Transitory shocks to GNP and the consumption-based term structure of interest rates

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January 6, 2002

Abstract

When relative risk aversion is constant, the yield curve should be completely flat if the growth rate of the economy follows an i.i.d. process, i.e., if shocks to GNP are permanent. We examine how the yield curve is affected by the existence of non-persistent shocks to the economy. When consumption exhibits mean reversion, the long-term macroeconomic risk is proportionally smaller than the short-term one. Because less risk entails a reduction in the equilibrium interest rate if the representative agent is prudent, it is intuitive that the presence of mean reversion should make the yield curve increasing, at least for long horizons where mean reversion seems to play a role. We provide a theoretical foundation for an upward-sloping curve when growth reverts to its trend.

1 Introduction

The term structure of interest rate has long been a lively topics for research. Cox, Ingersoll and Ross (1985a,b) have been the first to develop a general equilibrium model to predict the yield curve. Interest rates are determined by the interaction of the demand of liquidity by investors and the supply of liquidity by households who want to save for the future. Both investment decisions and saving decisions are influenced by anticipations of future events. Therefore, the schedule of interest rate provides a rich set of information about these anticipations. For example, when consumers expect an increase in their future incomes, they want to cash this benefit immediately by reducing their saving. This raises the equilibrium interest rate. This wealth effect relies on the standard assumption that consumers want to smooth their consumption over time.

Among the many difficulties to extract testable hypothesis about the relationship between the term structure and anticipations of the future economic activity, the most important one is due to uncertainty. Since Leland (1968), we know that uncertainty about future incomes raises the willingness to save of prudent consumers. This precautionary effect tends to reduce the interest rate. An interesting question is to examine how does the accumulation of risk for longer time horizons influence the determination of the corresponding interest rate. Because longer horizons mean larger expected consumption, people want to save less for these better times. On the contrary, longer horizons also mean more risk, which implies that consumers want to save more for these more uncertain times. Which of these wealth and precautionary effects will dominate the other? If the wealth effect dominates the precautionary effect, then the yield curve is increasing.

The simplest case is when the growth rate of the economy is i.i.d. over time. In this case, both the expected log consumption and its variance increases proportionally with the horizon. It implies that the wealth effect and the precautionary effect exactly compensate each other when the representative agent has a constant relative risk aversion (CRRA). CRRA and i.i.d. growth rates implies that the yield curve is completely flat. Gollier (2001b) examines an economy with i.i.d. growth rates, but without assuming CRRA. He shows that the shape of the yield curve depends upon some properties of up to the fifth derivative of the utility function. It is only when there is no risk of recession that increasing relative risk aversion is sufficient for an upward-sloping yield curve.

The assumption that the growth rate of consumption follows a pure random walk has been questioned in the literature. Campbell and Mankiw (1987) find that U.S. post-World War II quarterly GNP shows essentially no tendency to revert to its trend level after a disturbance, which seems to confirm that there is no serial correlation in the growth rate, or that shocks to GNP are permanent. But Cochrane (1988), on the other hand, provides evidence that there is a strong negative correlation in the growth rates of GNP over a time horizon of more than 10 years. In other words, shocks to GNP would be at least partially temporary. If this is the case, then assuming independence would generate an overestimation of the uncertainty for far distant futures if based on the accumulation of short-term risks. This will overestimate the negative precautionary effect on long-term interest rates. As a consequence, the presence of a negative serial correlation in the growth rate tends to induce an upward-sloping yield curve. This paper provides a solid theoretical foundation to this hypothesis.

There are several other factors than serial correlation in the growth rate that may affect the shape of the yield curve. For example, the anticipation of a deterministic acceleration of the growth of the economy generates an upward sloping yield curve, as tested for example by Estrella and Hardouvelis (1991). The anticipation of a deterministic reduction in the volatility of growth will have the same effect (Barsky (1989)). But those phenomena are temporary. They could not explain why the yield curve is most often upward shaped. On the contrary, we consider in this paper a stationary process for the growth rate.

Section 2 restates the standard Lucas trees economy that is used in this paper. In section 3, we use a second-order approximation to the consumption Euler equation to quantify the effect of serial correlation on the yield curve. The main section of the paper in section 4, where we prove that the presence of negatively correlated growth rates reduces long-term interest rates. Because we consider a general expected utility model that is more general than mean-variance, we need more structure to the statistical relationship of growth rates. We say that growth rates are negatively first-order stochastically correlated if an increase in the current growth rate shifts the distribution of the future growth rate upwards, in the sense of first-order stochastic dominance. In section 5, we examine the effect on long-term interest rates of a negative second-order stochastic correlation in growth rates. Some concluding remarks are presented in section 6.

2 Equilibrium risk-free rates

We consider the standard Lucas (1978) trees economy. There is a representative consumer who maximizes the sum of future expected utility discounted at rate $\delta = \beta^{-1} - 1$. Parameter δ measures the rate of pure preference for the present. It must be constant over time to guarantee the time consistency of the decision process. Let $\widetilde{z}_t$ denote consumption at date t. At date $t = 0$, it is equal to z_0, which is certain. The utility function u on consumption is assumed to be three times differentiable, increasing and concave.

