A Generalized Volatility Bound for Dynamic Economies

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Abstract. We develop a generalization of the Hansen-Jagannathan (1991) volatility bound that (i) incorporates the serial correlation properties of return data and (ii) allows us to calculate a *spectral* version of the bound. This generalization enables us to judge whether models match important aspects of the data in the long run, at business cycle frequencies, seasonal frequencies, etc.

Our generalization is related to the space that the model IMRS is projected onto. Instead of specifying the projection to be a linear combination of contemporaneous returns, we let the projection live in a linear space of current, past and future returns. We show that the spectrum of the model IMRS must exceed the spectrum of our bounding IMRS. Our evaluation does not require solving the model and is based solely on a frequency-by-frequency examination of the fundamental component of the model, namely, the Euler equation that links asset returns to the IMRS.

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I. Introduction

Empirical evaluation of dynamic structural models has a long history in economics. In dynamic general equilibrium frameworks with linear-quadratic preferences, Hansen and Sargent (1980) computed linear decision rules explicitly and linked theory with measurement using the likelihood function. As they showed, a frequency domain approximation to the likelihood may be used in the model assessment, in which case the fit of a model is judged by how well its spectral density matches the corresponding spectral density computed from the data. The procedure, unfortunately, rejects practically all models. In circumstances where closed-form expressions for decision rules aren't available, Hansen (1982) and Hansen and Singleton (1983) used the Generalized Method of Moments (GMM) to formally estimate and evaluate dynamic models using a subset of the model implications for the data. Tests based on the procedure are less demanding than those based on the full likelihood, though few models pass them. Hansen and Jagannathan (1991) proposed a still less restrictive test that generalizes the variance bounds developed by LeRoy and Porter (1981) and Shiller (1981). They showed how to use asset return data to derive a lower bound on the volatility of a representative consumer's intertemporal marginal rate of substitution (IMRS). A model is said to be consistent with the data if the volatility of the IMRS implied by the model is greater than the Hansen-Jagannathan (HJ) volatility bound.

While the HJ test dismisses many models for violating the volatility bound, many others do satisfy the bound. Cochrane and Hansen (1992), for instance, argued that for reasonable parameterizations of time non-separable preferences and of state non-separable preferences, the consumption based asset-pricing models do satisfy the HJ bound. Tallarini (2000) modified the preferences in the standard business cycle model to allow for non-separabilities across states and showed that the model is consistent with asset return data using the HJ bound.

In this paper, we develop a generalized volatility bound that (i) incorporates the serial correlation properties of return data and (ii) allows us to calculate a *spectral* decomposition of the bound. This enables us to judge whether models fail to match important aspects of the data in the long run, at business cycle frequencies, seasonal frequencies, high frequencies, etc. Our evaluation of models is also based solely on the Euler equation that links the asset returns to the IMRS. This Euler equation governs intertemporal decisions, and hence the propagation of economic fluctuations—so a spectral (i.e., temporal) bound is especially useful in evaluating the model. Specifically, we can identify the frequencies at which a model violates the necessary conditions.

Technically, to derive the bound, Hansen and Jagannathan projected the model IMRS onto the space of contemporaneous asset returns and utilized only a *necessary* condition associated with dynamic models, namely the intertemporal Euler equation. Our generalization involves projecting the model IMRS onto the space of current, past, and future returns. As in Hansen and Jagannathan (1991), the projection involves a covariance between the IMRS and returns, a covariance which is given by Euler equation of the model and which is the sole implication used in the derivation of our bound. We show that the *variance* of the model IMRS must exceed the *variance* of the projection. Because of the nature of our projection we also show that the *spectrum* of the model IMRS must exceed the *spectrum* of the projection. Using the serial correlation properties of returns (together with the mean and variance), we derive a lower bound on the spectrum of the model IMRS. Similar to GMM and the HJ bound, our procedure does not require one to solve the entire model.

