000)
spr_{t+1}	-0.018	0.000	-0.002	0.014	0.022	-0.001	0.499	0.278
	(-0.352)	(0.047)	(-0.062)	(0.616)	(0.562)	(-0.978)	(6.400)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	xnb	y	$(d-p)$	spr
rtb	0.598	0.236	-0.242	0.451	-0.512	-0.247	0.405
xr	-	7.582	-0.131	0.279	-0.215	-0.969	0.112
$xrcb$	-	-	1.327	0.712	-0.446	0.083	0.153
xnb	-	-	-	2.731	-0.766	-0.325	0.413
y	-	-	-	-	0.350	0.257	-0.899
$(d-p)$	-	-	-	-	-	7.932	-0.140
spr	-	-	-	-	-	-	0.250

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE D

Mean Asset Demands with Hypothetical Real Bonds
(Quarterly Sample: 1952Q2 - 1997Q4)

A: Nominal Bills, Stocks, and Real Consol Bond

State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	277.57	313.71
Real Consol Bond	228.90	246.49
Cash	-406.47	-460.20
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	137.74	243.58
Real Consol Bond	119.24	-507.49
Cash	-156.98	363.91
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	53.84	329.71
Real Consol Bond	53.44	-977.11
Cash	-7.28	747.40
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	11.89	212.08
Real Consol Bond	20.54	-480.32
Cash	67.57	368.24
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	-1.95	4.41
Real Consol Bond	9.68	89.18
Cash	92.27	6.41

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE D (ctd.)

Mean Asset Demands with Hypothetical Real Bonds

(Quarterly Sample: 1952Q2 - 1997Q4)

B: Nominal Bills, Stocks, Real Consol Bond and Nominal Bond		
State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	286.88	347.48
Real Consol Bond	355.95	628.27
Nominal Bond	-81.99	-239.64
Cash	-460.85	-636.11
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	144.20	281.46
Real Consol Bond	207.46	637.71
Nominal Bond	-56.93	-797.37
Cash	-194.74	-21.81
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	58.60	597.31
Real Consol Bond	118.37	496.39
Nominal Bond	-41.90	-1273.07
Cash	-35.07	279.38
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	15.79	-
Real Consol Bond	73.82	-
Nominal Bond	-34.38	-
Cash	44.77	-
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	1.67	-
Real Consol Bond	59.12	-
Nominal Bond	-31.90	-
Cash	71.11	-

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset. " - " indicates that the recursion for δ failed to converge