

The $6D$ bias and the equity premium puzzle

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Abstract

If decision costs lead agents to update consumption every D periods, then econometricians will find an anomalously low correlation between equity returns and consumption growth (Lynch 1996). We analytically characterize the dynamic properties of an economy composed of consumers who have such delayed updating. In our setting, an econometrician using an Euler equation procedure would infer a coefficient of relative risk aversion biased up by a factor of $6D$. Hence with quarterly data, if agents adjust their consumption every $D = 4$ quarters, the imputed coefficient of relative risk aversion will be *24 times* greater than the true value. High levels of risk aversion implied by the equity premium and violations of the Hansen-Jagannathan bounds cease to be puzzles. The neoclassical model with delayed adjustment explains the consumption behavior of shareholders. Once limited participation is taken into account, the model matches most properties of aggregate consumption and equity returns, including new evidence that the covariance between $\ln(C_{t+h}/C_t)$ and R_{t+1} slowly rises with h .

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1 Introduction

Consumption growth covaries only weakly with equity returns, which implies that equities are not very risky. However, investors have historically received a very large premium for holding equities. For twenty years, economists have asked why an asset with little apparent risk has such a large required return.¹

Grossman and Laroque (1990) argued that adjustment costs might answer the equity premium puzzle. If it is costly to change consumption, households will not respond instantaneously to changes in asset prices. Instead, consumption will adjust with a lag, explaining why consumption growth covaries only weakly with current equity returns. In the Grossman and Laroque framework, equities are risky, but that riskiness does not show up in a high contemporaneous correlation between consumption growth and equity returns. The comovement is only observable in the long-run.

Lynch (1996) and Marshall and Parekh (1999) have simulated discrete-time delayed adjustments models and demonstrated that these models can potentially explain the equity premium puzzle.² In light of the complexity of these models, both sets of authors used *numerical* simulations.

We propose a continuous-time generalization of Lynch's (1996) model. Our extension provides two new sets of results. First, our analysis is analytically tractable; we derive a complete analytic characterization of the model's dynamic properties. Second, our continuous-time framework generates effects that are up to six times larger than those in discrete time models.

We analyze an economy composed of consumers who update their consumption every D (as in *Delay*) periods. Such delays may be motivated by decision costs, attention allocation costs, and/or mental accounts.³ The core of the paper describes the consequences of such delays. In addition, we derive a sensible value of D based on a decision cost framework.

The “ $6D$ bias” is our key result. Using data from our economy, an econometrician estimating the coefficient of relative risk aversion (CRRA) from the consumption Euler equation would generate a multiplicative CRRA bias of $6D$. For example, if agents adjust their consumption every $D = 4$ quarters, and the econometrician uses quarterly aggregates in his analysis, the imputed coefficient of relative risk aversion will be 24 times greater than the true value. Once we take account of this $6D$ bias, the Euler equation tests are unable to reject the standard consumption model. High equity returns and associated violations of the Hansen-Jagannathan (1991) bounds cease to be puzzles.

The basic intuition for this result is quite simple. If households adjust their consumption every $D \geq 1$ periods, then on average only $\frac{1}{D}$ households will adjust each period. Consider only the households that adjust during the current period and assume that these households adjust consumption at dates spread uniformly over the period. Normalize the timing so the current period is the time interval $[0, 1]$. When a household adjusts at time $i \in [0, 1]$, it can only respond to equity returns that have already been realized by time i . Hence, the household can only respond to fraction i of within-period equity returns. Moreover, the household that adjusts at time i can only change consumption for the remainder of the period. Hence, only fraction $(1 - i)$ of this

¹For the intellectual history of this puzzle, see Rubinstein (1976), Lucas (1978), Shiller (1982), Hansen and Singleton (1983), Mehra and Prescott (1985), and Hansen and Jagannathan (1991). For useful reviews see Kocherlakota (1996) and Campbell (2000).

²See also related work by Caballero (1995), He and Modest (1999), Heaton and Lucas (1996), Luttmer (1995), and Lynch and Balduzzi (1999).

³See Gabaix and Laibson (2000b) for a discussion of decision costs and attention allocation costs. See Thaler (1992) for a discussion of mental accounts.

period's consumption is affected by the change at time i . On average the households that adjust during the current period display a covariance between equity returns and consumption growth that is biased down by factor

$$\int_0^1 i(1-i)di = \frac{1}{6}.$$

The integral is taken from 0 to 1 to average over the uniformly distributed adjustment times.

Since only fraction $\frac{1}{D}$ of households adjust in the first place, the aggregate covariance between equity returns and consumption growth is approximately $\frac{1}{6} \cdot \frac{1}{D}$ as large as it would be if all households adjusted instantaneously. The Euler equation for the instantaneous adjustment model implies that the coefficient of relative risk aversion is inversely related to the covariance between equity returns and consumption growth. If an econometrician used this Euler equation to impute the coefficient of relative risk aversion, and he used data from our delayed adjustment economy, he would impute a coefficient of relative risk aversion that was $6D$ times too large.

In section 2 we describe our formal model, motivate our assumptions, and present our key analytic finding. In section 2.2 we provide an heuristic proof of our results for the case $D \geq 1$. In section 3 we present additional results that characterize the dynamic properties of our model economy. In section 4 we close our framework by describing how D is chosen. In section 5 we consider the consequences of our model for macroeconomics and finance. In section 6 we discuss empirical evidence that supports the Lynch (1996) model and our generalization. The model matches most of the empirical moments of aggregate consumption and equity returns, including a new test which confirms the $6D$ prediction that the covariance between $\ln(C_{t+h}/C_t)$ and R_{t+1} should slowly rise with h . In section 7 we conclude.

2 Model and key result

Our framework is a synthesis of ideas from the continuous-time model of Merton (1969) and the discrete-time model of Lynch (1996). In essence we adopt Merton's continuous-time modeling approach and Lynch's emphasis on delayed adjustment.⁴

We assume that the economy has two linear production technologies: a risk free technology and a risky technology (i.e., equities). The risk free technology has instantaneous return r . The returns from the risky technology follow a geometric diffusion process with expected return $r + \pi$ and standard deviation σ .

We assume that consumers hold two accounts: a checking account and a balanced mutual fund. A consumer's checking account is used for day to day consumption, and this account holds only the risk free asset. The mutual fund is used to replenish the checking account from time to time. The mutual fund is professionally managed and is continuously rebalanced so that θ share of the mutual fund assets are always invested in the risky asset.⁵ The consumer is able to pick θ .⁶ In practice, the consumer picks a mutual fund that maintains the consumer's preferred value of θ . We call θ the equity share (in the mutual fund).

Every D periods, the consumer looks at her mutual fund and decides how much wealth to withdraw from the mutual fund to deposit in her checking account. Between withdrawal periods — i.e., from withdrawal date t to the next withdrawal date $t + D$ — the consumer spends from

⁴See Calvo (1983), Fischer (1977), and Taylor (1979) for earlier examples of delayed adjustment in macroeconomics.

⁵This assumption can be relaxed without significantly changing the quantitative results. In particular, the consumer could buy assets in separate accounts without any instantaneous rebalancing.

⁶The fact that θ does not vary once it is chosen is optimal from the perspective of the consumer in this model.

her checking account and does *not* monitor her mutual fund. For now we take D to be exogenous. Following a conceptual approach taken in Duffie and Sun (1990), we later calibrate D with a decision cost model (see section 4). Alternatively, D can be motivated with a mental accounting model of the type proposed by Thaler (1992).

Finally, we assume that consumers have isoelastic preferences and exponential discount functions:

$$U_{it} = E_t \int_{s=t}^{\infty} e^{-\rho(s-t)} \left(\frac{c_{is}^{1-\gamma} - 1}{1-\gamma} \right) ds.$$

Here i indexes the individual consumer and t indexes time.

We adopt the following notation. Let w_{it} represent the wealth in the mutual fund at date t . Between withdrawal dates, w_{it} evolves according to

$$dw_{it} = w_{it} ((r + \theta\pi)dt + \theta\sigma dz_t),$$

where z_t is a Wiener process. We can now characterize the optimal choices of our consumer. We describe each date at which the consumer monitors — and in equilibrium withdraws from — her mutual fund as a “reset date.” Formal proofs of all results are provided in the appendix.

Proposition 1 *On the equilibrium path, the following properties hold.*

1. *Between reset dates, consumption grows at a fixed rate $\frac{1}{\gamma}(r - \rho)$.*
2. *The balance in the checking account just after a reset date equals the net present value of consumption between reset dates, where the NPV is taken with the risk free rate.*
3. *At reset date τ , consumption is $c_{i\tau+} = \alpha w_{i\tau-}$, where α is a function of the technology parameters, preference parameters, and D .*
4. *The equity share in the mutual fund is*

$$\theta = \frac{\pi}{\gamma\sigma^2}. \tag{1}$$

Note that $c_{i\tau+}$ represents consumption immediately after reset and $w_{i\tau-}$ represents wealth in the mutual fund immediately before reset.

Claim 1 follows from the property that between reset dates the rate of return to marginal savings is fixed and equal to r . So between reset dates the consumption path grows at the rate derived in Ramsey’s (1928) original deterministic growth model:

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(r - \rho).$$

Claim 2 reflects the advantages of holding wealth in the balanced mutual fund. Instantaneous rebalancing of this fund makes it optimal to store ‘extra’ wealth — i.e., wealth that is not needed for consumption between now and the next reset date — in the mutual fund. So the checking account is exhausted between reset dates. Claim 3 follows from the homotheticity of preferences. Claim 4 implies that the equity share is equal to the same equity share derived by Merton (1969) in his instantaneous adjustment model. This exact equivalence is special to our institutional assumptions, but approximate equivalence is a general property of models of delayed adjustment

(see Rogers 2001 for numerical examples in a related model). Note that the equity share is increasing in the equity premium (π) and decreasing in the coefficient of relative risk aversion (γ) and the variance of equity returns (σ^2).

Combining claims 1-3 results implies that the optimal consumption path between date τ and date $\tau + D$ is $c_{it} = \alpha e^{\frac{1}{\gamma}(r-\rho)(t-\tau)} w_{i\tau-}$ and the optimal balance in the checking account just after reset date τ is

$$\int_{\tau}^{\tau+D} c_{is} e^{-r(s-\tau)} ds = \int_{\tau}^{\tau+D} \alpha e^{\frac{1}{\gamma}(r-\rho)(s-\tau)-r(s-\tau)} w_{i\tau-} ds.$$

Claim 3 implies that at reset dates optimal consumption is linear in wealth. The actual value of the propensity to consume, α , does not matter for the results that follow. Any linear rule — e.g., linear rules of thumb — will suffice. In practice, the optimal value of α in our model will be close to the optimal marginal propensity to consume derived by Merton,

$$\alpha = \frac{\rho}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\pi^2}{2\gamma\sigma^2}\right).$$

Merton's value is exactly optimal in our framework when $D = 0$.

2.1 Our key result: the $6D$ bias

In our economy, each agent resets consumption at intervals of D units of time. Agents are indexed by their reset time $i \in [0, D)$. Agent i resets consumption at dates $\{i, i + D, i + 2D, \dots\}$.

We assume that the consumption reset times are distributed uniformly.⁷ More formally, there exists a continuum of consumers whose reset indexes i are distributed uniformly over $[0, D)$. So the proportion of agents resetting their consumption in any time interval of length $\Delta t \leq D$ is $\Delta t/D$.

To fix ideas, suppose that the unit of time is a quarter of the calendar year, and $D = 4$. In other words, the span of time from t to $t + 1$ is one quarter of a year. Since $D = 4$, each consumer will adjust her consumption once every four quarters. We will often choose the slightly non-intuitive normalization that a quarter of the calendar year is *one* period, since quarterly data is the natural unit of temporal aggregation with contemporary macroeconomic data.

Call C_t the aggregate consumption between $t - 1$ and t .

$$C_t = \int_{i=0}^D \left[\int_{s=t-1}^t c_{is} ds \right] \frac{1}{D} di.$$

Note that $\left[\int_{s=t-1}^t c_{is} ds \right]$ is per-period consumption for consumer i .

Suppose that an econometrician estimates γ and β using a consumption Euler equation (i.e., the consumption CAPM). What will the econometrician infer about preferences?

Theorem 2 *Consider an economy with true coefficient of relative risk aversion γ . Suppose an econometrician estimates the Euler equation*

$$E_{t-1} \left[\widehat{\beta} \left(\frac{C_t}{C_{t-1}} \right)^{-\widehat{\gamma}} R_t^a \right] = 1$$

⁷The results change only a little when we relax the assumption of a uniform distribution. Most importantly, if reset dates were clumped at the end of periods — a natural assumption — then the implied bias would be infinite.

for two assets, the risk free bond and the stock market. In other words, the econometrician fits $\hat{\beta}$ and $\hat{\gamma}$ to match the Euler equation above for both assets. Then the econometrician will find

$$\hat{\gamma} = \begin{cases} 6D\gamma & \text{for } D \geq 1 \\ \frac{6}{3(1-D)+D^2}\gamma & \text{for } 0 \leq D \leq 1 \end{cases} \quad (2)$$

plus higher order terms characterized in subsequent sections.

Figure 1 plots $\hat{\gamma}/\gamma$ as a function of D . The formulae for the cases $0 \leq D \leq 1$ and $D \geq 1$ are taken from Theorem 2.

Insert Figure 1 about here

The two formulae paste at the crossover point, $D = 1$. Convexity of the formula below $D = 1$, implies that $\hat{\gamma}/\gamma \geq 6D$ for all values of D . The case of instantaneous adjustment (i.e., $D = 0$) is of immediate interest since it has been solved already by Grossman, Melino, and Shiller (1987). With $D = 0$ the only bias arises from time aggregation of the econometrician's data, *not* delayed adjustment by consumers. Grossman, Melino, and Shiller show that time aggregation produces a bias of $\hat{\gamma}/\gamma = 2$, matching our formula for $D = 0$.

The most important result is the equation for $D \geq 1$, $\hat{\gamma} = 6D\gamma$, which we call the $6D$ bias. For example, if each period (t to $t + 1$) is a quarter of a calendar year, and consumption is reset every $D = 4$ quarters, then we get $\hat{\gamma} = 24\gamma$. Hence γ is overestimated by a factor of 24. If consumption is revised every 5 years then we have $D = 20$, and $\hat{\gamma} = 120\gamma$.

Reset periods of four quarters or more are not unreasonable in practice. For an extreme case, consider the 30-year-old employee who accumulates balances in a retirement savings account (e.g., a 401(k)) and fails to recognize any fungibility between these assets and his pre-retirement consumption. In this case, stock market returns will effect consumption at a considerable lag ($D > 120$ quarters for this example).

However, such extreme cases are not necessary for the points that we wish to make. Even with a delay of only four quarters, the implications for the equity premium puzzle literature are dramatic. With a multiplicative bias of 24, econometrically imputed coefficients of relative risk aversion of 50 suddenly appear quite reasonable, since they imply *actual* coefficients of relative risk aversion of roughly 2.

In addition, our results do not rely on the strong assumption that all reset rules are time- and not state-contingent. In Appendix B we incorporate the realistic assumption that all households adjust immediately when the equity market experiences a large (Poisson) shock. In practice, such occasional state-contingent adjustments only slightly modify our results.

We can also compare the $6D$ bias analytically to the biases that Lynch (1996) numerically simulates in his original discrete time model. In Lynch's framework, agents consume every month and adjust their portfolio every T months. Lynch's econometric observation period is the union of F one-month intervals, so $D = T/F$. In Appendix C we show that when $D \geq 1$ Lynch's framework generates a bias which is bounded below by D and bounded above by $6D$. Specifically, an econometrician who naively estimated the Euler equation with data from Lynch's economy would find a bias of

$$\frac{\hat{\gamma}}{\gamma} = D \frac{6F^2}{(F+1)(F+2)} + \text{higher order terms.} \quad (3)$$

Holding D constant, the continuous time limit corresponds to $F \rightarrow \infty$, and for this case $\hat{\gamma}/\gamma = 6D$. The discrete time case where agents consume at every econometric period corresponds to $F = 1$, implying $\hat{\gamma}/\gamma = D$, which can be derived directly.

Finally, the $6D$ bias complements participation bias (e.g., Vissing 2000, Brav et al 2000). If only a fraction s of agents hold a significant share of their wealth in equities (say $s = \frac{1}{3}$), then the covariance between aggregate consumption and returns is lower by a factor s . As Theorem 8 demonstrates, this bias combines *multiplicatively* with our bias: if there is limited participation, the econometrician will find the values of $\hat{\gamma}$ in Theorem 2, divided by s . In particular, for $D \geq 1$, he will find:

$$\hat{\gamma} = \frac{6D}{s}\gamma. \quad (4)$$

This formula puts together three important biases generated by Euler equation (and Hansen-Jagannathan) tests: $\hat{\gamma}$ will be overestimated because of time aggregation and delayed adjustment (the $6D$ factor), and because of limited participation (the $1/s$ factor).