Because our bound involves projection of the model IMRS onto a larger space, it must be at least as tight as the HJ bound. Bounds tighter than the HJ bound are of course straightforward to find. Adding observable variables to the right hand side of the implicit HJ regression of the IMRS on returns permits construction of a tighter bound. The virtue of our list of additional variables (past and future returns) is that the residual in the projection must be orthogonal to returns at all leads and lags; this is what enables us to compute the frequency-by-frequency additive decomposition of our volatility bound. The issue that must be confronted in computing our projection is that there are many new covariances between the IMRS and returns that the model does not supply—those at leads and lags. To fill in these missing covariances, we make use of sample returns and the "sample" IMRS calculated from data (typically just consumption data) for specific model parameters. What is gained by this approach is that when returns and the model IMRS have serial correlation and comovement, the model-implied contemporaneous covariance carries implications for the sort of IMRS volatility that is permissible over long and short horizons. That is, whereas the HJ bound reflects the fact that the variance of the IMRS must exceed a particular minimum given the variance of returns and the Euler equation covariance, our bound reflects the fact that the stochastic process for the IMRS must be sufficiently variable at each frequency given the stochastic process for returns together with the Euler equation covariance. An additional attractive feature of our bound is that bound violations may be localized, and this may suggest modifications of the model to "fix" problems—e.g., at high frequencies, low frequencies, or business cycle frequencies.

We apply our bound to four asset pricing models. The time-separable model turns out to satisfy the bound at business cycle and lower frequencies. This can be accomplished with only slightly less risk aversion than the extraordinary amount necessary for the model to achieve the HJ bound. The state non-separable model of Epstein and Zin (1991) fares somewhat better, satisfying the bound at frequencies associated with yearly and longer cycles, indicating at least that the model is consistent with US data at business cycle horizons. We find that the habit formation model of Constantinides (1990) satisfies the bound at high frequencies while violating the bound at business cycle frequencies. We interpret this result as indicating that the habit model does a poor job at explaining the dynamic properties of return data. The richer habit formation model of Campbell and Cochrane (1999) fairs much better at business cycle frequencies.

In related work, Watson (1993) provided a measure based on the distance, in the frequency domain, between the autocovariance function of the variables simulated from an approximate solution of the model and that of the corresponding observed time series data. Our spectral bound, in contrast, is just an implication of the (nonlinear) intertemporal first order condition; thus unlike Watson's approach, ours does not require even an approximate solution to the model's system of equations. Cogley (2001) extended Watson's frequency domain measure of fit to the assessment of the specification error developed by Hansen and Jagannathan (1997). For practical purposes, his decomposition can be thought of as describing the frequency domain characteristics of the distance between a candidate IMRS and one that prices securities correctly. Our approach differs in that we work directly with the bound itself, therefore providing a description of a new *third* dimension, frequency, to the "mean, variance" picture of Hansen and Jagannathan (1991). This third dimension is useful when returns are serially correlated and/or preferences are temporally nonseparable. Alvarez and Jermann (2002) derive a lower bound on the size of the permanent component of the pricing kernel. That is, they are interested in bounding the volatility of the agent's marginal utility at the zero frequency, while we are interested in bounding the volatility of the agent's stochastic discount factor (or ratio of marginal utilities) across the entire spectrum. Daniel and Marshall (1999) utilize spectral analysis to document low frequency comovement of of asset returns and consumption growth. They also assess various asset pricing models using the HJ bound and return and IMRS data at different horizons, and find that the models better describe longer-horizon returns. In particular, their empirical analysis concludes that long-horizon return data (e.g. the return over a horizon of 2 to 3 years) supports time-non-separable utility functions. We also use the spectral properties of returns and consumption, but we evaluate models using a frequency domain decomposition of the relationship implied by theory between stochastic discount factors and asset returns.

In Section II of the paper, we review the derivation of the HJ bound and develop the generalized bound. In Section III applies the spectral bound to four popular asset-pricing models. Section IV extends the generalized bound to economies with frictions. Section V concludes.

II. The Hansen-Jagannathan Bound and the Spectral Bound

Let R denote the $n \times 1$ (gross) return vector of risky assets. Consider an intertemporal marginal rate of substitution (or "stochastic discount factor") m that prices the n assets according to

(1) $ERm = \iota$,

where ι is an $n \times 1$ vector of ones. Equation (1) is the unconditional version of the standard asset-pricing model Euler condition equating the expected marginal cost and marginal benefit of delaying consumption one period (e.g., Lucas 1978). The derivation presented in the next subsection differs somewhat from the original, but will provide a useful foundation for the generalization to the spectral bound presented in subsection II.2.