The equilibrium per period rate of return at date $t = 0$ for a zero-coupon bond maturing at date t is denoted $r(t) - 1$. To be in equilibrium, investing marginally in such an asset should leave the expected discounted utility of the representative agent unchanged. This condition is written as

$$r(k)^k \beta^k E u'\left(\widetilde{z}_k\right) = u'(z_0), \tag{1}$$

which is the standard consumption Euler equation. The consumption-based pricing formula is obtained by rewriting condition (1) as

$$r(k)^k = \frac{u'(z_0)}{\beta^k E u'(\widetilde{z}_k)}. \tag{2}$$

It is often useful to decompose $\widetilde{z}_k$ as $z_0 \prod_{t=1}^{k} \widetilde{x}_t$, where $\widetilde{x}_t$ is one plus the growth rate of consumption between date $t-1$ to date t. We hereafter normalize z_0 to unity. We assume that the support of $\widetilde{x}_t$ is bounded below by zero.

Suppose that the representative agent has a constant relative risk aversion (CRRA): $u(z) = z^{1-g}/(1-g)$. Some initial insights about the effect of time horizon on the interest rate can be obtained by assuming CRRA together with $\widetilde{z}_k$ being lognormally distributed. Let

$$\widetilde{y}_k = \log \widetilde{z}_k - \log z_0 = \sum_{t=1}^{k} \log \widetilde{x}_k$$

denote the log of cumulative consumption growth from $t = 0$ to date k. If $\widetilde{y}_t$ is normally distributed with mean $\mu(k)$ and variance $\sigma^2(k)$, it is well-known that

$$E\left(\widetilde{z}_k\right)^{-g} = E \exp\left(-g \widetilde{y}_k\right) = \exp\left(-g\left[\mu(k) - 0.5 g \sigma^2(k)\right]\right).$$

This implies in turn that

$$\log r(k) = -\log \beta + g \left[\frac{\mu(k)}{k} - 0.5g\frac{\sigma^2(k)}{k} \right]. \tag{3}$$

Because $\mu(k)$ is equal to $\log E\widetilde{z}_k - 0.5\sigma^2(k)$, this can also be written as

$$\log r(k) = -\log \beta + g \left[\frac{\log E\widetilde{z}_k}{k} - 0.5(g+1)\frac{Var(\log \widetilde{z}_k)}{k} \right]. \tag{4}$$

This formula is for example in Carroll (2001) and Gollier (2001a). If we don't assume that future wealth is lognormally distributed, the above formula is equivalent to a second-order Taylor approximation of the consumption Euler equation. It provides a better approximation than the more standard log-linearized Euler equation, which yields $\log r(k) \simeq -\log \beta + g \log E\widetilde{z}_k/k$. Term $(g \log E\widetilde{z}_k)/k$ represents the effect of the willingness to smooth consumption over time. It is increasing in the expected growth $E\widetilde{z}_k$, and in the degree g of relative aversion to consumption fluctuations over time. The second term expresses prudence. When the uncertainty on future incomes increases, prudent consumers want to save more. Following Kimball (1990), one can define an equivalent sure reduction in future income that has the same effect on savings than the increase in future risk. This precautionary premium is proportional to the degree of relative prudence $g+1 = -zu'''(z)/u''(z)$. The existence of this precautionary premium associated to future incomes reduces the equilibrium interest rate. This precautionary premium is increasing in risk, and with the relative degree of prudence $g+1$. The effect on savings of this equivalent sure reduction of future incomes is proportional to the relative degree g of aversion to consumption fluctuations over time. Therefore, the effect of risk on the equilibrium risk-free rate is proportional to $g(g+1)$. More intuition on this is provided in Gollier (2001a).

3 Some preliminary results

We now examine the term structure of interest rates. The benchmark case exhibits constant relative risk aversion with a pure random walk for consumption: growth rates $\widetilde{x}_1, \widetilde{x}_2, ...$ are assumed to be independent and identically distributed random variables. This is a situation where shocks to consumption are permanent. In such an economy, the asset pricing formula (2) can be written as

$$\beta r(k) = \left[E\left(\prod_{t=1}^{k} \widetilde{x}_t \right)^{-g} \right]^{-1/k} = \left[E\left(\widetilde{x}_1 \right)^{-g} \right]^{-1}, \tag{5}$$

which implies that the yield curve is completely flat in this case: $r(k)$ is constant. This can also be seen from approximation (3), where

$$\frac{\mu(k)}{k} = k^{-1} E \log \left(\prod_{t=1}^{k} \widetilde{x}_t \right) = E \log \widetilde{x}_1,$$

and

$$\frac{\sigma^2(k)}{k} = k^{-1} Var \left(\log \left(\prod_{t=1}^{k} \widetilde{x}_t \right) \right) = Var \left(\log \widetilde{x}_1 \right).$$

These two equations means that the expected growth of the economy and its variance are both proportional to time horizon. Thus, considering a longer time horizon has the *same* effect on the *total* equilibrium risk-free return than a proportional increase in the expected growth of the economy and in its variance. These increases generate a positive consumption-smoothing effect and a negative precautionary effect, both proportional to k. The total effect is thus also proportional to time horizon. These changes have thus no effect on the *per period* risk-free rate $r(k)$. In short, when relative risk aversion is constant and when shocks on consumption are permanent, longer time horizons generate a consumption smoothing effect and a precautionary effect on $r(k)$ that neutralize each other. Gollier (2001b) characterizes utility functions that yield an increasing or decreasing yield curve when consumption follows such an i.i.d. process. The present paper considers the other road which is to relax the assumption that shocks on consumption are permanent.