2.2 Argument for $D \geq 1$

In this section we present a heuristic proof of Theorem 2. A rigorous proof is provided in the appendix.

Normalize a generic period to be one unit of time. The econometrician observes the return of the stock market from 0 to 1:

$$\ln R_1 = r + \pi - \frac{\sigma^2}{2} + \sigma \int_0^1 dz_s. \quad (5)$$

where r is the risk-free interest rate, π is the equity premium, σ^2 is the variance of stock returns, and z is a Wiener process. The econometrician also observes aggregate consumption over the period:

$$C_1 = \int_{i=0}^D \left[\int_{s=0}^1 c_{is} ds \right] \frac{1}{D} di.$$

As is well-known, when returns and consumption are assumed to be jointly log normal, the standard Euler equation implies that⁸

$$\hat{\gamma} = \frac{\pi}{\text{cov} \left(\ln \frac{C_1}{C_0}, \ln R_1 \right)}. \quad (6)$$

We will show that when $D \geq 1$ the measured covariance between consumption growth and stock market returns $\text{cov}(\ln C_1/C_0, \ln R_1)$ will be $6D$ lower than the instantaneous covariance,

⁸ $E_{t-1} \left[\hat{\beta} \left(\frac{C_t}{C_{t-1}} \right)^{-\hat{\gamma}} R_t^a \right] = 1$ with $R_t^a = e^{\mu_a - \sigma_a^2/2 + \sigma_a \varepsilon_a}$. The a subscripts and superscripts denote asset-specific returns and standard deviations. As Hansen and Singleton (1983) showed,

$$\ln \hat{\beta} + \mu_a - \hat{\gamma} \left(\mu_c - \frac{\sigma_c^2}{2} + \hat{\gamma} \frac{\sigma_c^2}{2} \right) - \hat{\gamma} \sigma_{ac} = 0.$$

If we evaluate this expression for the risk-free asset and equities, we find that,

$$\pi = \hat{\gamma} \text{cov} \left(\ln \frac{C_t}{C_{t-1}}, \ln R_t \right).$$

Note that $\pi + r = \mu_a$.

$cov(d \ln C_t, d \ln R_t)/dt$, that arises in the frictionless CCAPM. As is well-known, in the frictionless CCAPM

$$\gamma = \frac{\pi}{cov(d \ln C_t, d \ln R_t)/dt}.$$

Assume that each agent consumes one unit in period $[-1, 0]$.⁹ So aggregate consumption in period $[-1, 0]$ is also one: $C_0 = 1$. Since $\ln C_1/C_0 \simeq C_1/C_0 - 1$, we can write

$$cov\left(\ln \frac{C_1}{C_0}, \ln R_1\right) \simeq cov(C_1, \ln R_1) \quad (7)$$

$$= \int_0^D cov(C_{i1}, \ln R_1) \frac{1}{D} di \quad (8)$$

with $C_{i1} = \int_0^1 c_{is} ds$ the time-aggregated consumption of agent i during period $[0, 1]$.

First, take the case $D = 1$. Agent $i \in [0, 1)$ changes her consumption at time i . For $s \in [0, i)$, she has consumption $c_{is} = \alpha w_{i\tau^-} e^{\frac{1}{\gamma}(r-\rho)(s-\tau)}$, where $\tau = i - D$.

Throughout this paper we use approximations to get analytic results. Let $\varepsilon \equiv \max(r, \rho, \theta\pi, \sigma^2, \sigma^2\theta^2, \alpha)$. When we use annual periods ε will be approximately .05.¹⁰ For quarterly periods, ε will be approximately .01. We can express our approximation errors in higher order terms of ε .

Since consumption in period $[-1, 0]$ is normalized to one, at time $\tau = i - D$, α times wealth will be equal to 1, plus small corrective terms; more formally:

$$\begin{aligned} \alpha w_{i\tau^-} &= 1 + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon) \\ \alpha w_{i\tau^+} &= 1 + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon). \end{aligned}$$

Here $O(\varepsilon)$ represents stochastic or deterministic terms of order ε , and $O_{<0}(\sqrt{\varepsilon})$ represents stochastic terms that depend only on equity innovations that happen *before* time 0. Hence these stochastic $O_{<0}(\sqrt{\varepsilon})$ terms are all orthogonal to equity innovations during period $[0, 1]$.

Drawing together our last two results, for $s \in [0, i)$,

$$\begin{aligned} c_{is} &= e^{\frac{1}{\gamma}(r-\rho)(s-\tau)} \alpha w_{i\tau^-} \\ &= (1 + O(\varepsilon)) (1 + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon)) \\ &= 1 + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon). \end{aligned}$$

Without loss of generality, set $z(0) = 0$. So the consumer i 's mutual fund wealth at date $t = i^-$ is

$$\begin{aligned} \alpha w_{i,t=i^-} &= e^{(r+\theta\pi-\theta^2\sigma^2/2)D+\theta\sigma(z(i)-z(i-D))} \alpha w_{i\tau^+} \\ &= [1 + \theta\sigma z(i) + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon)] [1 + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon)] \\ &= 1 + \theta\sigma z(i) + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon) \end{aligned}$$

The consumer adjusts consumption at $t = i$, and so for $s \in [i, 1]$ she consumes

$$\begin{aligned} c_{is} &= e^{\frac{1}{\gamma}(r-\rho)(s-i)} \alpha w_{i,t=i^-} \\ &= [1 + O(\varepsilon)] [1 + \theta\sigma z(i) + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon)] \\ &= 1 + \theta\sigma z(i) + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon) \end{aligned}$$

⁹This assumption need not hold exactly. Instead, consumption must be unity up to $O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon)$ terms. See notation below.

¹⁰For a typical annual calibration $r = .01$, $\rho = .05$, $\theta\pi = (.78)(.06)$, $\sigma^2 = (.16)^2$, $\sigma^2\theta^2 = \left(\frac{\pi}{\gamma\sigma}\right)^2 = \left(\frac{0.06}{3 \cdot 0.16}\right)^2$, and $\alpha = .04$.

The covariance of consumption and returns for agent i is

$$\begin{aligned}
\text{cov}(C_{i1}, \ln R_1) &= \int_0^1 \text{cov}(c_{is}, \ln R_1) ds \\
&= \int_0^i 0 ds + \int_i^1 \text{cov} \left(1 + \theta \sigma z(i) + O_{<0}(\sqrt{\varepsilon}) + O(\varepsilon), \sigma z(1) + r + \pi - \frac{\sigma^2}{2} \right) ds \\
&= \int_i^1 \left[\theta \sigma^2 \text{cov}(z(i), \sigma z(1)) + O(\varepsilon^{3/2}) \right] ds \\
&= \theta \sigma^2 i(1-i) + O(\varepsilon^{3/2}) \\
&\simeq \theta \sigma^2 i(1-i)
\end{aligned}$$

Here and below \simeq means “plus higher order terms in ε .”

The covariance contains the multiplicative factor i because the consumption change reflects only return information which is revealed between date 0 and date i . The covariance contains the multiplicative factor $(1-i)$ because the change in consumption occurs at time i , and therefore affects consumption for only the subinterval $[i, 1]$.

We often analyze “normalized” variances and covariances. Specifically, we divide the moments predicted by the $6D$ model by the moments predicted by the benchmark model with instantaneous adjustment and instantaneous measurement. Such normalizations highlight the “biases” introduced by the $6D$ economy.

For the case $D = 1$, the normalized covariance of aggregate consumption growth and equity returns is

$$\begin{aligned}
\frac{1}{\theta \sigma^2} \text{cov}(C_1, \ln R_1) &= \int_{i=0}^1 \frac{1}{\theta \sigma^2} \text{cov}(C_{i1}, R_1) \frac{di}{D} \\
&\simeq \int_0^1 i(1-i) di = \frac{1}{6}
\end{aligned}$$

which is the (inverse of the) $6D$ factor for $D = 1$.

Consider now the case $D \geq 1$. Consumer $i \in [0, D)$ resets her consumption at $t = i$. During period one (i.e., $t \in [0, 1]$) only agents with $i \in [0, 1]$ will reset their consumption. Consumers with $i \in (1, D]$ will not change their consumption, so they will have a zero covariance, $\text{cov}(C_{i1}, R_1) = 0$. Hence,

$$\frac{1}{\theta \sigma^2} \text{cov}(C_{i1}, R_1) \simeq \begin{cases} i(1-i) & \text{if } i \in [0, 1] \\ 0 & \text{if } i \in [1, D] \end{cases}$$

For $D \geq 1$ the covariance of aggregate consumption is just $1/D$ times what it would be if we had $D = 1$.

$$\begin{aligned}
\frac{1}{\theta \sigma^2} \text{cov}(\ln C_1 / C_0, R_1) &\simeq \int_0^D \frac{1}{\theta \sigma^2} \text{cov}(C_{i1}, R_1) \frac{di}{D} \\
&= \frac{1}{D} \int_0^1 \frac{1}{\theta \sigma^2} \text{cov}(C_{i1}, R_1) di \\
&\simeq \frac{1}{D} \int_0^1 i(1-i) di \\
&= \frac{1}{6D}
\end{aligned}$$

The $6D$ lower covariance of consumption with returns translates into a $6D$ higher measured CRRA $\hat{\gamma}$. Since $\theta = \frac{\pi}{\gamma\sigma^2}$ (equation 1) we get

$$\text{cov}(\ln C_1/C_0, \ln R_1) = \frac{\pi}{6D\gamma}.$$

The Euler equation (6) then implies

$$\hat{\gamma} = 6D\gamma$$

as anticipated.

Several properties of our result should be emphasized. First, holding D fixed, the bias in $\hat{\gamma}$ does not depend on either preferences or technology: $r, \pi, \sigma, \rho, \gamma$. This independence property will apply to all of the additional results that we report in subsequent sections. When D is endogenously derived, D itself will depend on the preference and technology parameters.

For simplicity, the derivation above assumes that agents with different adjustment indexes i have the same “baseline” wealth at the start of each period. In the long-run this wealth equivalence will not apply exactly. However, if the wealth disparity is moderate, the reasoning above will still hold approximately.¹¹ Numerical analysis with 50-year adult lives implies that the actual bias is very close to $6D$, the value it would have if all of the wealth levels were identical period-by-period.

3 General characterization of the economy

In this section we provide a general characterization of the dynamic properties of the economy described above. We analyze four properties of our economy: excess smoothness of consumption growth, positive autocorrelation of consumption growth, low covariance of consumption growth and asset returns, and non-zero covariance of consumption growth and lagged equity returns.

Our analysis focuses on first-order effects with respect to the parameters $r, \rho, \theta\pi, \sigma^2, \sigma^2\theta^2$, and α . Call $\varepsilon \equiv \max(r, \rho, \theta\pi, \sigma^2, \sigma^2\theta^2, \alpha)$. We assume ε to be small. Empirically, $\varepsilon \simeq .05$ with a period length of a year, and $\varepsilon \simeq .01$ with a period length of a calendar quarter. All the results that follow (except one¹²) are proved with $O(\varepsilon^{3/2})$ residuals. In fact, at the cost of more tedious calculations, one can show that the residuals are actually $O(\varepsilon^2)$.¹³

The following theorem is the basis of this section. All proofs appear in the appendix.

Theorem 3 *The autocovariance of consumption growth at horizon $h \geq 0$ can be expressed*

$$\text{cov}\left(\ln \frac{C_{h+t}}{C_{h+t-1}}, \ln \frac{C_t}{C_{t-1}}\right) = \theta^2 \sigma^2 \Gamma(D, h) + O\left(\varepsilon^{3/2}\right) \quad (9)$$

where

$$\Gamma(D, h) \equiv \frac{1}{D^2} [d(D+h) + d(D-h) - d(h) - d(-h)], \quad (10)$$

¹¹More precisely, it is only important that the *average* wealth of households that switch on date t not differ significantly from the *average* wealth of households that switch on any date $s \in [t-D, t+D]$. To guarantee this cross-date *average* similarity we could assume that each reset interval ends stochastically. This randomness generates “mixing” between populations of households that begin life with different reset dates.

¹²Equation 12 is proved to order $O(\sqrt{\varepsilon})$, but with more tedious calculations can be shown to be $O(\varepsilon)$.

¹³One follows exactly the lines of the proofs presented here, but includes higher order terms. Calculations are available from the authors upon request.

$$d(D) \equiv \sum_{i=0}^4 \binom{4}{i} \frac{(-1)^i}{2 \cdot 5!} |D + i - 2|^5, \quad (11)$$

and $\binom{4}{i} = \frac{4!}{i!(4-i)!}$ is the binomial coefficient.

The expressions above are valid for non-integer values of D and h . Functions $d(D)$ and $\Gamma(D, h)$ have the following properties, many of which will be exploited in the analysis that follows¹⁴.

- $d \in C^4$
- $d(D) = |D|/2$ for $|D| \geq 2$
- $d(0) = 7/30$
- $\Gamma(D, h) \sim 1/D$ for large D
- $\Gamma(D, h) \geq 0$
- $\Gamma(D, h) > 0$ iff $D + 2 > h$
- $\Gamma(D, h)$ is nonincreasing in h
- $\Gamma(D, 0)$ is decreasing in D , but $\Gamma(D, h)$ is hump-shaped for $h > 0$.
- $\Gamma(0, h) = 0$ for $h \geq 2$
- $\Gamma(0, 0) = 2/3$
- $\Gamma(0, 1) = 1/6$

Figure 2 plots $d(D)$ along with a second function which we will use below.

Insert Figure 2 about here

3.1 $\Gamma(D, 0)$

We begin by studying the implications of the autocovariance function, $\Gamma(D, h)$, for the volatility of consumption growth (i.e, by setting $h = 0$). Like Caballero (1995), we also show that delayed adjustment induces excess smoothness. Corollary 4 describes our quantitative result.

Corollary 4 *In the frictionless economy ($D = 0$), $\text{var}(dC_t/C_t)/dt = \sigma^2\theta^2$. In our economy, with delayed adjustment and time aggregation bias,*

$$\frac{\text{var}(\ln C_t/C_{t-1})}{\sigma^2\theta^2} = \Gamma(D, 0) \leq 2/3.$$

The volatility of consumption, $\sigma^2\theta^2\Gamma(D, 0)$, decreases as D increases.

The normalized variance of consumption, $\Gamma(D, 0)$, is plotted against D in Figure 3.

¹⁴ Γ is continuous, so $\Gamma(0, h)$ is intended as $\lim_{D \rightarrow 0} \Gamma(D, h)$.

Insert Figure 3 about here.

For $D = 0$, the normalized variance is $2/3$, well below the benchmark value of 1. The $D = 0$ case reflects the bias generated by time aggregation effects. As D rises above zero, delayed adjustment effects also appear. For $D = 0, 1, 2, 4, 20$ the normalized variance takes values .67, .55, .38, .22, and .04. For large D , the bias is approximately $1/D$.

Intuitively, as D increases, none of the short-run volatility of the economy is reflected in consumption growth, since only $1/D$ proportion of the agents adjust consumption in any single period. Moreover, the size of the adjustments only grows with \sqrt{D} . So the total magnitude of adjustment is falling with $1/\sqrt{D}$ and the variance falls with $1/D$.

3.2 $\Gamma(D, h)$ with $h > 0$

We now consider the properties of the (normalized) autocovariance function $\Gamma(D, h)$ for $h = 1, 2, 4, 8$. Figure 4 plots these respective curves, ordered from $h = 1$ on top to $h = 8$ at the bottom. Note that in the benchmark case — instantaneous adjustment and no time-aggregation bias — the autocovariation of consumption growth is zero. With only time aggregation effects, the one-period autocovariance is $\Gamma(0, 1) = 1/6$, and all h -period autocovariances with $h > 1$ are zero.

Insert Figure 4 about here.

3.3 Revisiting the equity premium puzzle

We can also state a formal and more general analogue of Theorem 2.