II.1 The Hansen-Jagannathan Bound

Suppose we compute the least squares projection of the IMRS onto the linear space spanned by a constant and contemporaneous returns. The projection is of the form

(2) $m = m_v + \varepsilon$

with

(3) $m_v = v + (R - ER)'\beta,$

for β in $\Re^n$, where $v = Em = Em_v$, and ε is orthogonal to a constant as well as contemporaneous returns. This implies $E\varepsilon = 0$, and $ER\varepsilon = 0$. Together with (1), this in turn implies $ERm = ERm_v = \iota$. Then

$$\begin{aligned} var(m) &= var(m_v) + var(\varepsilon) + 2cov(m_v,\varepsilon) \\ &= var(m_v) + var(\varepsilon) + Em_v\varepsilon. \end{aligned}$$

By construction of m_v, $Em_v\varepsilon = 0$. Thus, we have

$$var(m) = var(m_v) + var(\varepsilon) \geq var(m_v),$$

meaning that a lower bound on the variance of m is that of m_v. This is the so-called Hansen-Jagannathan (HJ) bound.

To find this lower bound, we need an expression for the vector β. First, subtract v from both sides of (3), multiply by (R-ER) and take expectations:

(4) $E(R - ER)(m_v - v) = E(R - ER)(R - ER)'\beta = \Omega\beta,$

where Ω is the covariance matrix of risky-asset returns (assumed to be positive definite). Second, a more convenient expression for the left-hand-side of (4) can be derived as follows. Because m_v satisfies the first order condition, we have

$$ER(m_v - v) = \iota - vER \text{ , so}$$

$$E(R - ER)(m_v - v) = \iota - vER.$$

Therefore, we can solve (4) for β as

(5) $\beta = \Omega^{-1}(\iota - vER).$

That is, an IMRS that satisfies the Euler equation (1) will yield a projection coefficient given by (5). Thus,

$$var(m_v) = E(m_v - v)'(m_v - v)$$

$= \beta' E(R - ER)(m_v - v)$ (from equation (3))

(6) $= \beta'\Omega\beta$ (from equation (4)).

Substituting for β from (5), we can write

(7) $\text{var}(m_v) = (\iota - vER)'\Omega^{-1}(\iota - vER)$.

Notice that the lower bound, $\text{var}(m_v)$, is a function of the mean of the model IMRS and hence is "model dependent." In this case, however, it is a simple matter to plot the bound *frontier* as a function of v, meaning that the bound may be depicted graphically without reference to any model.

Given the mean of the IMRS and mean returns, equation (7) indicates that the bound is tighter the less volatile are returns. That is, if returns are not very volatile, satisfying the bound will in general require a more volatile IMRS.

II.2 A Spectral Bound

We now derive a generalized volatility bound and provide an orthogonal decomposition by horizon – long run, short run, etc. To do this, we employ the same frequency-domain methods that are of great use in computing orthogonal decompositions of the *variance* of an economic time series. Our procedure begins with a projection similar to (2), except that now we will project the IMRS on the space spanned by contemporaneous R together with *future* and *past* R. The orthogonality of the least squares projection error with all leads and lags of returns implies that the spectral density of the IMRS is bounded below by the spectral density of the projection at each frequency. To determine this bound, we employ cross covariances between the *model* IMRS and returns; the result is a model-dependent spectral bound.

We begin with an assumption about the stationarity of returns that we will maintain throughout the paper.

Assumption 1. The IMRS $\{m_t\}$ and the vector of returns $\{R_t\}$ are jointly covariance stationary stochastic processes with spectral densities $S_m(e^{-i\omega})$ (a scalar function) and $S_R(e^{-i\omega})$ (a matrix function) that are invertible at all frequencies $\omega \in [-\pi, \pi]$.

We now compute the least squares projection of the IMRS onto the linear space spanned by a constant as well as current, *past, and future* returns. This projection is of the form

(8) $m_t = g_t + \xi_t$,

with

(9) $$g_{v,t} = v + \sum_{k=-\infty}^{\infty} (R_{t-k} - ER_{t-k})' b_k \ .$$

As before, $E\xi_t = 0$ and $Eg_t = v = Em_t$, but now $\{\xi_t\}$ is orthogonal to current, past and future R. Note that the projection in (3) is a special case of (9). Consequently, we have the following lemma:

Lemma 1. $\mathrm{var}(m_t) \geq \mathrm{var}(g_t) \geq \mathrm{var}(m_{v,t})$.