In what follows, we assume that the unconditional expectation of the log of the growth rate per period is time-independent. Otherwise, the expectation of an increase in growth in the future would raise the long-term interest rate in an obvious way. This would generate an upward sloping yield curve. We exclude this possibility by assuming that $E \log \widetilde{x}_t$ is time-independent, which implies that $\mu(k)/k$ is a constant $\mu = E \log \widetilde{x}_1$.

Following Cochrane (1988) and Cogley (1990), let us define the variance ratio as

$$V(k) = \frac{k^{-1} Var(\log \widetilde{z}_k - \log z_0)}{Var(\log \widetilde{z}_1 - \log z_0)} = \frac{k^{-1}\sigma^2(k)}{\sigma^2(1)}.$$

The variance ratio associated to time horizon k is proportional to the variance of cumulative growth at horizon k divided by the variance of one year growth. It measures an equivalent *per period* risk associated to various time horizons. In the case of serially independent growth rates, this variance ratio is uniformly equal to unity. A decreasing V means that the equivalent per period risk becomes smaller for longer time horizons. If V tends to zero as k tends to infinity, this means that shocks on GNP are only temporary.

With this ratio, we can rewrite approximation (3) as follows:

$$r(k) - \delta \simeq g\left[\mu - 0.5 g \sigma^2 V(k)\right], \tag{6}$$

where $\sigma^2 = \sigma^2(1)$ is the variance of the 1-period log growth. This equation tells us that the shape of the yield curve is strongly related to the shape of the variance ratio V. An increasing V implies a downward-sloping yield curve. The intuition is that when V is increasing, longer horizon means more risk. Prudent consumers will therefore want to raise their saving targeted for the long term. At equilibrium, this induces a reduction of the corresponding interest rate. Cochrane (1988) estimated $V(k)$ for $k = 1, ..., 30$ by using data on the log real per capita GNP in the United States, 1869-1986. Figure 1 summarizes his estimates. The per period risk attached to time horizons less than 3 years is increasing. This comes from the positive serial correlation of growth at high frequency. On the contrary, V is decreasing in k for time horizons longer than 3 years. It tends to roughly one-third. Long horizons entail only one-third per period risk than short horizons, when risk is measured by the variance of consumption. This means that shocks to U.S. GNP are mostly temporary. Cochrane estimated the standard deviation of the per period consumption to $\sigma = 6.1\%$.

It is then straightforward to calibrate equation (6) to predict the shape of the yield curve. Let us fix the expected growth rate of the economy to $\mu = 1.8\%$ per year, which is the average growth rate of real per capita consumption in the United States over the period 1889-1978 (Kocherlakota (1996)). In Figure 2, we draw the yield curve $r(k) - \delta$ computed from equation (6) for four different degrees of relative risk aversion: $g = 1, 2, 4$ and 6. The yield curve is decreasing for small horizons to reflect the increasing per period risk. It is then increasing because of the increasing per period risk that longer horizons entail. This horizon effect originates from the prudent behavior of the representative agent. We know that this precautionary effect is proportional to g^2. Thus, it is strong only for large degrees of risk aversion.

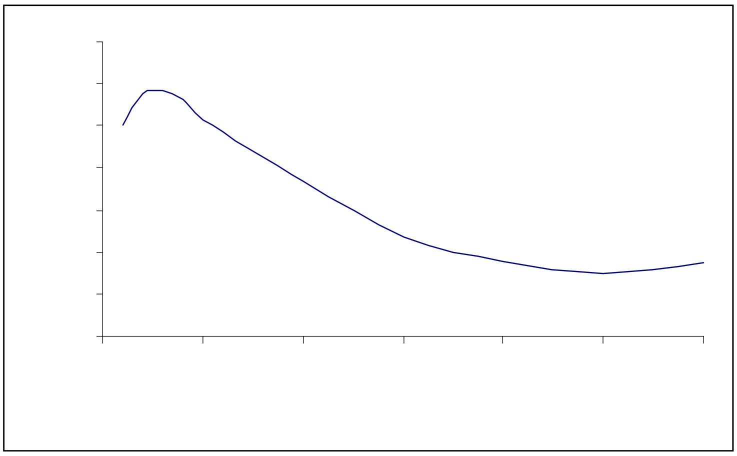

Figure 1: The variance ratio for the log real per capita GNP, 1869-1986. (Source: Cochrane (1988)).

Because $V(k)$ is less than unity for distant horizons, the long-term interest rate is larger than the short-term one, i.e., the yield curve is globally increasing. Notice that because parameter δ is hard to estimate, the level of the yield curve is not determined. Only its shape is relevant.

Cogley (1990) showed that the pattern of the variance ratio exhibits much difference across countries. In fact, the evidence indicates that the relative stability of long-term growth is unique to the United States. Using annual real per capita GDP growth, 1871-1985, he computed the variance ratio $V(20)$ for a twenty years horizon. He found 0.77 for Canada, which means that, as in the U.S., this country should experience a globally increasing yield curve, but with a smaller slope. He also found 0.97 for Sweden, 1.03 for the United Kingdom, and 1.09 for Denmark. The yield curve should be almost flat in these countries. But he also obtained 1.4 for Australia, 1.84 for France and 2.02 for Italy. In these countries, the per-period growth risk is decreasing with time horizon. It implies that the long-term interest rate should be smaller than the short-term one. For France, using Maddison (1991), we estimated $\mu = 1.97\%$ and $\sigma = 8.05\%$. For $g = 2$, it makes a risk-free rate $r(k)$ equaling $\delta + 1.32\%$ for the short term, and $\delta + 0.78\%$ for the long run. For $g = 4$, it generates $\delta + 0.67\%$ and $\delta - 0.41\%$ respectively for the short term and for the long term.