Proposition 5 *Suppose that consumers reset their consumption every h_a periods. Then the covariance between consumption growth and stock market returns at horizon h will be*

$$\text{cov}(\ln C_{[t,t+h]}/C_{[t-h,t]}, \ln R_{[t,t+h]}) = \frac{\theta\sigma^2 h}{b(D)} + O(\varepsilon^{3/2})$$

where, $D = h_a/h$, and

$$b(D) = \begin{cases} 6D & \text{for } D \geq 1 \\ \frac{6}{3(1-D)+D^2} & \text{for } 0 \leq D \leq 1 \end{cases} .$$

The associated correlation is

$$\text{corr}(\ln C_{[t,t+h]}/C_{[t-h,t]}, \ln R_{[t,t+h]}) = \frac{1}{b(D)\Gamma(D, 0)^{1/2}} + O(\varepsilon^{1/2}). \quad (12)$$

In the benchmark model with continuous sampling/adjustment, the covariance is just

$$\text{cov}(d \ln C_t, d \ln R_t)/dt = \theta\sigma^2.$$

Moreover, in the benchmark model with continuous sampling/adjustment, the covariance at horizon h is just

$$\text{cov}(\ln C_{[t,t+h]}/C_{[t-h,t]}, \ln R_{[t,t+h]}) = \theta\sigma^2 h.$$

So the effect introduced by the $6D$ model is captured by the factor $\frac{1}{b(D)}$, which appears in Proposition 5.

We compare this benchmark to the effects generated by our discrete observation, delayed adjustment model. As the horizon h tends to $+\infty$, the normalized covariance between consumption growth and asset returns tends to

$$\lim_{h \rightarrow \infty} \frac{\theta \sigma^2 h}{b(h_a/h)} \frac{1}{\theta \sigma^2 h} = \frac{1}{b(0)} = \frac{1}{2},$$

which is true for any fixed value of h_a . This effect is due exclusively to time aggregation. Delayed adjustment ceases to matter as the horizon length goes to infinity.

Proposition 5 covers the special case discussed in section two: horizon $h = 1$, and reset period $h_a = D \geq 1$. For this case, the normalized covariance is approximately equal to

$$\frac{\theta \sigma^2}{b(D)} \frac{1}{\theta \sigma^2} = \frac{1}{6D}.$$

Figure 5 plots the multiplicative covariance bias factor $1/b(h_a/h)$ as a function of h , for $h_a = 1$. In the benchmark case (i.e., continuous sampling and instantaneous adjustment) there is no bias; the bias factor is unity. In the case with only time aggregation effects (i.e., discrete sampling and $h_a = 0$) the bias factor is $1/b(0/h) = 1/2$.

Insert Figure 5 about here.

Hence, low levels of comovement show up most sharply when horizons are low. For $D \geq 1$ (i.e., $h_a/h \geq 1$), the covariance between consumption growth and stock returns is $6D$ times lower than one would expect in the model with continuous adjustment and continuous sampling.

We now characterize covariance between current consumption growth and lagged equity returns.

Theorem 6 *Suppose that consumers reset their consumption every $h_a = Dh$ periods. Then the covariance between $\ln C_{[t,t+1]}/C_{[t-1,t]}$ and lagged equity returns $\ln R_{[t+s_1,t+s_2]}$ ($s_1 < s_2 \leq 1$) will be*

$$\text{cov}(\ln C_{[t,t+1]}/C_{[t-1,t]}, \ln R_{[t+s_1,t+s_2]}) = \theta \sigma^2 V(D, s_1, s_2) + O(\varepsilon^{3/2}) \quad (13)$$

with

$$V(D, s_1, s_2) = \frac{e(s_1) - e(s_2) - e(s_1 + D) + e(s_2 + D)}{D} \quad (14)$$

where

$$e(x) = \begin{cases} \frac{3x^2 - |x|^3}{6} & \text{for } |x| \leq 1 \\ \frac{3|x| - 1}{6} & \text{for } |x| \geq 1 \end{cases} . \quad (15)$$

The following corollary will be used in the empirical section.

Corollary 7 *The covariance between $\ln C_{[s+h-1,s+h]}/C_{[s-1,s]}$ and lagged equity returns $\ln R_{[s,s+1]}$ will be*

$$\begin{aligned} \text{cov} \left(\ln \frac{C_{[s+h-1,s+h]}}{C_{[s-1,s]}}, \ln R_{[s,s+1]} \right) &= \theta \sigma^2 \frac{e(1+D) - e(1) - e(1-h+D) + e(1-h)}{D} \\ &+ O(\varepsilon^{3/2}). \end{aligned} \quad (16)$$

In particular, when $h \geq D + 2$, $\text{cov}(\ln C_{[s+h-1,s+h]}/C_{[s-1,s]}, \ln R_{[s,s+1]}) = \theta \sigma^2$; one sees full adjustment at horizons (weakly) greater than $D + 2$.

In practice, Theorem 6 is most naturally applied when the lagged equity returns correspond to specific lagged time periods: i.e., $s_2 = s_1 + 1$, $s_1 = 0, -1, -2, \dots$

Note that $V(D, s_1, s_2) > 0$ iff $s_2 > -D - 1$. Hence, the covariance in Theorem 6 is positive only at lags 0 through $D + 1$.

Figure 6 plots the normalized covariances of consumption growth and lagged asset returns for different values of D . Specifically, we plot $V(D, s, s + 1)$ against s for $D = .25, 1, 2, 4$, from right to left.

Insert Figure 6 about here.

Consider a regression of consumption growth on some arbitrary (large) number of lagged returns,

$$\ln C_{t+1}/C_t = \sum_{s=\underline{s}}^0 \beta_s \ln R_{t+1+s}.$$

One should find,

$$\beta_s = \theta V(D, s, s + 1).$$

Note that the sum of the normalized lagged covariances is one,

$$\frac{1}{\theta \sigma^2} \sum_{s=-\infty}^0 \text{cov}(\ln C_{[t,t+1]}/C_{[t-1,t]}, \ln R_{[t+s,t+s+1]}) = \sum_{s=-\infty}^0 V(D, s, s + 1) = 1.$$

This implies that the sum of the coefficients will equal the portfolio share of the stock market,¹⁵

$$\sum_{s=-D-1}^0 \beta_s = \theta. \tag{17}$$

3.4 Extension to multiple assets and heterogeneity in D .

We now extend the framework to the empirically relevant case of multiple assets with stochastic returns. We also introduce heterogeneity in D 's. Such heterogeneity may arise because different D 's apply to different asset classes and because D may vary across consumers.

Say that there are different types of consumers $l = 1, \dots, n_l$ and different types of asset accounts $m = 1 \dots n_m$. Consumers of type l exist in proportion p_l ($\sum_l p_l = 1$) and look at account m every D_{lm} periods. The consumer has wealth w_{lm} invested in account m , and has an associated marginal propensity to consume (MPC), α_{lm} . In most models the MPC 's will be the same for all assets, but for the sake of behavioral realism and generality we consider possibly different MPC 's.

For instance, income shocks could have a low $D = 1$, stock market shocks a higher $D = 4$, and shocks to housing wealth a $D = 40$.¹⁶ Account m has standard deviation σ_m , and shocks dz_t^m . Call $\rho_{mn} = \text{cov}(dz_{nt}, dz_{mt})/dt$ the correlation matrix of the shocks and $\sigma_{mn} = \rho_{mn} \sigma_m \sigma_n$ their covariance matrix.

¹⁵This is true in a world with only equities and riskless bonds. In general, it's more appropriate to use a model with several assets, including human capital, as in the next section.

¹⁶This example implies different short-run marginal propensities to consume out of wealth windfalls in different asset classes. Thaler (1992) describes one behavioral model with similar asset-specific marginal propensities to consume.

Total wealth in the economy is $\sum_{l,m} p_l w_{lm}$ and total consumption $\sum_{l,m} p_l \alpha_{lm} w_{lm}$. A useful and natural quantity is

$$\theta_{lm} = \frac{p_l \alpha_{lm} w_{lm}}{\sum_{l',m'} p_{l'} \alpha_{l'm'} w_{l'm'}} \quad (18)$$

A shock dz_{mt} in wealth account m will get translated at mean interval $\frac{1}{2} \sum_l p_l D_{lm}$ into a consumption shock $dC/C = \sum_l \theta_{lm} dz_{mt}$.

We can calculate the second moments of our economy.

Theorem 8 *In the economy described above, we have,*

$$\text{cov}(\ln C_t/C_{t-1}, \ln R_{[t+s_1, t+s_2]}^n) = \sum_{l,m} \theta_{lm} \sigma_{mn} V(D_{lm}, s_1, s_2) + O(\varepsilon^{3/2}) \quad (19)$$

and

$$\text{cov}(\ln C_{t+h}/C_{t+h-1}, \ln C_t/C_{t-1}) = \sum_{l,l',m,m'} \theta_{lm} \theta_{l'm'} \sigma_{mm'} \Gamma(D_{lm}, D_{l'm'}, h) + O(\varepsilon^{3/2}) \quad (20)$$

with

$$\Gamma(D, D', h) = \frac{1}{DD'} [d(D+h) + d(D'-h) - d(D'-D-h) - d(h)] \quad (21)$$

and V defined in (14), d defined in (11).

The function $\Gamma(D, t)$, defined earlier in (10), relates to $\Gamma(D, D', t)$ by $\Gamma(D, D, t) = \Gamma(D, t)$. Recall that $V(D, 0, 1) = 1/b(D)$. So a conclusion from (19) is that, when there are several types of people and assets, the bias that the econometrician would find is the harmonic mean of the individual biases $b(D_{lm})$, the weights being given by the ‘‘shares of variance.’’

As an application, consider the case with identical agents ($n_l = 1$, l is suppressed for this example) and different assets with the same MPC, $\alpha_m = \alpha$. Recall that $V(D, 0, 1) = 1/b(D)$. So, the bias $\hat{\gamma}/\gamma$ will be:

$$\frac{\hat{\gamma}}{\gamma} = \left(\sum_m \frac{\theta_m^2 \sigma_m^2}{\sum_{m'} \theta_{m'}^2 \sigma_{m'}^2} b(D_m)^{-1} \right)^{-1}. \quad (22)$$

Hence, with several assets, the aggregate bias is the weight mean of the biases, the mean being the harmonic mean, and the weight of asset m being the share of the total variance that comes from this asset. This allows us, in Appendix B, to discuss a modification of the model with differential attention to big shocks (jumps).

These relationships are derived exactly along the lines of the single asset, single type economy of the previous sections. Expression (19) is the covariance between returns, $\ln R_{[t+s_1, t+s_2]}^n = \sigma_n z_{[t+s_1, t+s_2]}^n + O(\varepsilon)$, and the representation formula for aggregate consumption,

$$\ln C_t/C_{t-1} = \sum_m \theta_m \sigma_m \int_{-1}^1 a(i) z_{[t-1+i-D_m, t-1+i]}^m di + O(\varepsilon^{3/2}), \quad (23)$$

where $a(i) = (1 - |i|)^+$. Equation (23) can also be used to calculate the autocovariance (20) of consumption, if one defines:

$$\Gamma(D, D', h) = \int_{i,j \in [-1,1]} a(i) a(j) \text{cov} (z_{[t-1+i-D, t-1+i]}, z_{[t-1+j+h-D', t-1+j+h]}) \frac{di}{D} \frac{dj}{D'}. \quad (24)$$

The closed form expression (21) of Γ is derived in the appendix.

3.5 Sketch of the proof

Proofs of the propositions appear in the appendix. In this subsection we provide intuition for those arguments. We start with the following representation formula for consumption growth.

Proposition 9 *We have,*

$$\ln C_{t+1}/C_t = \theta\sigma \int_{-1}^1 a(i)z_{[t+i-D, t+i]} \frac{1}{D} di + O(\varepsilon). \quad (25)$$

Note that the order of magnitude of $\theta\sigma \int_{-1}^1 a(i)z_{[t+i-D, t+i]} \frac{di}{D}$ is the order of magnitude of σ , i.e. $O(\sqrt{\varepsilon})$.

Assets returns can be represented as $\ln R_{[t+s_1, t+s_2]} = \sigma z_{[t+s_1, t+s_2]} + O(\varepsilon)$. So we get

$$\text{cov} \left(\ln \frac{C_t}{C_{t-1}}, \ln R_{[s+s_1, s+s_2]} \right) = \theta\sigma^2 \int_{-1}^1 a(i) \text{cov} \left(z_{[t-1+i-D, t-1+i]}, z_{[s+s_1, s+s_2]} \right) \frac{di}{D} + O(\varepsilon^{3/2}) \quad (26)$$

$$= \theta\sigma^2 \int_{-1}^1 a(i) \lambda([t-1+i-D, t-1+i] \cap [s+s_1, s+s_2]) \frac{di}{D} + O(\varepsilon^{3/2}). \quad (27)$$

Here $\lambda(I)$ is the length (the Lebesgue measure) of interval I . Likewise one gets $\text{cov}(\ln C_{h+t}/C_{h+t-1}, \ln C_t/C_{t-1})$

$$\begin{aligned} &= \theta^2\sigma^2 \int_{-1}^1 \int_{-1}^1 a(i)a(j) \text{cov}(z_{[h+t-1+i-D, h+t-1+i]}, z_{[t-1+j-D, t-1+j]}) \frac{di}{D} \frac{dj}{D} + O(\varepsilon^{3/2}) \\ &= \theta^2\sigma^2 \int_{-1}^1 \int_{-1}^1 a(i)a(j) \lambda([h+t-1+i-D, h+t-1+i] \cap [t-1+j-D, t-1+j]) \frac{di}{D} \frac{dj}{D} + O(\varepsilon^{3/2}). \end{aligned}$$

The bulk of the proof is devoted to the explicit calculation of this last equation and equation (27).

4 Endogenizing D

Until now, we have assumed that D is fixed exogenously. In this section we discuss how D is chosen, and provide a framework for calibrating D .

Because of delayed adjustment, the actual consumption path will deviate from the “first-best” instantaneously-adjusted consumption path. In steady-state, the welfare loss associated with this deviation is equivalent, using a money metric, to a proportional wealth loss of Λ_C , where¹⁷

$$\Lambda_c = \frac{\gamma}{2} E \left(\frac{\Delta C}{C} \right)^2 + \text{higher order terms.} \quad (28)$$

Here ΔC is the difference between actual consumption and first-best instantaneously adjusted consumption. If the asset is observed every D periods, we have

$$\Lambda_c = \frac{1}{4} \gamma \theta^2 \sigma^2 D + O(\varepsilon^2). \quad (29)$$

¹⁷This is a second-order approximation. See Cochrane (1989) for a similar derivation.

Equations (28) and (29) are derived in the appendix. We assume¹⁸ that each consumption adjustment costs q proportion of wealth, w . A sensible calibration of q would be $qw = (1\%)(\text{annual consumption}) = (.01)(.04)w = (4 \cdot 10^{-4})w$.

The NPV of costs as a fraction of current wealth is $q \sum_{n \geq 0} e^{-\rho n D}$, implying a total cognitive cost of

$$\Lambda_q = \frac{q}{1 - e^{-\rho D}}.$$

The optimal D minimizes both consumption variability costs and cognitive costs, i.e. $D^* = \arg \min \Lambda_c + \Lambda_q$.

$$D^* = \arg \min_D \frac{1}{4} \gamma \theta^2 \sigma^2 D + \frac{q}{1 - e^{-\rho D}}$$

so

$$\begin{aligned} \frac{1}{4} \gamma \theta^2 \sigma^2 &= q \rho \frac{e^{-\rho D}}{(1 - e^{-\rho D})^2} = \frac{q \rho}{(e^{\rho D/2} - e^{-\rho D/2})^2} \\ &= \frac{q \rho}{4 \sinh^2 \frac{\rho D}{2}} \end{aligned}$$

and we find for the optimal D

$$\begin{aligned} D^* &= \frac{2}{\rho} \arg \sinh \sqrt{\frac{q \rho}{\gamma \theta^2 \sigma^2}} \\ &\simeq \frac{2}{\theta \sigma} \sqrt{\frac{q}{\gamma \rho}} \end{aligned} \tag{30}$$

when $\rho D \ll 1$.

We make the following calibration choices: $q = 4 \cdot 10^{-4}$, $\sigma^2 = (0.16)^2$, $\gamma = 3$, $\rho = 0.01$, $\pi = 0.06$, and $\theta = \pi/(\gamma \sigma^2) = 0.78$. Substituting into our equation for D , we find

$$D \simeq 2 \text{ years.}$$

This calibration implies that D values of at least 1 year (or 4 quarters) are quite easy to defend. Moreover, our formula for D^* is highly sensitive to the value of θ . If a liquidity constrained consumer has only a small fraction of her wealth in equities — because most of her wealth is in other forms like human capital or home equity — then the value of D will be quite large. If $\theta = .05$ because of liquidity constraints, then $D^* \simeq 30$ years.