Proof: The first inequality follows because the projection of m_t onto the space spanned by current, past, and future $\{R_t\}$ involves error that is orthogonal to that space. The second inequality follows because that space is larger than the one spanned by current R_t alone. ■

It is clear from the lemma that the projection in (8) and (9) generally yields a tighter bound than the one proposed by Hansen and Jagannathan, though there are special cases where the two bounds coincide. We will illustrate some of these special cases in Section II.3.

The tighter bound in lemma 1 is also additively decomposable by frequency, giving rise to additional restrictions that the model IMRS must satisfy:

Proposition 1. $S_m(e^{-i\omega}) \geq S_g(e^{-i\omega})$, where $S_m(e^{-i\omega})$ is the spectral density of the stochastic process $\{m_t\}$ and $S_g(e^{-i\omega})$ is the spectral density of the process $\{g_t\}$.

Proof: (This is a simple application of ideas in Sargent 1987, Chapter 11.) Subtract means from both sides of (8), multiply the result by itself lagged j periods, and take expectations to yield

$\Gamma_m(j) = \Gamma_g(j) + \Gamma_\xi(j)$

where $\Gamma_m(j)$ denotes the autocovariance of $\{m_t\}$ at lag j (cross covariances between g_t and ξ_{t-j} are zero for all j by the least squares orthogonality condition.) Now take Fourier transforms (multiply by $e^{-i\omega j}$ and sum over all j) to get

$S_m(e^{-i\omega}) = S_g(e^{-i\omega}) + S_\xi(e^{-i\omega})$.

The non-negativity of the spectrum of $\{\xi_t\}$ yields the inequality result. ■

To find the bounding spectral density, we must compute the lag coefficients β_k in the projection (9). Equation (9) implies that the spectrum of g_t can be written as

(10) $$S_g(e^{-iw}) = b(e^{iw})' S_R(e^{-iw}) b(e^{-iw}) \ ,$$

where the Fourier transform, $b(e^{-iw})$, of the lag coefficients, is given by

(11) $$b(e^{-iw}) = S_R(e^{-iw})^{-1} S_{Rg}(e^{-iw}) \, .$$

Thus as a consequence of Proposition 1 and equations (10) and (11), we have

(12) $$S_m(e^{-iw}) \geq S_{Rg}(e^{iw})' S_R(e^{-iw})^{-1} S_{Rg}(e^{-iw}) \quad \text{, for all } w.$$

The right hand side of (12) is the new frequency-by-frequency or "spectral" bound. However, the cross-spectrum S_{Rg} involves covariances between returns and g_t, which are of course not given by the model. To make (12) operational, it is necessary to specify these covariances.

Define

$$ER_{t-j}g_t = \Phi_j, \text{ j = 0, ±1, ±2, …}$$

where $\Phi_0 = \iota$ is the standard Euler equation restriction; Φ_j, $j \neq 0$ are the cross-products between g_t and returns at lags *other* than the contemporaneous one. Just as with the static HJ bound, we could display a frontier (in this case, an infinite-dimensional one) as a function of the unknown mean v and covariances Φ_j, $j \neq 0$, but this would not be practical. Instead, we calculate a *model-dependent* bound by using a property of the projection in (8) and (9) to calculate the missing covariances. Because g_t is the projection of m_t onto current, past and future R_t,

$$E(R_{t-j}-ER_{t-j})(m_t-v) = E(R_{t-j}-ER_{t-j})(g_t-v),$$

so that we may replace the unobserved covariances between $\{R_t\}$ and $\{g_t\}$ by the "observed" covariances between returns and the model IMRS, thereby making the bound model dependent. Recall that for stationary R_t, $ER_{t-j} = ER_t = ER$. Let

$$\Psi_j \equiv \Phi_j - vER$$

so that

$$E(R_{t-j} - ER_{t-j})(m_t - v) = \Psi_j .$$

Note that the Euler equation (1) implies $\Psi_0 = \iota - vER$; Ψ_j, $j \neq 0$ are the covariances between m_t and returns at lags other than the contemporaneous one. These covariances are determined once a model IMRS is specified.