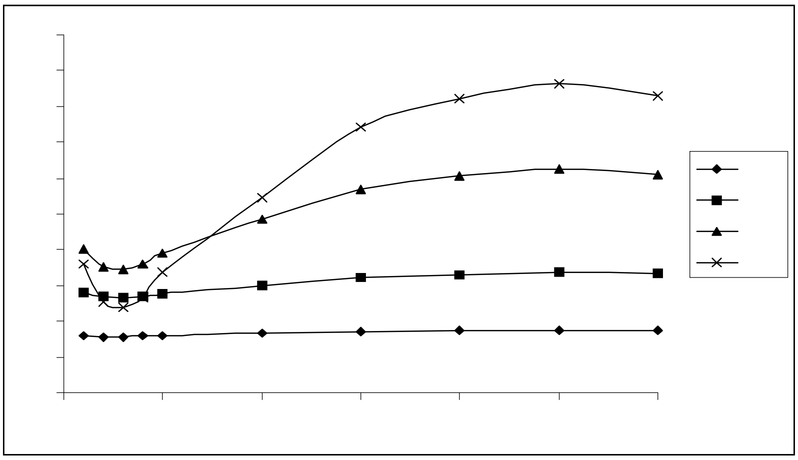

Figure 2: The risk-free rate $r(k) - \delta$ as a function of time horizon k.

4 First-order stochastic correlation in growth rates

In the previous section, we used approximations to estimate the effect of temporary shocks to GNP on the shape of the yield curve. Our purpose there was mostly illustrative. We suggested that a negative serial correlation $(V(2) < V(1))$ in the growth rate of the economy tends to raise the long term rate above the short term rate. More specifically, we want to understand the relationship that exists between the long term interest rate and the degree of serial correlation in the growth of the economy. We want to do this without relying on second-order approximations. Carroll (2001) provides good arguments for why the second-order approximation (??) should not be taken too seriously.

We hereafter compare the interest rates for respectively a one-period (short-term) horizon and a two-period (long-term) horizon. Observe that the variance ratio for the two-period horizon is such that

$$V(2) = \frac{Var(\log \widetilde{x}_1 + \log \widetilde{x}_2)}{2Var(\log \widetilde{x}_1)} = 0.5\left[1 + \frac{Var(\log \widetilde{x}_2)}{Var(\log \widetilde{x}_1)}\right] + \rho,$$

where ρ is the index of correlation between the first period log growth and the second period log growth. We see that a variance ratio smaller than unity can come from two different phenomena. The first one would be a deterministic downward trend in the volatility of consumption: $Var(\log \widetilde{x}_2) < Var(\log \widetilde{x}_1)$. We hereafter assume that the process is stationary, which excludes this possibility. The second possible phenomenon is the presence of a negative serial correlation in the growth rate of the economy.

In the general expected utility model, one need to get more structure to the underlying stochastic process. The sign of the covariance usually is not enough, except in the special case of mean-variance. In what follows, we will consider two forms of statistical relationship between $\widetilde{x}_1$ and $\widetilde{x}_2$: first–order stochastic correlation (FSC) and second-order stochastic correlation (SSC). This section is devoted to FSC.

Consider two random variables $\widetilde{x}_1$ and $\widetilde{x}_2$. G denotes the cumulative distribution function of $\widetilde{x}_1$: $G(x) = \Pr[\widetilde{x}_1 < x]$. Let F be the cumulative distribution of $\widetilde{x}_2$ conditional to x_1: $F(x_2 \mid x_1) = \Pr[\widetilde{x}_2 < x_2 \mid x_1]$. We propose the following definition.

Definition 1 *Consider a pair of random variables $(\widetilde{x}_1, \widetilde{x}_2)$ with marginal cdf $G(.)$ for $\widetilde{x}_1$ and conditional cdf $F(. \mid .)$ for $\widetilde{x}_2$. We say that there is a positive FSC correlation between $\widetilde{x}_1$ and $\widetilde{x}_2$ if F is decreasing in x_1 for all x_2.*

In other words, an increase in x_1 generates a first-order stochastic dominant shift in the conditional distribution of $\widetilde{x}_2$. Figure 3 illustrates this statistic relation between $\widetilde{x}_1$ and $\widetilde{x}_2$. This is a strong form of positive correlation. Milgrom (1981) uses this concept to define the notion of a good news. An example of stochastic process that satisfies the FSC property is an AR(1): $\widetilde{x}_2 = kx_1 + \widetilde{\varepsilon}$. An economic growth process that reverts to its secular trend is compatible with a negative FSC correlation of the growth rates.

Consider an economy characterized by a serially correlated process $(\widetilde{x}_1, \widetilde{x}_2)$ for per-period growth rates. The long-term interest rate in such an economy equals

$$r(2) = \beta^{-1} \left[Eu'(\widetilde{x}_1 \widetilde{x}_2) \right]^{-1}.$$

We want to compare this rate to the one that would prevail in an economy with the same marginal distributions for $\widetilde{x}_1$ and $\widetilde{x}_2$, but with no serial correlation.