Note that formula (30) would work for other types of shocks than stock market shocks. With several accounts indexed by m , people would pay attention to account m at intervals of length

$$D_m = \frac{2}{\rho} \arg \sinh \sqrt{\frac{q_m \rho}{\gamma \theta_m^2 \sigma_m^2}} \tag{31}$$

¹⁸This would come from a utility function

$$U = E \left[\left(1 - q \sum_{i \geq 0} e^{-\rho \tau_i} \right)^{1-\gamma} \int_0^\infty e^{-\rho s} \frac{c_{is}^{1-\gamma}}{1-\gamma} ds \right]$$

if the adjustment to consumption are made at dates $(\tau_i)_{i \geq 0}$. A session of consumption planning at time t lowers utility by a consumption equivalent of $q e^{-\rho t}$.

with $q_m w_m$ representing the cost of evaluating asset m , and θ_m generalized as in equation (18). Equation (31) implies sensible comparative statics on the frequency of reappraisal. Thus we get a mini-theory of the allocation of attention across accounts.¹⁹

5 Consequences for macroeconomics and finance

5.1 Simple calibrated macro model

To draw together the most important implications of this paper, we describe a simple model of the US economy. We use our model to predict the variability of consumption growth, the autocorrelation of consumption growth, and the covariance of consumption growth with equity returns.

Assume the economy is comprised of two classes of consumers: stockholders and non-stockholders.²⁰ The consumers that we model in section 2 are stockholders. Non-stockholders do not have any equity holdings, and instead consume earnings from human capital. Stockholders have aggregate wealth S_t and non-stockholders have aggregate wealth N_t . Total consumption is given by the weighted sum

$$C_t = \alpha (S_t + N_t).$$

Recall that α is the marginal propensity to consume.

So, consumption growth can be decomposed into

$$dC/C = s dS/S + n dN/N.$$

Here s represents the wealth of shareholders divided by the total wealth of the economy and $n = 1 - s$ represents the wealth of non-shareholders divided by the total wealth of the economy. So s and n are wealth shares for shareholders and non-shareholders respectively. We make the simplifying approximation that s and n are constant in the empirically relevant medium-run.

Using a first-order approximation,

$$\ln(C_t/C_{t-1}) = s \ln(S_t/S_{t-1}) + n \ln(N_t/N_{t-1}).$$

If stockholders have loading in stocks θ , the ratio of stock wealth to total wealth in the economy is,

$$\Theta = s\theta. \tag{32}$$

To calibrate the economy we begin with the observation that human capital claims about 2/3 of GDP, Y . In this model, human capital is the discounted net present value of labor income accruing to the current cohort of non-stockholders. We assume that the expected duration of the remaining working life of a typical worker is 30 years, implying that the human capital of the current work-force is equal to

$$H = \int_0^{30} e^{-rt} \frac{2}{3} Y dt = \frac{2(1 - e^{-30r})}{3r} Y \simeq 17Y,$$

¹⁹See Gabaix and Laibson (2000a,b) for a broader theoretical and empirical analysis of attention allocation.

²⁰This is at a given point in time. A major reason for non-participation is that relatively young agents have most of their wealth in human capital, against which they cannot borrow to invest in equities (see Constantinides, Donaldson and Mehra 2000).

where Y is aggregate income. Capital income claims 1/3 of GDP. Assuming that it has the riskiness (and the returns) of the stock market, the amount of capital is

$$K = \frac{1}{3(r + \pi)}Y \simeq 5Y$$

so that the equity share of total wealth is

$$\Theta = \frac{K}{K + H} \simeq .22.$$

By assuming that all capital is identical to stock market capital, we implicitly increase the predicted covariance between stock returns and consumption growth. A more realistic model would assume a more heterogeneous capital stock, and hence a lower covariance between stock returns and consumption growth.

In this model economy, we work with data at the quarterly frequency. We assume $\sigma = .16/\sqrt{4}$, $\pi = .06/4$, $r = .01/4$, and $\gamma = 3$, so the equity share (equation (1) above) is $\theta = \pi/(\gamma\sigma^2) = .78$. Then equation (32) implies $s = .28$. In other words, 28% of the wealth in this economy is owned by shareholders. All of stockholders' claims are in either stock or risk-free bonds. To keep things simple, we counterfactually assume that N and S are uncorrelated.

We have to take a stand on the distribution of D 's in the economy. We assume that D values are uniformly distributed from 0 to $\bar{D} = 120$ quarters (i.e., 30 years). We adopt this distribution to capture a wide range of investment styles. Extremely active investors will have a D value close to 0, while passive savers may put their retirement wealth in a special mental account, effectively ignoring the accumulating wealth until after age 65 (Thaler 1992). We are agnostic about the true distribution of D types, and we present this example for illustrative purposes. Any wide range of D values would serve to make our key points.

To keep the focus on stockholders, we assume that non-stockholders adjust their consumption instantaneously in response to innovations in labor income — i.e., at intervals of length 0.

Theorem 3 implies that the quarterly volatility of aggregate consumption growth is,

$$\sigma_C^2 = n^2\Gamma(0, 0)\sigma_N^2 + \Theta^2\sigma^2 \int \int_{D, D' \in [0, \bar{D}]} \Gamma(D, D', 0) \frac{dDdD'}{\bar{D}^2}.$$

We assume that the quarterly standard deviation of growth in human capital is $\sigma_N = .01$.²¹ Our assumptions jointly imply that $\sigma_C = .0063$.²² Most of this volatility comes from variation in the consumption of *non-stockholders*. Stockholders generate relatively little consumption volatility because they represent a relatively small share of total consumption and because they only adjust consumption every D periods. This adjustment rule smooths out the response to wealth innovations, since only fraction $\frac{1}{D}$ of stockholders adjust their consumption during any single period and the average adjustment is of magnitude \sqrt{D} .

Our model's implied quarterly consumption volatility — $\sigma_C = 0.0063$ — lies below its empirical counterpart. We calculate the empirical σ_C using the cross-country panel dataset created by

²¹We calibrate σ_N from post-War U.S. data on wage growth. From 1959-2000 the standard deviation of per-capita real wage growth at the quarterly frequency has been .0097 (National Income and Product Accounts, Commerce Department, Bureau of Economic Analysis). If wages follow a random walk, then the standard deviation of growth in human capital, σ_N , will equal the standard deviation in wage growth.

²²Figure 3 plots the function $\Gamma(D, 0)$. Note that $\Gamma(0, 0) = 2/3$ and that $\Gamma(D, 0) \simeq 1/D$ for large D . In the decomposition of σ_C^2 above, $n^2\Gamma(0, 0)\sigma_N^2 = .34 \cdot 10^{-4}$ and $\Theta^2\sigma^2 \int \int_{D, D' \in [0, \bar{D}]} \Gamma(D, D', 0) \frac{dDdD'}{\bar{D}^2} = 0.049 \cdot 10^{-4}$.

Campbell (1999).²³ We estimate $\sigma_C = 0.0106$ by averaging across all of the countries in Campbell’s dataset: Australia, Canada, France, Germany, Italy, Japan, Netherlands, Spain, Sweden, Switzerland, United Kingdom, and U.S.²⁴ Part of the gap between our theoretical standard deviation and the empirical standard deviation may reflect measurement error, which should systematically raise the standard deviation of the empirical data. In addition, most of the empirical consumption series include *durables*, which should raise the variability of consumption growth (Mankiw 1982). By contrast, the US consumption data omits durables and for the US we calculate $\sigma_C = 0.0054$, closely matching our theoretical value.

Next, we turn to the first-order autocorrelation of consumption growth, applying again Theorem 3:

$$\begin{aligned}\rho_C &= \text{corr}(\ln C_t/C_{t-1}, \ln C_{t-2}/C_{t-1}) \\ &= (\sigma_C^2)^{-1} \left[n^2 \sigma_N^2 \Gamma(0, 1) + \Theta^2 \sigma^2 \int \int_{D, D' \in [0, \bar{D}]} \Gamma(D, D', 1) \frac{dD dD'}{\bar{D}^2} \right].\end{aligned}$$

Using our calibration choices, our model implies $\rho_C = 0.34$.²⁵ This theoretical prediction lies well above the empirical estimate of -0.11 , found by averaging across the country-by-country autocorrelations in the Campbell dataset. Here too, both measurement error and the inclusion of durables are likely to bias the empirical correlations down. Again, the US data, which omits durables, comes much closer to matching our theoretical prediction. In the US data, $\rho_C = 0.22$.

We turn now to the covariation between aggregate consumption growth and equity returns $\text{cov}(\ln C_t/C_{t-1}, \ln R_t)$. We find

$$\text{cov}(\ln C_t/C_{t-1}, \ln R_t) = \Theta \sigma^2 \int_{D \in [0, \bar{D}]} V(D, 0, 1) \frac{dD}{\bar{D}} = 0.13 \cdot 10^{-4}$$

assuming that in the short-run the consumption growth of non-stockholders is uncorrelated with the consumption growth of stockholders. The covariance estimate of $0.13 \cdot 10^{-4}$ almost matches the average covariance in the Campbell dataset, $0.14 \cdot 10^{-4}$. This time, however, the U.S. data does not “outperform” the rest of the countries in the Campbell dataset. For the U.S., the covariance is $0.60 \cdot 10^{-4}$. However, all of these covariances come much closer to matching our model than to matching the benchmark model with instantaneous adjustment/measurement. The benchmark model with no delayed adjustment predicts that the quarterly covariance will be $\theta \sigma^2 \simeq 50 \cdot 10^{-4}$.

What would an econometrician familiar with the consumption-CAPM literature conclude if he observed quarterly data from our $6D$ economy, but thought he were observing data from the benchmark economy? First, he might calculate,

$$\hat{\gamma} = \frac{\pi}{\text{cov}(\ln C_t/C_{t-1}, \ln R_t)} \simeq 1000,$$

and conclude that the coefficient of relative risk aversion is over one-thousand. If he were familiar with the work of Mankiw and Zeldes (1991), he might restrict his analysis to stockholders and calculate,

$$\hat{\gamma} = \frac{\pi}{\text{cov}(\ln S_t/S_{t-1}, \ln R_t)} \simeq 300.$$

²³We thank John Campbell for sharing this dataset with us.

²⁴We use quarterly data from the Campbell dataset. The quarterly data begins in 1947 for the US, and begins close to 1970 for most of the other countries. The dataset ends in 1996.

²⁵The respective effects are $n^2 \sigma_N^2 \Gamma(0, 1) = .077 \cdot 10^{-4}$ and $\Theta^2 \sigma^2 \int \int_{D, D' \in [0, \bar{D}]} \Gamma(D, D', 1) \frac{dD dD'}{\bar{D}^2} = .048 \cdot 10^{-4}$.

Finally, if he read Mankiw and Zeldes carefully, he would realize that he should also do a continuous time adjustment (of the type suggested by Grossman et al 1987), leading to another halving of his estimate. But, after all of this hard work, he would still end up with a biased coefficient of relative risk aversion: $300/2 = 150$. For this economy, the true coefficient of relative risk aversion is 3!

These observations suggest that the literature on the equity premium puzzle should be reappraised. Once one takes account of delayed adjustment, high estimates of γ no longer seem anomalous. If workers in mid-life take decades to respond to innovations in their retirement accounts, we should expect naive estimates of γ that are far too high.

Defenders of the Euler equation approach might argue that economists can go ahead estimating the value of γ and simply correct those estimates for the biases introduced by delayed adjustment. However, we do not view this as a fruitful approach, since the adjustment delays are difficult to observe or calibrate.

For an active stock trader, knowledge of personal financial wealth may be updated daily, and consumption may adjust equally quickly. By contrast, for the typical employee who invests in a 401(k) plan, retirement wealth may be in its own mental account,²⁶ and hence may not be integrated into current consumption decisions. This generates lags of decades or more between stock price changes and consumption responses. Without precise knowledge of the distribution of D values, econometricians will be hard pressed to accurately measure γ using the Euler equation approach.

In summary, our model tells us that high imputed γ values are not anomalous and that high frequency properties of the aggregate data can be explained by a model with delayed adjustment. Hence, the equity premium may not be a puzzle.

Finally, we wish to note that our delayed adjustment model is complementary to the theoretical work of other authors who have analyzed the equity premium puzzle.²⁷ Our qualitative approach has some similarity with the habit formation approach (e.g., Constantinides 1990, Abel 1990, Campbell and Cochrane 1999). Habit formation models imply that slow adjustment is optimal because households prefer to smooth the growth rate (not the level) of consumption. In our $6D$ model, slow adjustment is only optimal because decision costs make high frequency adjustment too expensive.

6 Review of related empirical evidence

In this section, we review two types of evidence that lend support to our model. In this first subsection we review survey evidence which suggests that investors know relatively little about high frequency variation in their equity wealth. In the second subsection we show that equity innovations predict future consumption growth.

6.1 Knowledge of equity prices

Consumers can't respond to high frequency innovations in equity values if they don't keep close tabs on the values of their equity portfolios. In this subsection, we discuss survey evidence that suggests that consumers may know relatively little about high frequency variation in the value of their equity wealth.²⁸ We also discuss related evidence that suggests that consumers may not

²⁶See Thaler (1992).

²⁷For other proposed solutions to the equity premium puzzle see Kocherlakota (1996), Bernartzi and Thaler (1995), and Barberis et al (2000).

²⁸We are grateful to Karen Dynan for pointing out much of this evidence to us.

adjust consumption in response to business cycle frequency variation in their equity holdings. All of this evidence is merely suggestive, since survey responses may be unreliable.

The 1998 Survey of Consumer Finances (SCF) was conducted during the last six months of 1998, a period of substantial variation in equity prices. In July the average value of the Wilshire 5000 equity index was 10,770. The index dropped to an average value of 9,270 in September, before rising back to an average value of 10,840 in December. Kennickell et al (2000) analyze the 1998 SCF data to see whether self-reported equity wealth covaries with movements in stock market indexes. Kennickell et al find that the SCF equity measures are uncorrelated with the value of the Wilshire index on the respondents' respective interview dates. Only respondents that were active stock traders (≥ 12 trades/year) showed a significant correlation between equity holdings and the value of the Wilshire index.

Dynan and Maki (2000) report related results. They analyze the responses to the Consumer Expenditure Survey (CEX) from the first quarter of 1996 to the first quarter of 1999. During this period, the U.S. equity markets rose over 15% during almost every 12 month period. Nevertheless, when respondents were surveyed for the CEX, one third of *stockholders* reported no change in the value of their securities during the 12 month period before their respective interviews.²⁹

Starr-McCluer (2000) analyzes data from the Michigan Survey Research Center (SRC) collected in the summer of 1997. One of the survey questions asked, "Have you [Has your family] changed the amount you spend or save as a result of the trend in stock prices during the past few years?" Among all stockholder respondents, 85.0% said "no effect." Among stockholder respondents with most of their stock outside retirement accounts, 83.3% said "no effect." Even among stockholders with large portfolios ($\geq \$250,000$), 78.4% said "no effect."

6.2 The effect of lagged equity returns on consumption growth

Dynan and Maki (2000) analyze household level data on consumption growth from the CEX, and ask whether lagged stock returns affect future consumption growth. They break their results down for non-stockholders and stockholders. For stockholders with at least \$10,000 in securities a 1% innovation in the value of equity holdings generates a 1.03% increase in consumption of nondurables and services. However, this increase in consumption occurs with a lag. One third of the increase occurs during the first nine-months after the equity price innovation. Another third occurs 10 to 18 months after the price innovation. Another quarter of the increase occurs 19 to 27 months after the price innovation and the rest of the increase occurs 28 to 36 months after the price innovation.

We now turn to evidence from aggregate data. We look for a relationship between equity returns and future consumption growth. Specifically, we evaluate $Cov(\ln [C_{t+h}/C_t], \ln R_{t+1})$ for $h = 1, 2, \dots, 25$.

Under the null hypothesis of $D = 0$, the quarterly covariance between equity returns and consumption growth is predicted to be,

$$\begin{aligned} Cov(\ln [C_{t+1}/C_t], \ln R_{t+1}) &= \frac{\Theta\sigma^2}{2} \\ &= \frac{(.22)(.16/\sqrt{4})^2}{2} \\ &= 0.0007. \end{aligned}$$

²⁹For the purposes of this survey a change in the value of equity securities includes changes due to price appreciation, sales, and/or purchases.

The effects of time aggregation bias are incorporated into this prediction. An equity innovation during period $t + 1$ only affects consumption after the occurrence of the equity innovation. So the predicted covariance, $Cov(\ln[C_{t+1}/C_t], \ln R_{t+1})$, is half as great as it would be if consumption growth were measured instantaneously.