Now we build the cross spectral density S_{Rg} from the Ψ_j sequence as

$$(13) \qquad S_{Rg}(e^{-i\omega}) = \sum_{j=-\infty}^{-1} \Psi_j e^{-i\omega j} + \iota - vER + \sum_{j=1}^{\infty} \Psi_j e^{-i\omega j} .$$

Thus an operational spectral bound is obtained by replacing S_{Rg} on the right-hand side of (12) by the expression in (13). The variance of g_t of course is given by the integral of the spectral bound:

$$(14) \qquad \text{var}(g_t) = (2\pi)^{-1}\int_{-\pi}^{\pi} S_{Rg}(e^{i\omega})' S_R(e^{-i\omega})^{-1} S_{Rg}(e^{-i\omega}) d\omega .$$

II.3 Relationship of the Spectral Bound to other assessment devices

II.3.1 Relationship to the HJ Bound. In the previous subsection, we described a procedure wherein sample data are used to estimate covariances between returns and the IMRS at leads and lags. Different ways of supplying those covariances, which are not restricted by the model—will lead to different bounds. One alternative way of specifying these covariances is analogous to the one utilized by Watson (1993). In his case, the missing covariances were between simulated model time series and actual data. Unlike in our case, there is no natural way in Watson's to achieve temporal alignment of model simulations and actual data, so Watson assumed that the missing covariances were such that residual error (between model and data) was as small as possible. In our case, the analogous assumption would involve choosing the missing covariances between returns and the IMRS to be such that the resulting bound is as weak as possible. It turns out that doing so leads to the HJ bound:

Proposition 2. The variance minimizing choice of the covariances $\{\Psi_j\}$, j = ±1, ±2, … in (13) leads to the HJ bound.

Proof: To minimize (13) by choice of $\{\Psi_j\}$, note that

$$\partial S_{R\gamma}(e^{-i\omega})/\partial\Psi_j = e^{-i\omega j}I \ .$$

The first-order conditions for choosing $\{\Psi_j\}$, j = ±1, ±2, ±3, … are given by

$$(2\pi)^{-1}\int_{-\pi}^{\pi}[e^{-i\omega j}\beta(e^{i\omega}) + e^{i\omega j}\beta(e^{-i\omega j})]d\omega = 0 \quad j = \pm 1, \pm 2, \ldots .$$

Because $\int_{-\pi}^{\pi} e^{i\omega k}d\omega$ = 0 unless k = 0, this is equivalent to

$\beta_j = 0$ for j = ±1, ±2, ±3 ….

This leaves

$$\text{var}(\gamma_t) = (2\pi)^{-1}\int_{-\pi}^{\pi}\beta' S_R(e^{-i\omega})\beta\, d\omega = \beta'\Omega\beta = (\iota - \nu ER)'\Omega^{-1}(\iota - \nu ER),$$

which is the same as the HJ bound. ■

According to Proposition 2, the HJ bound is decomposable by frequency if one makes the variance-minimizing assumption about those covariances not pinned down by the Euler equation. Thus when one assumes (possibly in the face of contrary evidence) that the projection of $\{m_t\}$ onto current, past, and future $\{R_t\}$ involves only current $\{R_t\}$, one of course recovers the HJ bound. Furthermore, such a projection will not have the property that the projection error is orthogonal to past and future returns, so the spectrum of the model IMRS will not necessarily exceed the spectrum of the projection.

II.3.2 Relationship to Cogley's Pricing Error Decomposition. Cogley (2001) provides a frequency decomposition not of the variance bound (which was introduced in Hansen and Jagannathan 1991), but of the pricing error associated with a particular model (introduced by Hansen and Jagannathan 1997). As he notes (p. 489), his approach can be adapted to yield a bound frequency by frequency.

Cogley begins with a definition of the pricing error as the difference between a model IMRS M_t and the "true" discount factor Δ_t:

$M_t = \Delta_t + u_t$

where in Cogley's case M_t is the *nominal* discount factor at time t. Denoting the gross nominal excess return by R_{xt}^{n}, the Euler equation (1) can be rearranged to yield the equity premium

$$ER_{xt}^{n} = -\text{cov}(\Delta_t, R_{xt}^{n}) / EB_{ft-1}$$

where B_{ft} is the price of a nominal risk-free bond at time t. (Cogley works with nominal returns and discount factors precisely to pin down the mean of the IMRS by this bond price.) Related to the equity premium is what Cogley dubs the "innovation" premium:

$$ER_{xt}^{n} = -\text{cov}(B_{ft-1}, R_{xt}^{n}) / EB_{ft-1} - \text{cov}(e_{\Delta t}, e_{xt}) / EB_{ft-1}$$

where $\varepsilon_{\Delta t}$ and ε_{xt} are innovations in the true discount factor and excess returns relative to an information set at time t-1 that includes at least all lags of Δ_t and R_{xt}^{n}. The key observation Cogley makes is that because $\varepsilon_{\Delta t}$ and ε_{xt} are innovations, they are uncorrelated with each other at all leads and lags, and their cross-spectrum is therefore flat. Cogley then calculates the decomposition of the pricing error by minimizing the spectrum of the pricing error u_t frequency by frequency, respecting the cross-spectrum constraint. The minimized spectrum is

$$g_{uu}(w) = \frac{| g_{M-B_f, e_x}(w) - \text{cov}(e_{\Delta t}, e_{xt}) / 2p |^2}{g_{e_x e_x}(w)}$$

where $g_{M-B_f, e_x}(w)$ is the cross-spectrum between the model IMRS (less the mean of the true IMRS) and the return innovations, and $g_{e_x e_x}(w)$ is the spectrum of return innovations. To convert this to a frequency-by-frequency volatility bound, note that *if* the IMRS were constant, then the spectrum of u_t and that of Δ_t would coincide, and the spectrum of any candidate IMRS would need to exceed that of Δ_t.

The key difference between our approach and Cogley's is in the implicit projections: because he focuses on the innovation premium, he employs a *one-sided* projection on the infinite past of true IMRS's and excess returns; we utilize a *two-sided* projection on returns alone. Where we use data to supply missing covariances between the model IMRS and returns at all leads and lags, Cogley uses data to supply covariances between the IMRS and return innovations at all leads and lags. Of course, this calculation will depend upon the model used to measure the return innovations. Further, it is necessary that the return innovations be relative to an information set that includes past *true* discount factors Δ_t; this makes it difficult to make the pricing error decomposition and the associated bound operational in general.

II.4 Illustrations of the Spectral Bound

Example 1: Flat $b(e^{-iw})$. This is precisely the case of the previous subsection, but here we suppose that rather than assuming lead-lag covariances to be zero, the projection in (8) and (9) has the property that $\beta_0 \neq 0$ and $\beta_k = 0$ for k = ±1, ±2, … and that the data reflect this. That is, suppose that in projecting the model IMRS on current, past, and future returns, *only* current returns are relevant. Then $\beta(e^{-i\omega}) = \beta_0$. (Note from (11) that a flat $\beta(e^{-i\omega})$ implies that the cross-spectrum S_{Rg} is *proportional* to the spectral density of returns.) The Euler equation (1) implies that β_0 coincides with the value of β (given in (5)) associated with the static HJ bound. Thus we can compute the spectral bound directly from (10) as

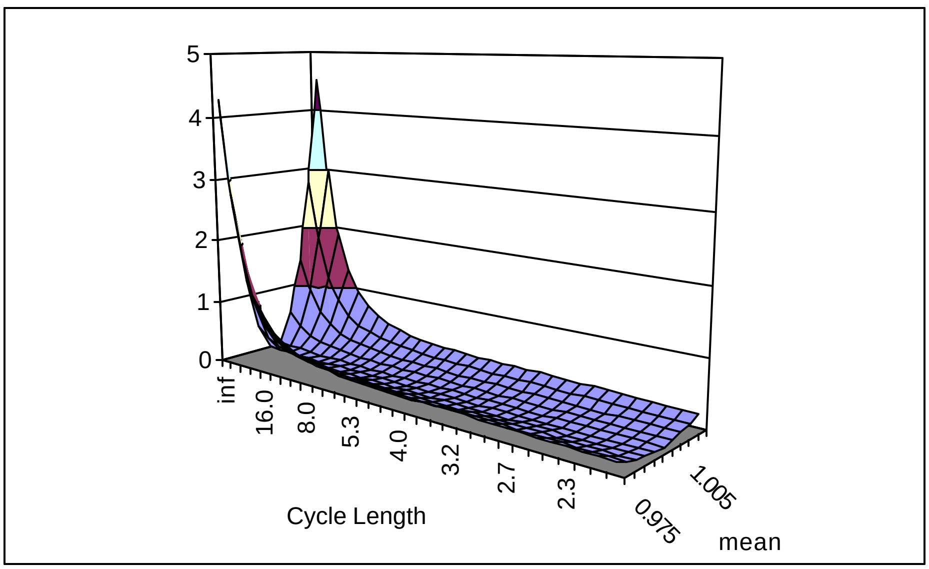
(15) $$S_g(e^{-iw}) = b\,'S_R(e^{-iw})b \ .$$