Definition 2 *Consider a pair of random variables $(\widetilde{x}_1, \widetilde{x}_2)$ with cdf $(G(.), F(. \mid .))$. We say that $(\widetilde{y}_1, \widetilde{y}_2)$ is the corresponding decorrelated pair of $(\widetilde{x}_1, \widetilde{x}_2)$ if*

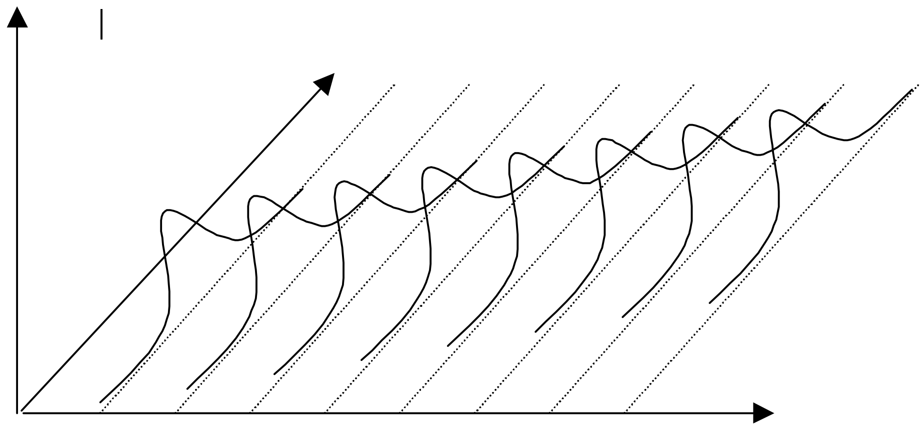

Figure 3: $\widetilde{x}_2$ is positively first-order stochastically correlated to $\widetilde{x}_1$.

$\widetilde{y}_1$ and $\widetilde{y}_2$ are independent and have the same marginal distributions than $(\widetilde{x}_1, \widetilde{x}_2)$: $\widetilde{y}_1$ is distributed as G and $\widetilde{y}_2$ is distributed as $\widehat{F}$, with $\widehat{F}(x_2) = \int F(x_2 \mid x_1) dG(x_1)$.

None of the marginal moments of the two random variables $\widetilde{x}_1$ and $\widetilde{x}_2$ is affected by our decorrelation technique. In such a decorrelated economy, the long-term interest rate would equal

$$\widehat{r}(2) = \beta^{-1} \left[Eu'(\widetilde{y}_1 \widetilde{y}_2) \right]^{-1}.$$

Interest rate $\widehat{r}(2)$ would be what one would obtain from the calibrated model by assuming independence and by using the observed standard deviation of annual growth rates as the estimation of the variance of $\widetilde{x}_2$. We want to determine conditions under which $r(2)$ is smaller than $\widehat{r}(2)$, when $\widetilde{x}_2$ exhibits positive FSC correlation with respect to $\widetilde{x}_1$. This would be true if

$$Eu'(\widetilde{x}_1 \widetilde{x}_2) \geq Eu'(\widetilde{y}_1 \widetilde{y}_2). \tag{7}$$

The following Lemma is useful to examine this problem.

Lemma 3 *Consider a differentiable function h defined in a domain of R^2, together with any pair of random variables $(\widetilde{x}_1, \widetilde{x}_2)$ that satisfies positive first-order correlation and whose support is in the domain of h. Let $(\widetilde{y}_1, \widetilde{y}_2)$ denote* *the decorrelated pair associated to* $(\widetilde{x}_1, \widetilde{x}_2)$*. Then,*

$$Eh(\widetilde{x}_1, \widetilde{x}_2) \geq Eh(\widetilde{y}_1, \widetilde{y}_2) \tag{8}$$

if and only if h *is supermodular, i.e., if* $\partial h/\partial x_2$ *is increasing in* x_1*.*

Proof: See the appendix.[1]

Applying this lemma to condition (7) requires using function $h(x_1, x_2) = u'(x_1 x_2)$. It is supermodular if $x_1 u''(x_1 x_2)$ is increasing in x_1, or if

$$P^r(z) = \frac{zu'''(z)}{-u''(z)} \geq 1$$

for all z in the domain of u. $P^r(z)$ is the degree of relative prudence evaluated at consumption level z.

Proposition 4 *The presence of positive first-order stochastic correlation in the growth rate of the economy reduces the long-term risk-free rate if relative prudence is uniformly larger than unity. Otherwise, it is possible to find a positive FSC process that raises the long-term risk-free rate.*

Similarly, the long-term risk-free rate is increased by negative FSC correlation if relative risk aversion is larger than unity. Observe that this Proposition is in line with our simulations of the previous section. Indeed, we assumed there that relative risk aversion g was constant ($u(z) = z^{1-g}/(1-g)$). With such a utility function, relative prudence equals a constant $P^r(z) = g + 1$, which must be larger than unity for the representative agent to be risk-averse ($g > 0$). Thus, when relative risk aversion is constant, positive (negative) FSC correlation always reduces (raises) the long-term risk-free rate.