This time-aggregation bias vanishes once we extend the consumption growth horizon to two or more periods. So, if $D = 0$ and $h \geq 2$,

$$\begin{aligned} Cov(\ln[C_{t+h}/C_t], \ln R_{t+1}) &= \Theta\sigma^2 \\ &= (.22)(.16/\sqrt{4})^2 \\ &= 0.0014. \end{aligned}$$

Hence the $D = 0$ assumption implies that the profile of $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$ for $h \geq 2$ should be flat.

Figure 7 plots the empirical values of $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$, for $h \in \{1, 2, \dots, 25\}$.³⁰ We use the cross-country panel dataset created by Campbell (1999).³¹ Figure 7 plots the value of $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$, averaging across all of the countries in Campbell’s dataset: Australia, Canada, France, Germany, Italy, Japan, Netherlands, Spain, Sweden, Switzerland, United Kingdom, and USA.³² Figure 7 also plots the average value of $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$, averaging across all of the countries with large stock markets. Specifically, we ordered the countries in the Campbell dataset by the ratio of stock market capitalization to GDP in 1993. The top half of the countries were included in our “large stock market” subsample: Switzerland (.87), United Kingdom (.80), USA (.72), Netherlands (.46), Australia (.42), and Japan (.40).

Insert Figure 7 about here.

Two properties of the empirical covariances stand out. First, the empirical covariances slowly rise as the consumption growth horizon, h , increases. Contrast this slow increase with the counterfactual prediction for the $D = 0$ case that the covariance should plateau at $h = 2$. Second, the empirical covariances are much lower than the covariance predicted by the $D = 0$ case. For example, at a horizon of 4 quarters, the average empirical covariance is roughly .0002, far smaller than the theoretical prediction of .0014.

Figure 7 also plots the predicted³³ covariance profile implied by the $6D$ model.³⁴ To generate this prediction we assume that D values are uniformly distributed from 0 years to 30 years, as discussed in the previous section.

³⁰See Hall (1978) for early evidence that lagged stock returns predict future consumption growth. See Lettau and Ludvigson (2001) for a VAR approach that implies that lagged stock returns do not predict future consumption growth. Future work should attempt to reconcile our results with those of Lettau and Ludvigson.

³¹We thank John Campbell for giving this dataset to us.

³²Specifically, we calculate $Cov(\ln R_{t+1}, \ln[C_{t+h}/C_t])$ for each country and each h -quarter horizon, $h \in \{1, 2, \dots, 25\}$. We then average across all of the countries in the sample. We use quarterly data from the Campbell dataset. The quarterly data begins in 1947 for the US, and begins close to 1970 for most of the other countries. The dataset ends in 1996.

³³Corollary 7 gives:

$$Cov(\ln[C_{t+h}/C_t], \ln R_{t+1}) = \Theta\sigma^2 \int_{D \in [0, \bar{D}]} (e(1+D) - e(1) - e(1-h+D) + e(1-h)) \frac{dD}{DD}.$$

³⁴The following approximation for the covariances provides intuition for the orders of magnitude. In normalized units,

$$\frac{1}{\Theta\sigma^2} cov(\ln C_{t+h}/C_t, R_{t+1} | D) \simeq \max\left(\frac{h}{D}, 1\right).$$

The 6D model predicts that the covariance, $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$, slowly rises with the horizon h . To understand this effect, recall that the 6D economy slowly adjusts to innovations in the value of equity holdings. Some consumers respond quickly to equity innovations, either because these consumers have low D values, or because they have a high D value and are coincidentally coming up to a reset period. Other consumers respond with substantial lags. For our illustrative example, the full response will take 30 years. For low h values, the 6D model predicts that the covariance profile will be close to zero. As h goes to infinity, the covariance profile asymptotes to the prediction of the instantaneous adjustment model, so $\lim_{h \rightarrow \infty} Cov(\ln[C_{t+h}/C_t], \ln R_{t+1}) = \Theta\sigma^2 = .0014$. Figure 7 shows that our illustrative calibration of the 6D model does a fairly good job of matching the empirical covariances.

This analysis has shown that the empirical data is completely inconsistent with the standard assumption of instantaneous adjustment. Lagged equity returns affect consumption growth at very long horizons: $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$ rises slowly with h , instead of quickly plateauing at $h = 2$. This slow rise is a key test of the 6D framework.

We conclude from Figure 7 that the 6D model successfully predicts the profile of $Cov(\ln[C_{t+h}/C_t], \ln R_{t+1})$ for $h = 1, 2, \dots, 25$. However, the 6D model fails to predict the profile of a closely related object, the normalized ‘‘Euler covariance,’’

$$\frac{1}{h} Cov\left(\ln[C_{t+h}/C_t], \sum_{i=1}^h \ln R_{t+i}\right).$$

This h -period covariance generalizes the one-period Euler covariance, $Cov(\ln[C_{t+1}/C_t], \ln R_{t+1})$.^{35,36}

The standard model with $D = 0$ predicts that the h -period normalized Euler covariance will equal $\frac{2h-1}{2h}\Theta\sigma^2$ for all (integer) values of h . The factor $\frac{2h-1}{2h}$ captures time aggregation bias, which becomes proportionately less important as the horizon increases. By contrast, the 6D model predicts that, if the D 's are uniformly distributed between 0 and \bar{D} (e.g., $\bar{D} = 30$ years or 120 quarters),

When the D 's are uniformly distributed in $[0, \bar{D}]$,

$$\begin{aligned} \frac{1}{\Theta\sigma^2} cov(\ln C_{t+h}/C_t, R_{t+1}) &= \int_0^{\bar{D}} cov(\ln C_{t+h}/C_t, R_{t+1} | D) \frac{dD}{\bar{D}} \\ &\simeq \int_0^{\bar{D}} \max\left(\frac{h}{D}, 1\right) \frac{dD}{\bar{D}} \\ &= \frac{h}{\bar{D}} \left(1 + \ln \frac{\bar{D}}{h}\right) \end{aligned}$$

This approximation turns out to be quite good for $h \geq 2$.

³⁵We thank Monika Piazzesi, whose insightful discussion of this paper at the NBER Macroeconomics Annual Conference led us to add analysis of the covariance Euler equation to this final draft.

³⁶The ‘‘Euler covariances’’ link the equity premium to the coefficient of relative risk aversion. Consider the h -period Euler equation for a discrete-time model with instantaneous adjustment, $E_{t-1} \left[\hat{\beta} \left(\frac{C_{t+h}}{C_t}\right)^{-\hat{\gamma}} \exp\left(\sum_{i=1}^h \ln R_{t+i}^a\right) \right] = 1$ (for all assets a). Manipulation of this equation implies,

$$\hat{\gamma} = \frac{\pi}{Cov\left(\left[\sum_{i=1}^h \ln R_{t+i}\right], \ln[C_{t+h}/C_t]\right) / h},$$

where π is the 1-period equity premium.

the h -period normalized Euler covariance should approximately³⁷ equal $\frac{h}{4\bar{D}} \left(3 - 2 \ln \frac{h}{\bar{D}}\right) \Theta \sigma^2$ for $h < \bar{D}$. For both the standard model ($D = 0$) and the $6D$ model, the normalized Euler covariance should monotonically rise with h , but this rise should be much steeper for the standard model.

The empirical data matches neither prediction. In the twelve-country Campbell data, an initial rise in the Euler covariance from $h = 1$ to $h = 7$, is subsequently reversed for larger values of h . For $h > 20$, the Euler covariances are very small in magnitude, with some negative point estimates.³⁸

This result seems to contradict the encouraging results plotted in Figure 7. To understand this tension, we assume stationarity and decompose the h -period Euler covariance:

$$\begin{aligned} \text{Cov} \left(\ln [C_{t+h}/C_t], \sum_{i=1}^h \ln R_{t+i} \right) &= \sum_{i=1}^h \text{Cov} (\ln [C_{t+i}/C_t], \ln R_{t+1}) \\ &\quad + \sum_{i=1}^{h-1} \text{Cov} (\ln [C_t/C_{t-i}], \ln R_{t+1}). \end{aligned}$$

The h -period Euler covariance (i.e., left hand side) is zero for large h 's, and the first sum in the right hand side is positive (this is the object plotted in Figure 7). It follows that the second term on the right-hand-side should be negative,

$$\sum_{i=1}^{h-1} \text{Cov} (\ln [C_t/C_{t-i}], \ln R_{t+1}) < 0,$$

which can be verified in our sample.³⁹ In words, lagged consumption growth negatively predicts the current stock return. Such predictability explains why the Euler covariance does not follow

³⁷We use the approximation above:

$$\frac{1}{\Theta \sigma^2} \text{cov} (\ln C_{t+h}/C_t, \ln R_{t+1}) \simeq \frac{h}{\bar{D}} \left(1 + \ln \frac{\bar{D}}{h}\right)$$

to get

$$\begin{aligned} \frac{1}{h\Theta \sigma^2} \text{cov} (\ln C_{t+h}/C_t, \ln R_{t+1} + \dots + \ln R_{t+h}) &= \frac{1}{h\Theta \sigma^2} \sum_{h'=1}^h \text{cov} (\ln C_{t+h'}/C_t, \ln R_{t+1}) \\ &\simeq \frac{1}{h} \int_0^h \frac{h'}{\bar{D}} \left(1 + \ln \frac{\bar{D}}{h'}\right) dh' \\ &= \frac{h}{4\bar{D}} \left(3 - 2 \ln \frac{h}{\bar{D}}\right) \text{ for } h \leq \bar{D} \\ &= 1 - \frac{\bar{D}}{4h} \text{ for } h > \bar{D} \end{aligned}$$

³⁸See Cochrane and Hansen (1992) for an early empirical analysis of the multi-period Euler equation. Daniel and Marshall (1997, 1999) report that consumption Euler equations for aggregate data are not satisfied at the quarterly frequency but improve at the two-year frequency. Our results are consistent with theirs, but we find that this relatively good performance deteriorates as the horizon is lengthened.

³⁹For quarterly horizons $h \in \{5, 10, 15, 20, 25\}$, the average value of

$$\sum_{i=1}^{h-1} \text{Cov} (\ln [C_t/C_{t-i}], \ln R_{t+1})$$

is $\{-0.9, -2.0, -4.6, -2.8, -3.6\} \cdot 10^{-4}$ for all of the countries in the Campbell dataset and $\{-1.2, -2.4, -5.0, -3.0, -3.2\} \cdot 10^{-4}$ for the countries with large stock markets.

the profile predicted by the $6D$ model. Of course, this predictability is inconsistent with any model in which the stock market follows a martingale. Alternative frameworks, like Campbell and Cochrane's (2000) model of habit formation, Barberis, Huang, and Santos' (2001) Prospect Theory model of asset pricing, or animal spirits models, are needed to explain why lagged consumption growth negatively forecasts future stock returns.

7 Conclusion

Grossman and Laroque (1990) argue that adjustment costs might explain the equity premium puzzle. Lynch (1996) and Marshall and Parekh (1999) have successfully numerically simulated discrete-time delayed adjustments models which confirm Grossman and Laroque's conjecture. We have described a continuous-time generalization of Lynch's (1996) model. We derive a complete analytic characterization of the model's dynamic properties. In addition, our continuous-time framework generates effects that are up to six times larger than those in discrete time models.

We analyze an economy composed of consumers who update their consumption every D periods. Using data from our economy, an econometrician estimating the coefficient of relative risk aversion (CRRA) from the consumption Euler equation would generate a multiplicative CRRA bias of $6D$. Once we take account of this $6D$ bias, the Euler equation tests are unable to reject the standard consumption model.

We have derived closed form expressions for the first and second moments of this delayed adjustment economy. The model matches most of the empirical moments of aggregate consumption and equity returns, including a new test which confirms the $6D$ prediction that the covariance between $\ln(C_{t+h}/C_t)$ and R_{t+1} should slowly rise with h . The $6D$ model fails long-horizon Euler equation tests, but this failure is due to the interesting empirical regularity that high lagged consumption growth predicts low future equity returns.

Future work should test the new empirical implications of our framework, including the rich covariance lag structure that we have derived. Most importantly, our model implies that standard Euler equation tests should be viewed very skeptically. Even small positive values of D (e.g., $D = 4$ quarters) dramatically bias the inferences that economists draw from Euler equations and the related Hansen-Jagannathan bounds.

8 Appendix A: Proofs

We use approximations to get analytic results. Let $\varepsilon \equiv \max(r, \rho, \theta\pi, \sigma^2, \sigma^2\theta^2, \alpha)$. For annual data $\varepsilon \simeq .05$. We shall use the notation $f(\varepsilon) = O_{<t}(\varepsilon^k)$, for $k \in \mathbb{R}$, to mean that f is measurable with respect to the information known at time t , and there is $\varepsilon_0 > 0$ and a constant $A > 0$ such that for $\varepsilon \leq \varepsilon_0 A$, we have $E_0[f^2]^{1/2} \leq A|\varepsilon^k|$. More concisely the norms are in the L_2 sense. For instance:

$$\begin{aligned} e^{rt+\sigma z(t)} &= 1 + \sigma z(t) + rt + \sigma^2 z(t)^2/2 + O_{<s}(\varepsilon^{3/2}) \\ &= 1 + \sigma z(t) + O_{<s}(\varepsilon) = 1 + O_{<s}(\varepsilon^{1/2}). \end{aligned}$$

We will often replace $O_{<t}(\varepsilon^k)$ by $O(\varepsilon^k)$ when there is a clear smallest time t such that f is measurable with respect to the information known at time t . For instance we would say $e^{rt+\sigma z(t)} = 1 + \sigma z(t) + O(\varepsilon)$ to mean $e^{rt+\sigma z(t)} = 1 + \sigma z(t) + O_{<t}(\varepsilon)$.

Also, we shall often use the function:

$$a(i) \equiv (1 - |i|)^+. \quad (33)$$

Finally, for z a generic standard Brownian motion, we call $z_{[i,j]} = z(j) - z(i)$, and remark:

$$\text{cov}(z_{[i-D,i]}, z_{[j-D',j]}) = \min\left(\left(D - (i - j)^+\right)^+, \left(D' - (j - i)^+\right)^+\right) \quad (34)$$

as both are equal to the measure of $[i - D, i] \cap [j - D', j]$.

8.1 Proof of Proposition 1

Call $v(w) = E \int_0^\infty e^{-\rho t} c_t^{1-\gamma} / (1 - \gamma) dt$ the expected value of the utils from consumption under the optimal policy, assuming the first reset date is $t = 0$. So $v(\cdot)$ is the value function that applies at reset dates. Say that the agent puts S in the checking account, and the rest, $w - S$, in the mutual fund. Call M the (stochastic) value of the mutual fund at time D . By homotheticity, we have $v(w) = v \cdot w^{1-\gamma} / (1 - \gamma)$. We have:

$$\begin{aligned} v(w) &= \int_0^D e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho D} E[v(w')] \quad \text{with} \\ w' &= M + S e^{rD} - \int_0^D c_t e^{r(D-t)} dt. \end{aligned} \quad (35)$$

Optimizing over c_t for $t \in [0, D)$, we get $c_t^{-\gamma} = E[v'(w')] e^{(r-\rho)(D-t)}$, so that consumption growth is that of the Ramsey model: $c_t = \alpha w e^{\frac{r-\rho}{\gamma} t}$ for some α (by the implicit function theorem one can show that it is a continuous function of D , and it has Merton's value when $D = 0$). To avoid bankruptcy, we need $S \geq S_0 = \int_0^D c_t e^{-rt} dt$. Imagine that the consumer starts by putting aside the amount S_0 . Then, he has to manage optimally the remaining amount, $w - S_0$. Given some strategy, he will end up with a stochastic wealth w' , and he has to solve the problem of maximizing $v E[w'^{1-\gamma} / (1 - \gamma)]$. But this is a finite horizon Merton problem with utility derived from terminal wealth, whose solution is well-known: the whole amount $w - S_0$ should be put in a mutual fund with constant rebalancing, with a proportion of stocks $\theta = \pi / (\gamma \sigma^2)$. In particular, only the amount S_0 is put in the checking account. \square

8.2 Proof of Proposition 9

The basis of our calculations is the representation formula for consumption, Proposition 9. To prove it we shall need the following

Lemma 10 *We have*

$$w_{it+s} = w_{it} (1 + \theta \sigma (z(t+s) - z(t)) + O(\varepsilon)). \quad (36)$$

Proof. If the agent doesn't check her portfolio between t and $t + s$, we have

$$\begin{aligned} w_{it+s} &= w_{it} e^{(r+\theta\pi-\theta^2\sigma^2/2)s+\sigma\theta(z(t+s)-z(t))} \\ &= w_{it} (1 + \sigma\theta (z(t+s) - z(t)) + O(\varepsilon)). \end{aligned} \quad (37)$$

When the agent checks her portfolio at time τ , she puts a fraction $f = \int_0^D \alpha e^{-rt + \left(\frac{r-\rho}{\gamma}\right)t} dt = O(\varepsilon)$ in the checking account, so that

$$w_{i\tau^+} = w_{i\tau^-} (1 - f) \quad (38)$$

$$= w_{i\tau^-} (1 + O(\varepsilon)). \quad (39)$$

Pasting together (37) and (39) at different time intervals we see that (37) holds between two arbitrary dates (i.e., possibly including reset dates) t and $t + s$, and the lemma is proven. ■

We can now proceed to the

Proof of Proposition 9. Say that $i \in [0, D)$ has her latest reset point before $t - 1$ at $t_i = t - 1 - i$. The following reset points are $t_i + mD$ for $m \geq 1$, and for $s \geq t - 1$ we have (the first $O(\varepsilon)$ term capturing the deterministic increase of consumption between reset dates)

$$\begin{aligned} c_{is}/\alpha &= \left(w_{it_i} + \sum_{m \geq 1} (w_{it_i+mD} - w_{it_i+(m-1)D}) \mathbf{1}_{s \geq t_i+mD} \right) (1 + O(\varepsilon)) \\ &= w_{it_i} + \sum_{m \geq 1} w_{it_i} (\theta \sigma z_{[t_i+(m-1)D, t_i+mD]} + O(\varepsilon)) \mathbf{1}_{s \geq t_i+mD} + O(\varepsilon) \end{aligned}$$

so that, using the notation $\zeta_{im} \equiv w_{it_i} \theta \sigma z_{[t_i+(m-1)D, t_i+mD]}$,

$$\begin{aligned} \int_{t_i}^T c_{is}/\alpha ds + O(\varepsilon) &= (T - t_i) w_{it_i} + \sum_{m \geq 1} \zeta_{im} \int_{t_i}^T \mathbf{1}_{s \geq t_i+mD} ds \\ &= (T - t_i) w_{it_i} + \sum_{m \geq 1} \zeta_{im} (T - (t_i + mD))^+, \end{aligned}$$

and we get

$$\begin{aligned} c_{it+1} - c_{it} + O(\varepsilon) &= \left(\int_{t_i}^{t+1} -2 \int_{t_i}^t + \int_{t_i}^{t-1} \right) (c_{is} ds) \\ &= \alpha \sum_{\substack{m \geq 1 \\ \tau_{im} = t_i + mD}} \zeta_{im} ((t+1 - \tau_{im})^+ - 2(t - \tau_{im})^+ + (t-1 - \tau_{im})^+) \\ &= \alpha \sum_{m \geq 1} \zeta_{im} a (t - (t_i + mD)) \text{ since } (x+1)^+ - 2x^+ + (x-1)^+ = a(x) \\ &= \alpha \sum_{m \geq 1} \zeta_{im} a (1 + i - mD) \end{aligned}$$

because $t_i = t - 1 - i$.

Let $w_{t-D-1} = w_{i_0, t-D-1}$, which implies that $w_{i, t-D-1} = w_{t-D-1} (1 + O(\varepsilon))$, for all i . Note that i_0 is an arbitrarily selected index value. We now get the expression for consumption growth

$$\begin{aligned} C_{t+1} - C_t &= \int_0^D (c_{it+1} - c_{it}) \frac{di}{D} \\ &= \alpha \sum_{m \geq 1} \int_0^D w_{t-D-1} \theta \sigma z_{[t-1-i+(m-1)D, t-1-i+mD]} a (1 + i - mD) \frac{di}{D} + O(\varepsilon). \end{aligned}$$

Defining $j \equiv D - 1 - i$, and noting that the above expressions “paste” together, we have

$$\frac{C_{t+1} - C_t}{\alpha w_{t-D-1}} = \int_{-1}^1 \theta \sigma z_{[t+j-D, t+j]} a(j) \frac{dj}{D} + O(\varepsilon).$$

One can likewise calculate

$$\frac{C_t}{\alpha w_{t-D-1}} = 1 + O(\sqrt{\varepsilon})$$

so

$$\ln C_{t+1}/C_t = \int_{-1}^1 \theta \sigma z_{[t+j-D, t+j]} a(j) \frac{dj}{D} + O(\varepsilon).$$

□

8.3 Proof of Theorem 2

Use Proposition 9, $\ln R_{t+1} = \sigma z_{[t, t+1]} + O(\varepsilon)$, to get,

$$\begin{aligned} \text{cov}(\ln C_{t+1}/C_t, \ln R_{t+1}) &= \theta \sigma^2 \int_{-1}^1 a(i) \text{cov}(z_{[t+i-D, t+i]}, z_{[t, t+1]}) \frac{di}{D} \\ &\quad + O_{<t}(\varepsilon^{3/2}) \text{ with} \\ \int_{-1}^1 a(i) \text{cov}(z_{[t+i-D, t+i]}, z_{[t, t+1]}) \frac{di}{D} &= \int_0^1 a(i) \min(D, i) \frac{di}{D} \text{ by (34)} \\ &= \frac{3(1-D) + D^2}{6} \text{ if } D \leq 1 \\ &= \frac{1}{6D} \text{ if } D \geq 1. \end{aligned}$$

Using (1) and (6), this leads to the expression (2).

8.4 Proof of Theorem 3

First we need the

Lemma 11 *We have, with d defined in (11), for $D \in \mathbb{R}$,*

$$\int_{\mathbb{R}} a(i) a(i+D) di = d''(D).$$

Proof of the lemma 11

Define, for $D \in \mathbb{R}$,

$$g(D) \equiv \int_{\mathbb{R}} a(i) a(i+D) di. \tag{40}$$

First, note that g is even because a is. In addition, for $D \geq 2$, $g(D) = 0$: for the integrand to be non-zero in (40), we need both $|i| < 1$ and $|i+D| < 1$, which is impossible for $D \geq 2$.

For a general D , we derive (in the sense of the theory of distributions, with Dirac's δ function⁴⁰) g over D , starting from (40):

$$\begin{aligned} g^{(4)}(D) &= \int_{\mathbb{R}} a(i)a^{(4)}(i+D)di \\ &= \int_{\mathbb{R}} a''(i)a''(i+D)di \text{ by integration by parts} \\ &= \sum_{j=0}^4 \binom{4}{j} (-1)^j \delta(j-2+D) \end{aligned}$$

by direct calculation (or combinatorial insight) using $a''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1)$. We now integrate $g^{(4)}(D)$, which gives:

$$\begin{aligned} g(D) &= \sum_{j=0}^4 \binom{4}{j} \frac{(-1)^j}{2 \cdot 3!} |j-2+D|^3 + \sum_{j=0}^3 b_j D^j \\ &= d''(D) + \sum_{j=0}^3 b_j D^j \end{aligned}$$

where the b_j are integration constants. But the condition $g(D) = 0$ for $D \geq 2$ forces the b_j 's to be 0, which concludes the proof. \square

The rest of the proof is in two steps. First we prove (41)-(42), then we calculate this expression of $p(D, t)$.

Step 1.

Using (25) at t and $t+h$, we get

$$\text{cov}(\ln C_{t+1}/C_t, \ln C_{t+1+h}/C_{t+h}) = \theta^2 \sigma^2 \Gamma(D, h) + O(\varepsilon^{3/2})$$

with

$$\begin{aligned} \Gamma(D, h) &= \text{cov} \left(\int_{-1}^1 a(i) z_{[t+i-D, t+i]} \frac{di}{D}, \int_{-1}^1 a(j) z_{[t+h+j-D, t+h+j]} \frac{dj}{D} \right) \\ &= \int_{-1}^1 \int_{-1}^1 a(i) a(j) \text{cov}(z_{[t+i-D, t+i]}, z_{[t+h+j-D, t+h+j]}) \frac{di}{D} \frac{dj}{D} \end{aligned}$$

so using (34) we get

$$\Gamma(D, h) = \frac{p(D, h)}{D^2} \tag{41}$$

with

$$p(D, h) \equiv \int \int_{i, j \in [-1, 1]} a(i) a(j) (D - |i - j - h|)^+ di dj \tag{42}$$

Step 2.

⁴⁰Dirac's δ function is equal to 0 everywhere except at 0, where $\delta(0) = \infty$.

Our next step is to calculate $p(D, h)$. Start with the case $D \geq h + 2$: then $(D - |i - j - h|)^+ = D - |i - j - h|$ as $|i - j - h| \leq 1 + 1 + h \leq D$, and given $\int \int_{i, j \in [-1, 1]} a(i)a(j) didj = \left(\int_{i \in [-1, 1]} a(i) di \right) \left(\int_{i \in [-1, 1]} a(j) dj \right) = 1$, we get

$$\begin{aligned} p(D, h) &= D - A(h) \text{ for } D \geq h + 2 \text{ with} \\ A(h) &= \int \int_{i, j \in \mathbb{R}} |i - j - h| a(i)a(j) didj. \end{aligned} \tag{43}$$

Going back to a general $D > 0$, we get from (42):

$$\begin{aligned} p''(D) &= \int \int_{i, j \in \mathbb{R}} a(i)a(j) \delta(D - |i - j - h|) didj \\ &= \int_{\mathbb{R}} a(i) (a(i + D - h) + a(i - D - h)) di \\ &= \int_{\mathbb{R}} a(i) (a(i + D - h) + a(i + D + h)) di \end{aligned}$$

because a is even and by an application of change in variables. So from Lemma 11 $p''(D) = d''(D - h) + d''(D + h)$ and

$$p(D, h) = d(D + h) + d(D - h) + d_0 + d_1 D$$

for some real numbers d_0, d_1 . Equation (43) gives us $d_1 = 0$, since $d'(x) = 1/2$ for $x > 2$. Finally, $p(0) = 0$ gives $A(h) = -d_0 = d(h) + d(-h)$, which concludes the proof.

8.5 Proof of Corollary 4

$\Gamma(D, 0)$ is monotonic by direct calculation from the result in Theorem 3. Theorem 3 also implies

$$\Gamma(D, 0) = \frac{2}{3} - \frac{D^2}{6} + \frac{D^3}{20} \text{ for } D \in [0, 1].$$

Alternatively, this result can be obtained more directly from the calculation at the end of the proof of Theorem 3.

8.6 Proof of Proposition 5

Extend the argument used to prove Theorem 2. To calculate the correlation coefficient, use the variance results from Corollary 4.

8.7 Proof of Theorem 6

Because $V(s_1, s_2) = V(s_1, 1) - V(s_2, 1)$, it is enough to fix $s_2 = 1$. We use the notation $s = s_1$. Recall (25), so that

$$\text{cov}(\ln C_{[t, t+1]}/C_{[t-1, t]}, \ln R_{[t+s, t+1]}) = \frac{\theta \sigma^2}{D} W(s) + O(\varepsilon^{3/2})$$

with

$$\begin{aligned}
W(s) &= D \int_{-1}^1 a(i) \text{cov}(z_{[t+i-D, t+i]}, z_{[t+s, t+1]}) \frac{di}{D} \\
&= \int_{-1}^1 a(i) (i - \max(i - D, s))^+ di.
\end{aligned} \tag{44}$$

So, using the Heaviside function — $H(x) \equiv 1$ if $x \geq 0$, 0 if $x < 0$ (so that $H' = \delta$) —

$$\begin{aligned}
W'(s) &= - \int a(i) H(i - \max(i - D, s)) H(s - i + D) di \\
&= - \int a(i) H(i - s) H(s - i + D) di
\end{aligned}$$

and

$$\begin{aligned}
W''(s) &= \int a(i) (\delta(i - s) H(s - i + D) - \delta(s - i + D) H(i - s)) di \\
&= a(s) - a(s + D).
\end{aligned}$$

Introducing the e function defined in (15), which satisfies $e'' = a$, we get:

$$W(s) = e(s) - e(s + D) + W_0 + W_1 s \tag{45}$$

for some constants W_0, W_1 . Observe that for $s \geq 1$, (44) gives $W(s) = 0$, so (45) gives us $W_1 = 0$ (and $W_0 = D/2$). This allows to conclude the proposition.

8.8 Proof of Corollary 7

Immediate application of the preceding Theorem.

8.9 Proof of Theorem 8

The expression (23) is derived exactly like in Proposition 9. The only new work is to calculate $\Gamma(D, D', h)$. Using (34) we get:

$$\begin{aligned}
\Gamma(D, D', h) &= \frac{p(D, D', h)}{DD'} \text{ with} \\
p(D, D', h) &= \int_{i, j \in [-1, 1]} a(i)a(j) \min\left(\left(D - (i - j - h)^+\right)^+, \left(D' - (j - i + h)^+\right)^+\right) didj.
\end{aligned}$$

To calculate p , we derive (again, $H(x) = 1_{x \geq 0}$ is Heaviside's function)

$$p_{D'} = \int a(i)a(j) H\left(\left(D - (i - j - h)^+\right)^+ - \left(D' - (j - i + h)^+\right)^+\right) H\left(D' - (j - i + h)^+\right) didj$$

and

$$\begin{aligned}
p_{D'D'} &= \int a(i)a(j) H\left(\left(D - (i - j - h)^+\right)^+ - \left(D' - (j - i + h)^+\right)^+\right) \delta\left(D' - (j - i + h)^+\right) didj \\
&\quad - \int a(i)a(j) \delta\left(\left(D - (i - j - h)^+\right)^+ - \left(D' - (j - i + h)^+\right)^+\right) H\left(D' - (j - i + h)^+\right) didj \\
&= \int a(i) (a(i + D' - h) - a(i + D' - D - h)) di.
\end{aligned}$$

So Lemma 11 gives:

$$p = d(D' - h) - d(D' - D - h) + e_0 + e_1 D'$$

where e_0, e_1 are functions of D and h . As $p = 0$ for $D' = 0$, we get $e_0 = -d(-h) + d(-D - h) = -d(h) + d(D + h)$, as d is even. As we should have $p(D, D, h) = p(D, h)$ for p in (42), we can conclude $e_1 = 0$ and deduce the value of e_0 , so Theorem 8 is proven.

8.10 Derivation of the utility losses

A fully rigorous derivation, e.g. of the type used by Rogers (2001), is possible here. Such a derivation begins with the Bellman equation (35), and then uses a Taylor expansion to derive an expression for v of the type $v = v_0 + v_1 D + O(v^2)$. This approach is tedious and not very instructive about the economic origins of the losses, which is why we present the following more heuristic proof.

Equation (28) is standard (e.g., see Cochrane 1989). For completeness's sake, though, let us mention a way to derive it. We want to calculate $U(C) - U(C')$, where $C = (c_t)_{t \geq 0}$ is the optimum vector of (stochastic) consumption flows, $U(C) = E \left[\int_0^\infty e^{-\rho t} u(c_t) \right]$, and C' is another vector that can be bought with the same Arrow-Debreu prices p . For C and C' close, we have

$$\begin{aligned} \Delta U &\equiv U(C') - U(C) \\ &= U'(C)(C' - C) + (C' - C)' \cdot U''(C) \cdot (C' - C)/2 + O((C' - C)^3). \end{aligned}$$

As by optimality of C , $U'(C) = \lambda p$ for some p , and $pC = pC' = \text{initial wealth} = W$, we have $U'(C)(C - C') = 0$. Expressing U'' finally gives

$$\Delta U = \frac{1}{2} E \left[\int_0^\infty e^{-\rho t} u''(c_t) (c_t - c'_t)^2 dt \right].$$

A change ΔW in the initial wealth creates, by homotheticity of the optimal policy, a change in consumption $\Delta c_t / c_t = \Delta W / W$, hence a change in utility

$$\Delta U = E \left[\int_0^\infty e^{-\rho t} u'(c_t) c_t \frac{\Delta W}{W} dt \right].$$

So the suboptimality of plan C' is equivalent to a wealth loss of (using $u'(c) = c^{-\gamma}$).

$$\begin{aligned} \Lambda_c &\equiv -\frac{\Delta W}{W} = -\frac{1}{2} \frac{E \left[\int_0^\infty e^{-\rho t} u''(c_t) c_t^2 \left(\frac{c_t - c'_t}{c_t} \right)^2 dt \right]}{E \left[\int_0^\infty e^{-\rho t} u'(c_t) c_t dt \right]} \\ &= \frac{\gamma}{2} \left\langle \left(\frac{c_t - c'_t}{c_t} \right)^2 \right\rangle \end{aligned}$$

where the weights in the mean $\langle \cdot \rangle$ are given by $\langle X_t \rangle = E \left[\int_0^\infty e^{-\rho t} c_t^{1-\gamma} X_t dt \right] / E \left[\int_0^\infty e^{-\rho t} c_t^{1-\gamma} dt \right]$.