It is easy to exhibit utility functions that are concave but whose relative prudence is not larger than unity. For example, the simplest departure of CRRA with $u(z) = (z + k)^{1-g}/(1-g)$, $k > 0$, implies a relative prudence $P^r(z) = (1 + g)z/(z + k)$. For such a concave utility function, relative prudence tends to zero with z. At early stages of its development, this economy may have an upward sloping yield curve even if growth rates are positively FSC correlated. How is this possible? After all, our intuition that a positive correlation in growth rates has a negative effect on long-term interest

[1] We can also prove that if h is supermodular, then condition (8) is satisfied if and only if $(\widetilde{x}_1, \widetilde{x}_2)$ exhibits positive FSC.

rate comes from the precautionary effect. Which opposite effect comes into the picture that requires that the precautionary effect be strong enough to guarantee the result? This second effect comes from an implicit wealth effect.

Observe that in spite of the fact that the decorrelation does not affect the expected *per-period* growth rate, the expected *cumulative* growth rate is increased by the presence of positive FSC correlation. This can be checked by using function $h(x_1, x_2) = x_1 x_2$ in the Lemma. The intuition is that the positive FSC reduces (raises) the expected second period growth rate when the first-period growth rate is small (large). In expectation, this raises the cumulative growth rate. This implicit increase in expected future incomes reduces the willingness to save for the long term, and it requires an increase in the corresponding interest rate. Therefore, one needs a sufficiently strong precautionary effect to dominate this opposite wealth effect.

To sum up, this section confirmed the intuition obtained earlier in the paper. We showed here that there is some difficulty to define properly the comparative statics exercise. When addressing the question of the effect of correlated growth rates, one needs to determine which ceteris paribus assumption to consider. In definition 2, we extended in a straightfoward way what we did in the calibration exercise of section 3. More precisely, we maintained the expected per period growth unchanged, and we examined the effect of adding some correlation in the growth rates (thereby affecting $V(2)$). An alternative technique would have been to maintain the expected cumulative growth rate unchanged, but that would have force us to modify the expected per period growth rate. None of these two methods is totally satisfactory.

This problem can be overcome by considering another type of restriction on the growth process. Let us define $\widetilde{x}_t$ as the dollar increase in consumption from date $t-1$ to date t, which implies that $\widetilde{z}_k = z_0 + \Sigma_{t=1}^{k} \widetilde{x}_t$. We compare the term structure of two economies. The first economy has permanent shocks to its GNP. Its $\widetilde{x}_t$ are independent. In the second economy, shocks either have a temporary component ($\widetilde{x}_2$ is negatively FSC to $\widetilde{x}_1$), or are unstable ($\widetilde{x}_2$ is positively FSC to $\widetilde{x}_1$). Using the Lemma 1 with function $h(x_1, x_2) = u'(x_1 + x_2)$ directly yields the following result.

Proposition 5 *The presence of positive (negative) first-order stochastic correlation in the growth level* $\widetilde{z}_t - z_{t-1}$ *of the economy reduces (raises) the long-term risk-free rate if relative prudence is nonnegative.*

In other words, the fact that shocks to the economy have a temporary component has a negative impact on long-term interest rates, as suggested

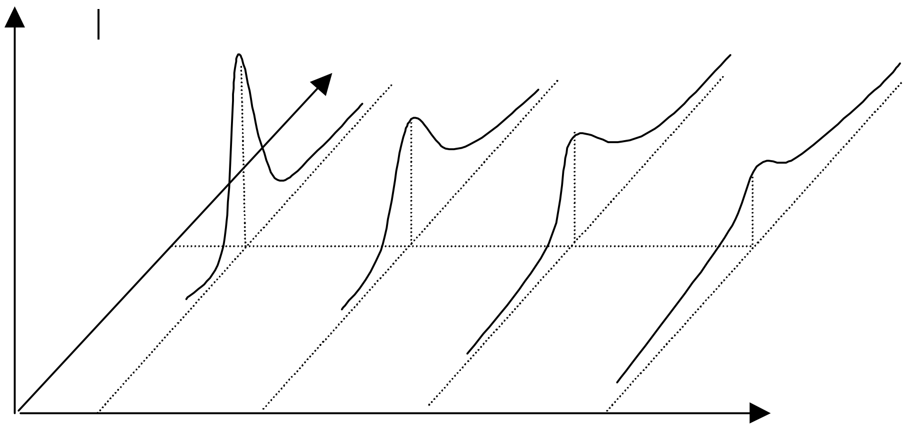

Figure 4: $\widetilde{x}_2$ is positively second-order stochastically correlated to $\widetilde{x}_1$.

by the intuition. When absolute growth rather than relative growth is decorrelated according to Definition 2, the expected cumulative growth remains unaffected. Therefore, only the precautionary effect is at play to determine the effect of such serial correlations in shocks to the economy.

5 Second-order stochastic correlation in growth rates

A natural extension of this work is to examine economies where growth rates $\widetilde{x}_1$ and $\widetilde{x}_2$ are statistically related according to the positive second-order stochastic correlation (SSC) property. We consider economies where an increase in the first period growth rate reduces the risk associated to the second period growth rate in the sense of Rothschild and Stiglitz (1970). In other words, the volatility of the economy is increased after a boom, as in Figure 4. An example of such process is $\widetilde{x}_2 = \mu + x_1\widetilde{\varepsilon}$, with $E\widetilde{\varepsilon} = 0$ and $\widetilde{\varepsilon}$ independent of $\widetilde{x}_1$.[2]

[2] Cochrane (2001, section 19.4) considered a simple term structure model with CRRA preferences and an AR(1) process for the log of consumption growth: $\widetilde{x}_2 \mid x_1 = x_1^{\rho}\widetilde{\varepsilon}$,with $\widetilde{\varepsilon}$ independent of $\widetilde{x}_1$. This yields some form of negative SSC.