This proves equation (28).

We now derive $\langle \Delta c_t^2 / c_t^2 \rangle$, with $\Delta c_t = c'_t - c_t$. With latest reset at time τ

$$\begin{aligned} \frac{\Delta c_t}{\alpha} &= \frac{c'_t - c_t}{\alpha} = (w'_\tau - w_t) (1 + O(\varepsilon)) \\ &= (w_\tau - w_t + w'_\tau - w_\tau) (1 + O(\varepsilon)). \end{aligned}$$

Now application of Lemma (10) gives (sparing the reader the tedious derivation),

$$\begin{aligned}\langle (w_\tau - w_t)^2 / w_t^2 \rangle &= E \left[\int_0^D \left(\int_0^t \theta \sigma dz_s \right)^2 \frac{dt}{D} \right] + O(\varepsilon^2) \\ &= \theta^2 \sigma^2 D / 2 + O(\varepsilon^2).\end{aligned}$$

Defining Ψ such that $E[c_t^{1-\gamma}] = c_0^{1-\gamma} e^{(\rho-\Psi)t}$, with $\Psi > 0$, we get

$$\begin{aligned}\langle (w'_\tau - w_\tau)^2 / w_t^2 \rangle &= \langle \alpha^2 \theta^2 \sigma^2 t D \rangle \\ &= \alpha^2 \theta^2 \sigma^2 D \frac{\int_0^\infty e^{-\Psi t} t dt}{\int_0^\infty e^{-\Psi t} dt} \\ &= \alpha^2 \theta^2 \sigma^2 D / \Psi \\ &= \theta^2 \sigma^2 D O(\varepsilon) = O(\varepsilon^2).\end{aligned}$$

The cross term $\langle (w_\tau - w_t)(w'_\tau - w_\tau) \rangle = 0$.

So we have the important (and general in these kinds of problems) fact that the first order contribution to the welfare loss is the direct impact of the delayed adjustment – the $w_\tau - w_t$ term – whereas the indirect impact (where a suboptimal choice of consumption creates modifications in future wealth) is second order. In other terms

$$\begin{aligned}\langle \Delta c_t^2 / c_t^2 \rangle &= \langle \Delta c_t^2 / c_t^2 \rangle_{\text{without modification of the wealth process}} + O(\varepsilon^2) \\ &= \langle (w_\tau - w_t)^2 / w_t^2 \rangle + O(\varepsilon^2) \\ &= \theta^2 \sigma^2 D / 2 + O(\varepsilon^2).\end{aligned}$$

Using (28) we get (29).

9 Appendix B: Model with immediate adjustment in response to large changes in equity prices

Suppose that people pay greater attention to “large” movements in the stock markets (because they are more salient, or because it is more rational to do so). How does our bias change? We propose the following tractable way to answer this question. Say that the returns in the stock market are:

$$dR_t = (\mu + r) dt + \sigma dz_t + dj_t$$

where j_t is a jump process with arrival rate λ . For instance, such jumps may correspond to crashes, or to “sharp corrections”, though we need not have $E[dj_t] < 0$. To be specific, when a crash arrives the return falls by J (to fix ideas, say $J = .1 - .3$). To model high attention to crashes, we say that consumption adjusts to dz_t shocks every D periods, and adjusts to dj shocks immediately ($D = 0$ for those Poisson events).

Call σ_B^2 the variance of Brownian shocks and $\sigma_J^2 = E[dj_t^2]/dt = \lambda J^2$ the variance of jump shocks. The total variance of the stock market is: $\sigma^2 = \sigma_{tot}^2 = \sigma_B^2 + \sigma_J^2$, assuming for simplicity that the two types of shocks are independent. The equity premium is $\pi = \mu - \lambda J$. By writing down the standard value function for the Merton problem, one sees that the optimal equity share, θ , is now the solution of a non-linear equation

$$\pi - \gamma \sigma_B^2 \theta - \lambda J ((1 - \theta J)^{-\gamma} - 1) = 0.$$

For tractability, we use the approximation $J \ll 1$ (which is reasonable, since a typical value for J is .1 to .25). We get the analogue of the simple formula (1):

$$\theta = \frac{\pi}{\gamma \sigma_{tot}^2}.$$

plus higher order terms in J . One can show that formula (22), which was derived in the case of assets with Brownian shocks, carries over to the case of a mix of Brownian shocks and jumps. Thus we get, to a first order:

$$\frac{\hat{\gamma}}{\gamma} = \left(\frac{\sigma_B^2}{\sigma_{tot}^2} \frac{1}{b(D)} + \frac{\sigma_J^2}{\sigma_{tot}^2} \frac{1}{b(0)} \right)^{-1}$$

with $b(0) = 2$ and $\sigma_{tot}^2 = \sigma_B^2 + \sigma_J^2$. Thus, the new bias is the harmonic mean of the $b(D) = 6D$ (if $D \geq 1$) bias for “normal” Brownian shocks, and the shorter $b(0) = 2$ bias of the Brownian shocks.

As a numerical illustration, say a “jump” corresponds to a monthly change in the stock market of more than $J = 25\%$ in absolute value. This corresponds, empirically, to an estimate of $\lambda = .53\%/year$ (5 months since 1925), i.e. a crash every 14 years. Then $\sigma_J^2/\sigma_{tot}^2 = \lambda J^2/\sigma^2 = .014$. Take $D = 4$ quarters as a baseline. The new $\hat{\gamma}/\gamma$ becomes 20.6 which is close to the old ratio of 24.

10 Appendix C: Expression of the bias in the Lynch setup when $D \geq 1$.

In Lynch’s (1996) discrete-time setup, agents consume every month and adjust their portfolio every T months. The econometric observation period is time-aggregated periods of F months, so $D = T/F$.

Say consumer $i \in \{1, \dots, T\}$ adjusts her consumption at $i + nT$, $n \in \mathbb{Z}$. Say the econometrician looks at period $\{1, \dots, F\}$. The aggregate per capita consumption over this period is

$$C_F = \frac{1}{T} \sum_{i=1}^T \sum_{s=1}^F c_{is}. \quad (46)$$

The returns are

$$\ln R_F = \sum_{s=1}^F r_s, \quad (47)$$

where $r_s = \ln R_s$. Call $C_{iF} = \sum_{s=1}^F c_{is}$ the consumption of agent i in the period.

For $i > F$, $cov(C_{iF}, \ln R_F) = 0$ because agent i did not adjust her consumption during the period.

For $1 \leq i \leq F$, $c_{it} = 1 + O(\varepsilon)$ (normalizing) when $t < i$, and $c_{it} = 1 + \theta \sum_{s=1}^i r_s + O(\varepsilon)$ when $t \geq i$, where the $O(\varepsilon)$ terms incorporate the deterministic part of consumption growth. The stochastic part, in r_s , has the order of magnitude $\sigma = O(\varepsilon^{1/2})$, and dominates those terms. Information about stock returns up to i will affect only consumption from time i to F , so, calling ΔC_{iF} the difference in total consumption between a given period of length F and the previous one:

$$\begin{aligned} cov(\Delta C_{iF}, \ln R_F) &= cov \left((F+1-i)\theta \sum_{s=1}^i r_s, \sum_{s=1}^F r_s \right) \\ &= \theta \sigma^2 i (F+1-i) \text{ for } 1 \leq i \leq F. \end{aligned}$$

So:

$$\begin{aligned}
cov(\Delta C_F, \ln R_F) &= \frac{1}{T} \sum_{i=1}^T \theta \sigma^2 i (F+1-i) 1_{1 \leq i \leq F} \\
&= \frac{\theta \sigma^2}{T} \sum_{i=1}^F (F+1) i - i^2 \\
&= \frac{\theta \sigma^2}{T} \left((F+1) \frac{F(F+1)}{2} - \frac{F(F+1)(2F+1)}{6} \right) \\
&= \theta \sigma^2 \frac{F(F+1)(F+2)}{6T}.
\end{aligned}$$

But given that the mean per period consumption $c_{it} = 1 + O(\varepsilon^{1/2})$, the aggregate consumption is $C_F = F + O_{<0}(\varepsilon^{1/2})$, and

$$\begin{aligned}
cov(\Delta C_F/C_F, \ln R_F) &\simeq cov(\Delta C_F, \ln R_F)/F \\
&= \theta \sigma^2 \frac{(F+1)(F+2)}{6T}.
\end{aligned}$$

The naive econometrician would predict $cov(\Delta C_F/C_F, \ln R_F) = \theta \sigma^2 F$. The econometrician estimating $\hat{\gamma} = \pi F / cov(\Delta C_F/C_F, \ln R_F)$ will get a bias of (with $D = T/F$ and as $\theta = \pi / (\gamma \sigma^2)$)

$$\frac{\hat{\gamma}}{\gamma} = D \frac{6F^2}{(F+1)(F+2)}. \tag{48}$$

Holding D constant, the continuous time limit corresponds to $F \rightarrow \infty$, and we find the value: $\hat{\gamma}/\gamma = 6D$. The discrete time case where agents would consume at every econometric period corresponds to $F = 1$, and then one gets $\hat{\gamma}/\gamma = D$, which can be easily derived directly.

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Figure 1: Ratio of estimated γ to true γ