Definition 6 *Consider a pair of random variables* $(\widetilde{x}_1, \widetilde{x}_2)$ *with marginal cdf* $G(.)$ *for* $\widetilde{x}_1$ *and conditional cdf* $F(. \mid .)$ *for* $\widetilde{x}_2$*. We say that there is a positive SSC correlation between* $\widetilde{x}_1$ *and* $\widetilde{x}_2$ *if* $q(x_2 \mid x_1) = \int^{x_2} F(y \mid x_1)dy$ *is decreasing in* x_1 *for all* x_2*, and if* $E\left[\widetilde{x}_2 \mid x_1\right]$ *is independent of* x_1*.*

We want to determine the effect of such statistical relationship in growth rates over time on the long-term interest rate. As in the previous section, we compare an economy $(\widetilde{x}_1, \widetilde{x}_2)$ with positive SSC with another one $(\widetilde{y}_1, \widetilde{y}_2)$ in which growth rates have been decorrelated according to definition 2. The following Lemma is helpful to solve this problem.

Lemma 7 *Consider a twice differentiable function* h *defined in a domain of* R^2*, together with any pair of random variables* $(\widetilde{x}_1, \widetilde{x}_2)$ *that satisfies positive second-order correlation and whose support is in the domain of* h*. Let* $(\widetilde{y}_1, \widetilde{y}_2)$ *denote the decorrelated pair associated to* $(\widetilde{x}_1, \widetilde{x}_2)$*. Then,*

$$Eh(\widetilde{x}_1, \widetilde{x}_2) \geq Eh(\widetilde{y}_1, \widetilde{y}_2)$$

if and only if $-\partial h/\partial x_2$ *is supermodular, i.e., if* $\partial^2 h/\partial x_2^2$ *is decreasing in* x_1*.*

Proof: See the appendix.

Applying this to the term structure with $r(2) = 1/\beta Eu'(\widetilde{x}_1\widetilde{x}_2)$ and $h(x_1, x_2) = u'(x_1 x_2)$, we obtain the following Proposition. It relies on the relative degree of temperance $T^r(z) = -zu''''(z)/u'''(z)$, which must be uniformly larger than 2 to satisfy the condition that -$\partial h/\partial x_2$ be supermodular.

Proposition 8 *The presence of positive second-order stochastic correlation in the growth rate of the economy reduces the long-term risk-free rate if relative temperance* $-zu''''(z)/u'''(z)$ *is uniformly larger than two. Otherwise, it is possible to find a positive SSC process that raises the long-term risk-free rate.*

Notice that in the special case of constant relative risk aversion g, relative temperance is also a constant equaling $2+g$. In consequence, if the risk-averse representative agent has a CRRA utility function, second-order correlation always reduces the long-term interest rate. Cox, Ingersoll and Ross (1985a,b) considered a model where higher growth rates are more volatile, which is compatible with a *negative* second-order stoachastic correlation. The above proposition can thus explain why this kind of term structure models produce a positively sloped yield curve.

We can also apply Lemma 2 by defining the growth process in an additive way: $\widetilde{z}_k = z_0 + \Sigma_{t=1}^{k} \widetilde{x}_t$. It yields the following result.

Proposition 9 *The presence of positive (negative) second-order stochastic correlation in the growth level* $\widetilde{z}_t - z_{t-1}$ *of the economy reduces (raises) the long-term risk-free rate if temperance is positive, i.e., if* u'''' *is uniformly negative.*

It is interesting to compare the different moments of the cumulative growth in the two economies $(\widetilde{x}_1, \widetilde{x}_2)$ and $(\widetilde{y}_1, \widetilde{y}_2)$ in the presence of a positive SSC in the $\widetilde{z}_t - z_{t-1}$. Both the mean and the variance of $\widetilde{x}_1 + \widetilde{x}_2$ and $\widetilde{y}_1 + \widetilde{y}_2$ are the same, as shown by using Lemma 2 with $h(x_1, x_2) = x_1 + x_2$ or $(x_1 + x_2)^2$. Thus, there is no consumption smoothing effect and no precautionary effect. Only the third moment is affected. More precisely, the positive SSC correlation reduces the skewness of the cumulative growth, as seen by using function $h(x_1, x_2) = (x_1 + x_2)^3$. This reduction in the third moment of $\widetilde{z}_2$ raises the willingness to save measured by $Eu'(\widetilde{z}_2)$ if u'''' is negative. This reduces the equilibrium long-term interest rate.

6 Conclusion

A correct assessment of how much Society should invest for its own future is central to economic analysis. Many of us are now cooperating with various organizations to analyze environmental projects whose costs and benefits are spreads over hundreds of years, in particular those linked to global warming and nuclear waste disposals. We know that the most important parameter when using cost-benefit analysis for such long-lasting projects is by far the discount rate. We as a profession have not been very good in proposing an agreed-upon discount rate for the long term. Weitzman (2001) asked to more than 2000 professional Ph.D.-level economists about their own recommendation for the discount rate to be used for far distant real cash-flows. He reported a sample mean at around 4% per year, which is quite larger than the secular real short-term interest rate of 1% (Kocherlakota (1996)). Economists seems to favor an upward-sloping discount yield curve.