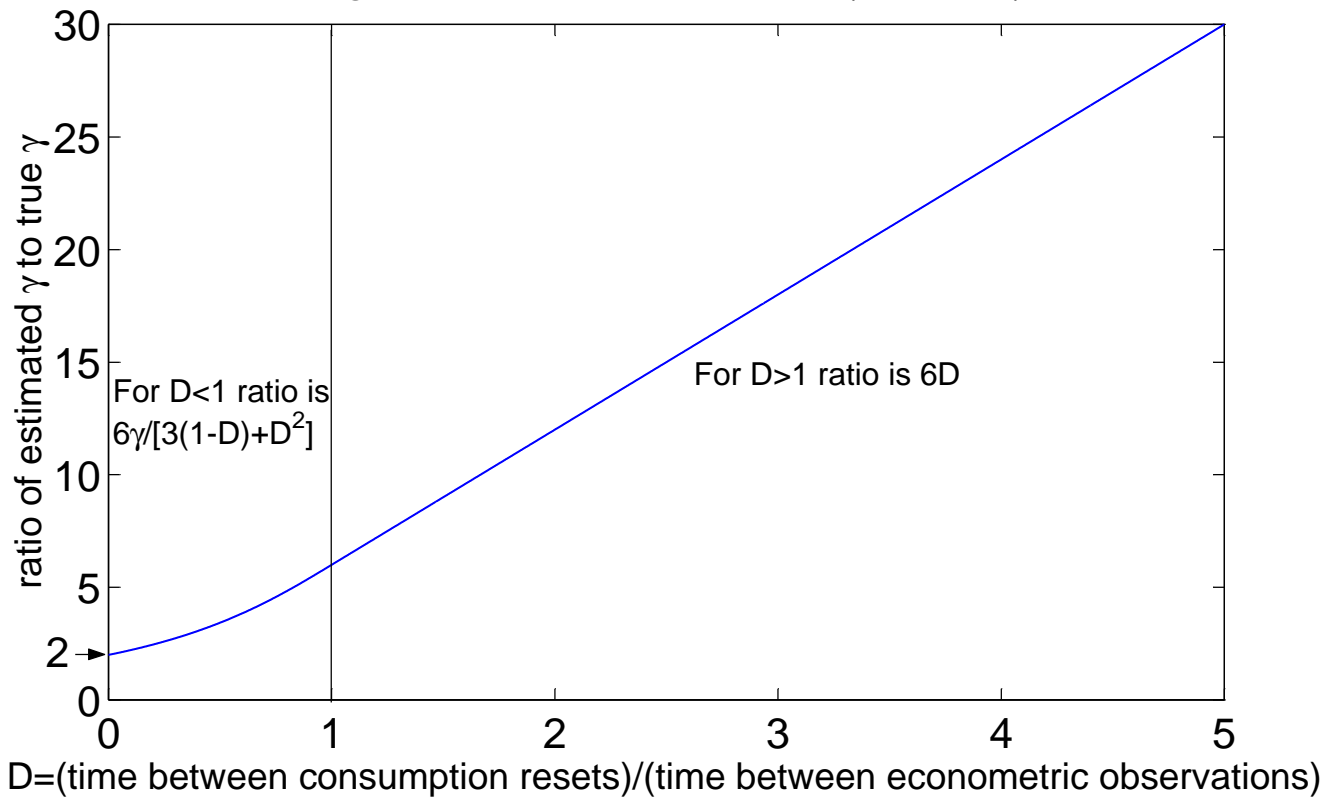


Figure 2: The $d(x)$ and $e(x)$ functions

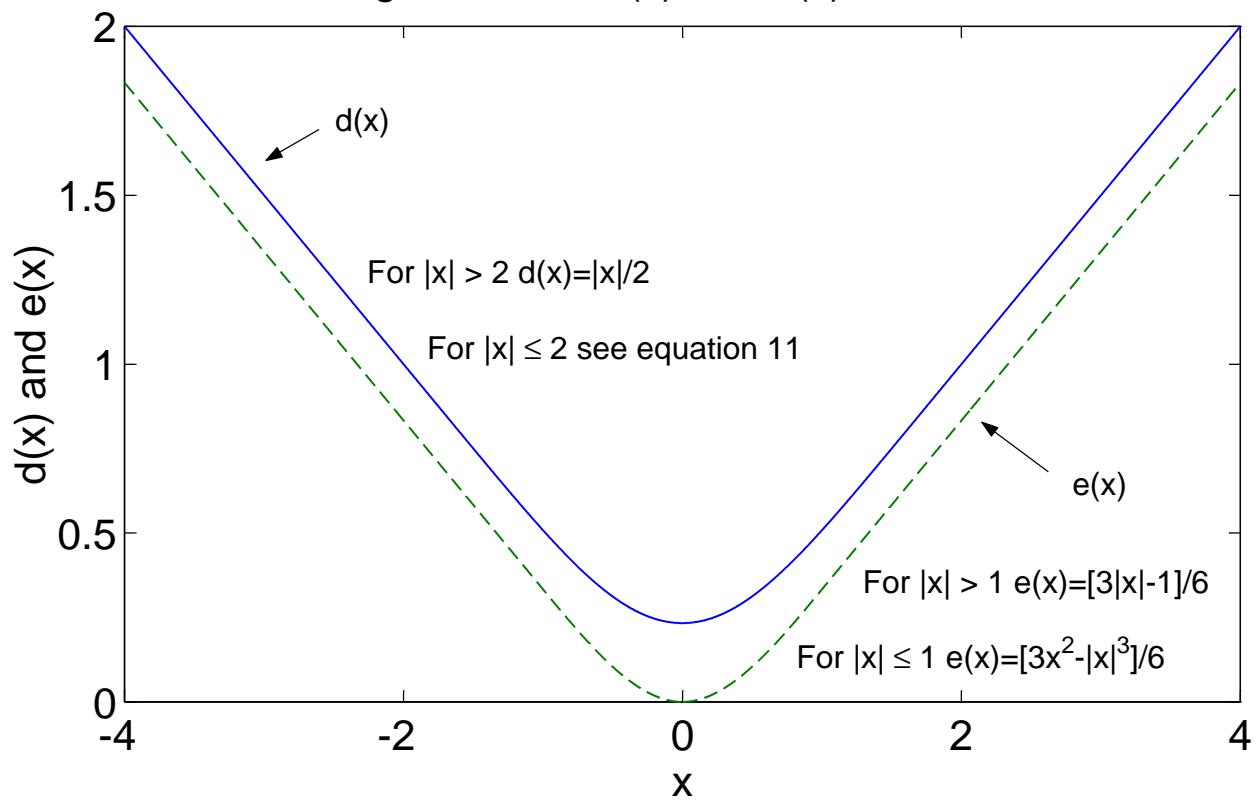


Figure 3: The normalized variance of consumption growth, $\Gamma(D,0)$

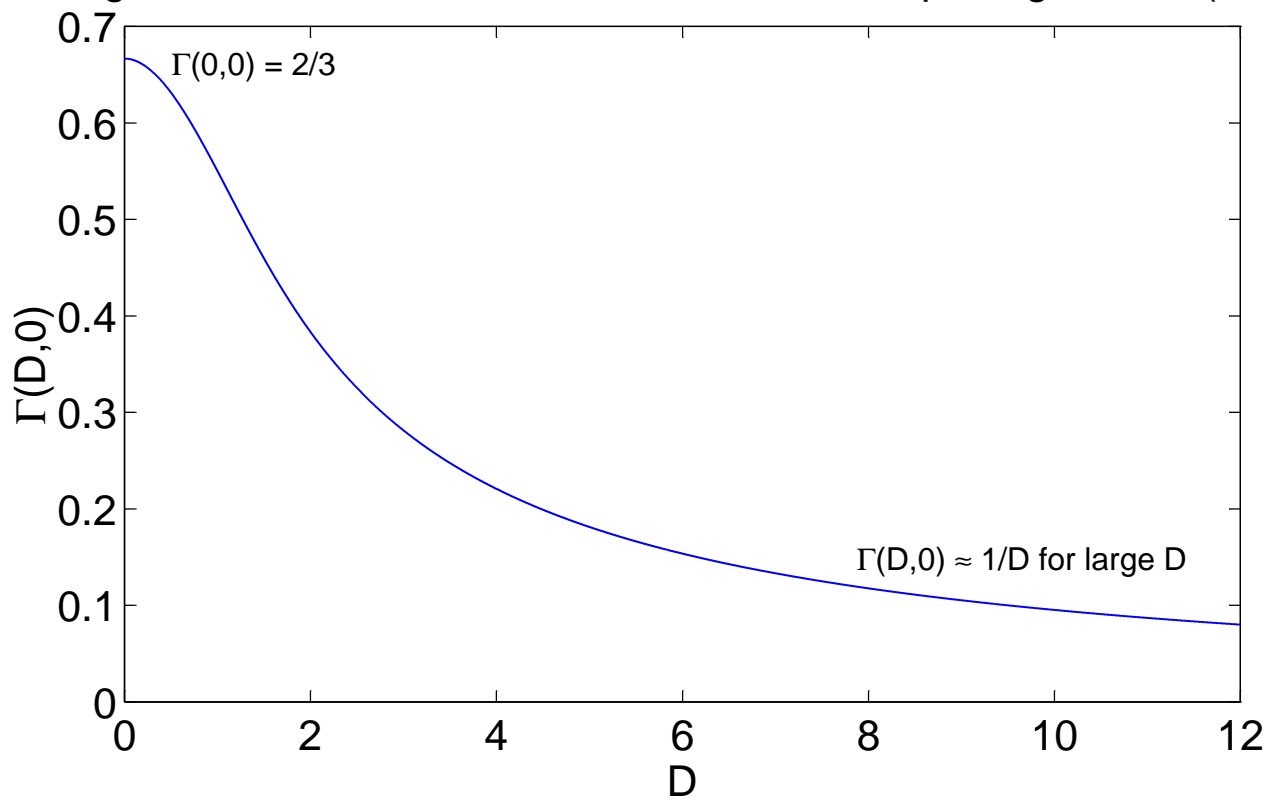


Figure 4: Normalized autocovariance, $\Gamma(D,h)$, with $h = 1, 2, 4, 8$

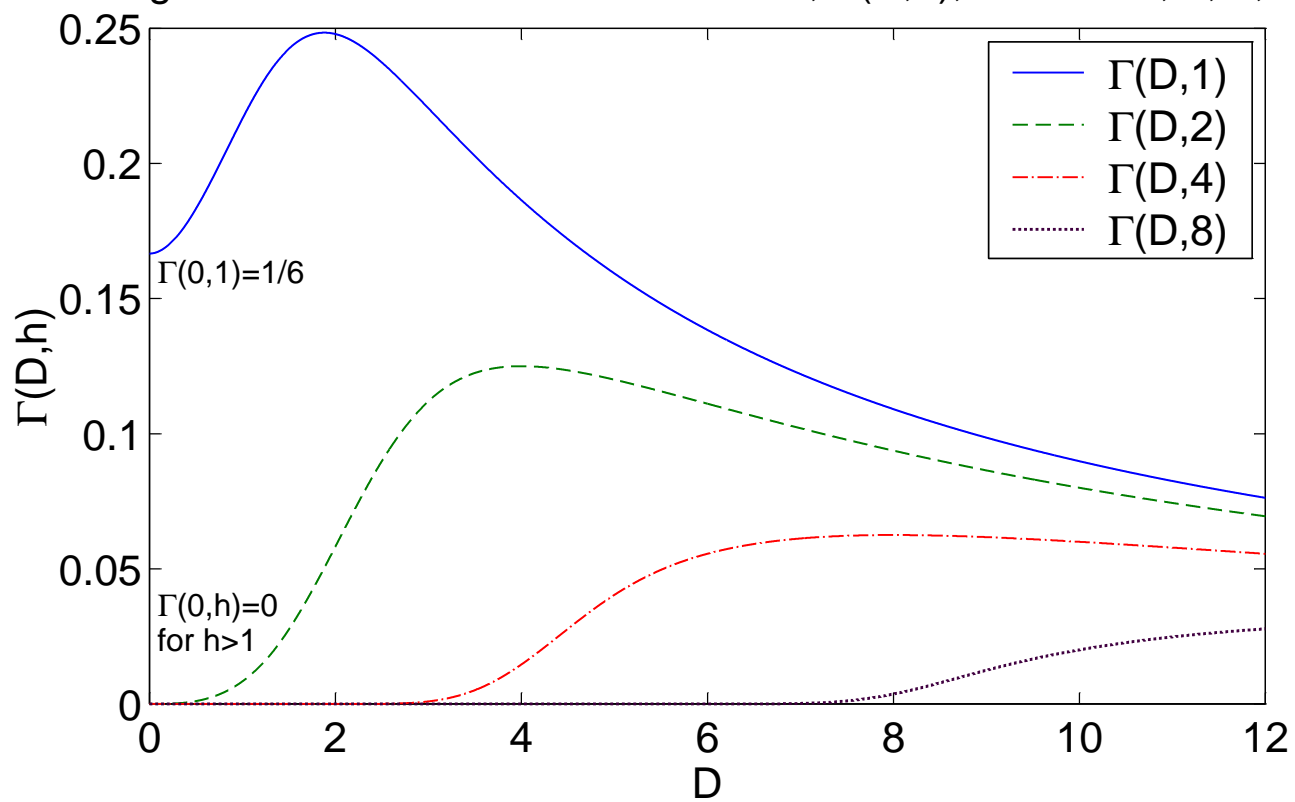


Figure 5: Multiplicative covariance bias factor $1/b(1/h)$

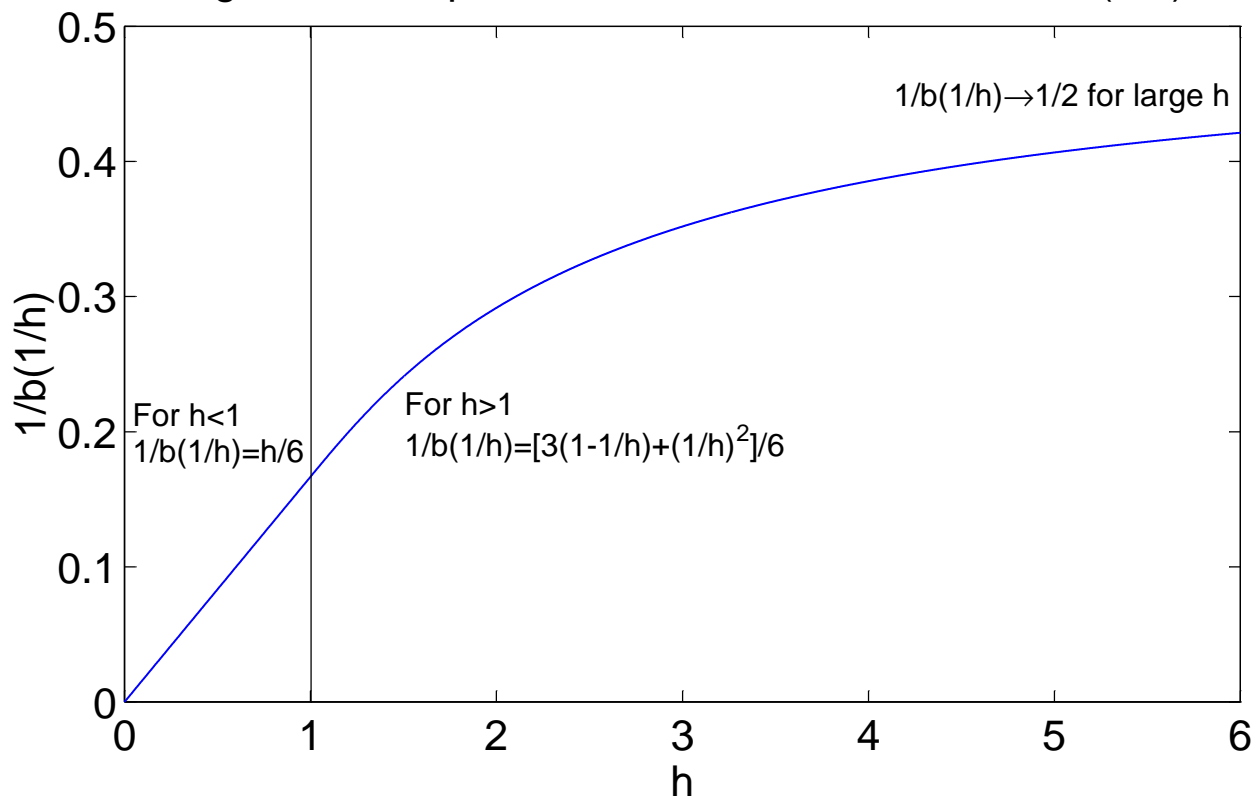


Figure 6: Normalized covariance of consumption growth and lagged asset returns, $V(D,s,s+1)$, for $D = .25, 1, 2, 4$

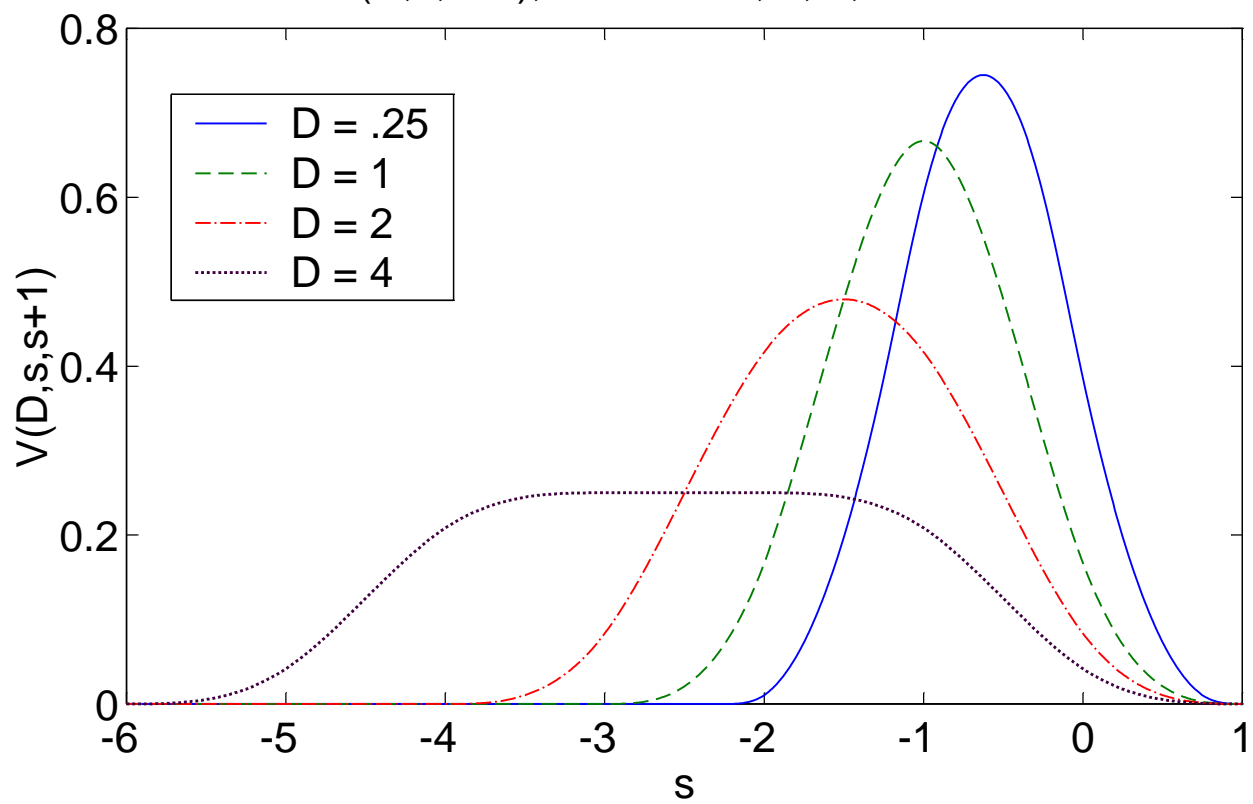
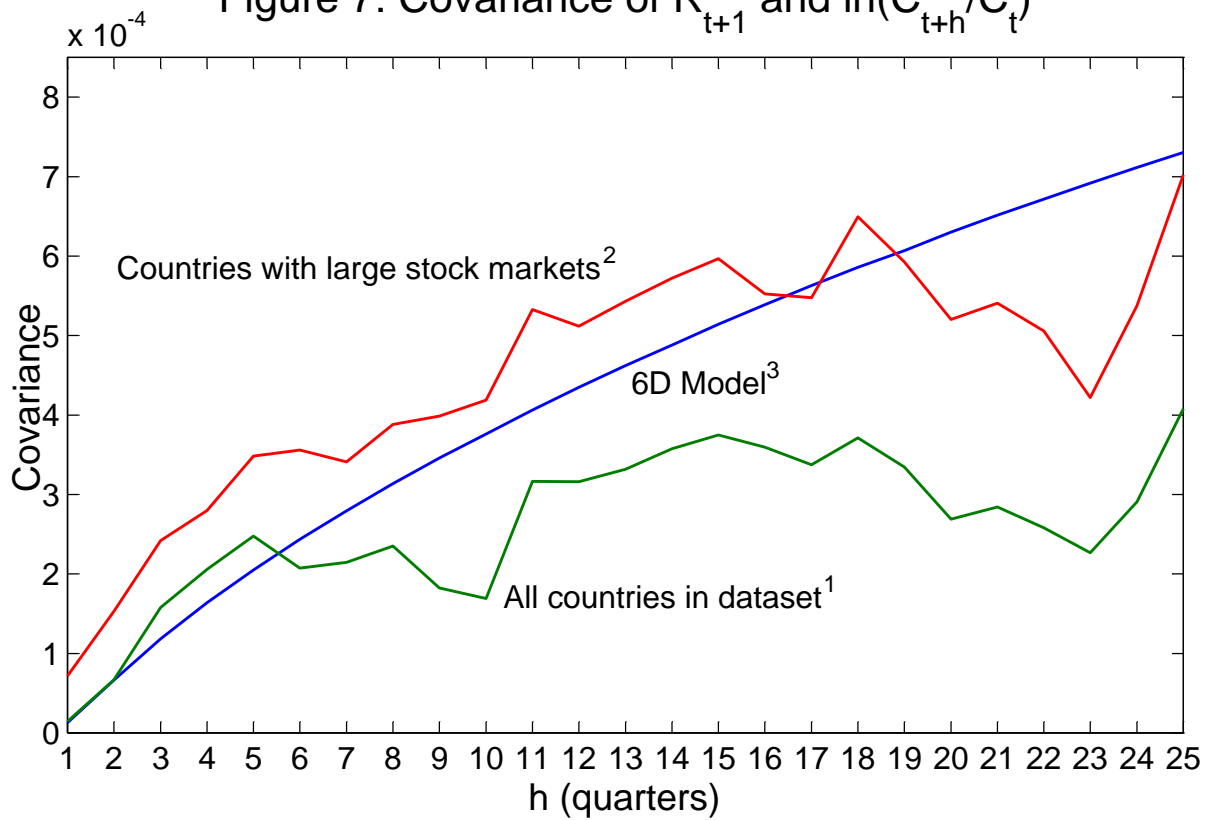


Figure 7: Covariance of R_{t+1} and $\ln(C_{t+h}/C_t)$



¹Dataset is from Campbell (1999). Full dataset includes Australia, Canada, France, Germany, Italy, Japan, Netherlands, Spain, Sweden, Switzerland, United Kingdom, and USA.

²To identify countries with large stock markets, we ordered the countries by the ratio of stock market capitalization to GDP (1993). The top half of the countries were included in our large stock market subsample: Switzerland (.87), United Kingdom (.80), USA (.72), Netherlands (.46), Australia (.42), and Japan (.40).

³We assume that households have D values that are uniformly distributed from 0 years to 30 years.