This paper can justify this recommendation on the basis of the existence of a negative serial correlation in the growth rate of the economy over many years. If growth rates tend to revert to the mean over long periods, then long-term risks are proportionally smaller than short-term ones. This makes the negative precautionary effect less important for long-term interest rate, thereby generating an upward-sloping yield curve. However, as shown by Cogley (1990), the serial correlation of growth rates seems to be negative only for the United States. On the contrary, for countries like Australia, France and Italy, long-term risks appear to be proportionally larger than short-term ones, which revert the recommendation.

Our calibrations suggest that the existence of correlated growth rates has a sizeable effect on the shape of the yield curve, in particular for larger degrees of relative risk aversion. Technically, we showed that the sign of the effect depends upon whether relative prudence is larger or smaller than unity, with the intuitive effect when it is larger than unity. This condition is satisfied if , for example, we believe in the hypothesis that relative risk aversion is approximately constant with respect to consumption.

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Proof of Lemma 1

Define function K as: $K(x_2 \mid x_1) = F(x_2 \mid x_1) - \widehat{F}(x_2)$. For sufficiency, we need to prove that

$$X = Eh(\widetilde{x}_1, \widetilde{x}_2) - Eh(\widetilde{y}_1, \widetilde{y}_2) = \iint h(x_1, x_2) dK(x_2 \mid x_1) dG(x_1)$$

is positive. For all x_1, integration by parts yields

$$\int h(x_1, x_2) dK(x_2 \mid x_1) = -\int \frac{\partial h(x_1, x_2)}{\partial x_2} K(x_2 \mid x_1) dx_2. \tag{9}$$

It implies that

$$X = \int \left[\int -\frac{\partial h(x_1, x_2)}{\partial x_2} K(x_2 \mid x_1) dG(x_1) \right] dx_2,$$

or equivalently,

$$X = \int E\left[-\frac{\partial h(\widetilde{x}_1, x_2)}{\partial x_2} K(x_2 \mid \widetilde{x}_1) \right] dx_2. \tag{10}$$

Observe now that for any x_2, $-\partial h/\partial x_2$ is decreasing in x_1 because h is supermodular. Moreover, K is decreasing in x_1 for all x_2 because of the assumption on positive FSC. Therefore for all x_2, the covariance rule[3] implies that

$$E\left[-\frac{\partial h(\widetilde{x}_1, x_2)}{\partial x_2} K(x_2 \mid \widetilde{x}_1) \right] \geq E\left[-\frac{\partial h(\widetilde{x}_1, x_2)}{\partial x_2} \right] E\left[K(x_2 \mid \widetilde{x}_1) \right] = 0.$$

Since the integrand in (10) is positive for all x_2, so is the integral X. This proves the sufficiency part of the Proposition. For necessity, suppose by contradiction that $-\partial h/\partial x_2$ be increasing in x_1 in a neighborhood A of some $(\overline{x}_1, \overline{x}_2)$. Using a pair of random variables satisfying positive FSC whose support is in A would generate $X \leq 0$, a contradiction. ■

[3] $Ef(\widetilde{x})g(\widetilde{x}) \geq Ef(\widetilde{x})Eg(\widetilde{x})$ for all $\widetilde{x}$ if $f'(x)g'(x)$ is nonnegative for all x. See Gollier (2001, section 6.4) for a formal proof.

Proof of Lemma 2

We limit the proof to suffiency. Let $k(x_2 \mid x_1)$ denote $\int^{x_2} K(y \mid x_1)dy$. Integrating by parts the integral in the right-hand side of equation (9) yields

$$\int h(x_1,x_2)dK(x_2 \mid x_1) = -\frac{\partial h(x_1,x_2)}{\partial x_2}k(+\infty \mid x_1) + \int \frac{\partial^2 h(x_1,x_2)}{\partial x_2^2}k(x_2 \mid x_1)dx_2 \tag{11}$$

for all x_1. By construction, we have that

$$k(+\infty \mid x_1) = \int \left(F(y \mid x_1) - \widehat{F}(y)\right) dy = E\left[\widetilde{x}_2 \mid x_1\right] - E\left[\widetilde{x}_2\right] = 0$$

since the expectation of $\widetilde{x}_2$ is independent of x_1. Thus we can use (11) to write

$$X = Eh(\widetilde{x}_1,\widetilde{x}_2) - Eh(\widetilde{y}_1,\widetilde{y}_2) = \int E\left[\frac{\partial^2 h(\widetilde{x}_1,x_2)}{\partial x_2^2}k(x_2 \mid \widetilde{x}_1)\right] dx_2.$$

Because $k(x_2 \mid x_1) = q(x_2 \mid x_1) - \int^{x_2} \widehat{F}(y)dy$, the assumption of positive SSC means that k is decreasing in x_1 for all x_2. Because $\partial^2 h/\partial x_2^2$ is decreasing in x_1 by assumption, the covariance rule applied for each possible x_2 implies that

$$X \geq \int E\left[\frac{\partial^2 h(\widetilde{x}_1,x_2)}{\partial x_2^2}\right] E\left[k(x_2 \mid \widetilde{x}_1)\right] dx_2.$$

Because $Ek(x_2 \mid \widetilde{x}_1)$ is zero for any x_2 by construction, we obtain that X is nonpositive.■