Using (5), we can write

(16) $$S_g(e^{-iw}) = (i - vER)'\Omega^{-1}S_R(e^{-iw})\Omega^{-1}(i - vER) \ .$$

Note that this is the frequency-domain analogue of equation (6). As with the HJ bound, a frontier as a function of v may be traced out; because the spectral density matrix is positive definite, the spectral bound at frequency ω will have the "U" shape characteristic of the HJ bound frontier. Figure 1 illustrates a spectral bound for this example.[1]

Figure 1: Spectral Bound



Integrating both sides of equation (16) over ω yields

$$\text{var}(g_t) = (\iota - vER)'\Omega^{-1}(\iota - vER).$$

Here we have used the fact that the variance-covariance matrix of returns, Ω, is given by

[1] We illustrate the spectra for frequencies $\omega = 0$ to π. Spectra are symmetric about $\omega = 0$, and we follow convention in plotting them only for nonnegative frequencies. In the figures that illustrate spectra we plot cycle length on the horizontal axis rather than frequency since cycle length is easier to interpret. To convert frequencies to cycle lengths we used the formula $\omega \times$ cycle length $= 2\pi$. Also, with the exception of Figure 4, the vertical axes of all spectra depicted in the paper are in units of 2π.

$$\Omega = (2p)^{-1} \int_{-p}^{p} S_R(e^{-iw})dw \; .$$

Clearly var(g_t) = var(m_v), the HJ bound. Figure 1 illustrates the fact that the height of the spectral bound varies with both frequency (ω) and the mean (v) of the model IMRS. Figure 2 illustrates the HJ bound; this is the same as the integral (along the frequency dimension) of the spectral bound in Figure 1 for the various v's. Figure 3 is the spectral bound at one mean (v = 0.992). The area under the curve in Figure 3 is equal to the height of the bound in Figure 2 at v = 0.992.

Figure 2: HJ Bound

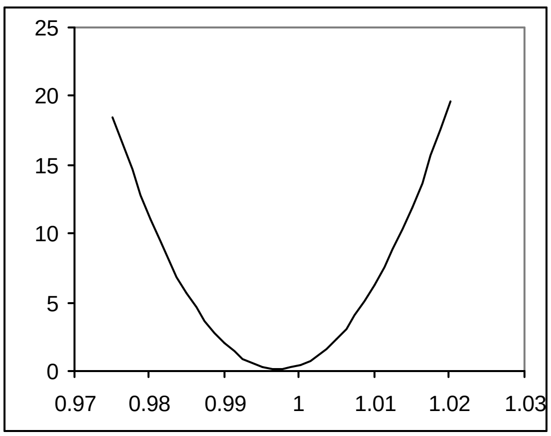
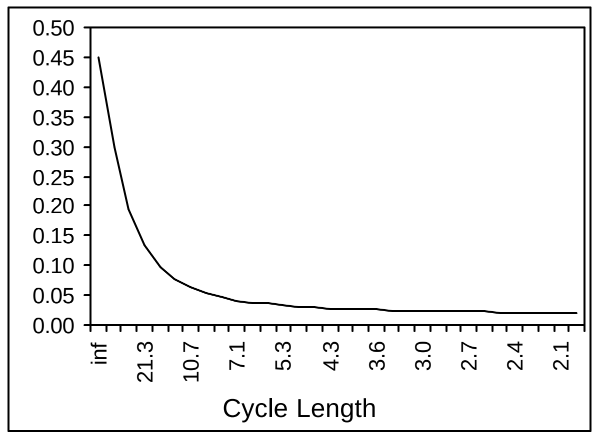


Figure 3: Spectral Bound for v = 0.99



Equation (16) illustrates how the spectral bound may help detect departures from the Euler equation restrictions. Given the variance of returns, the spectral bound in this case is tightest at frequencies at which returns are most volatile. That is, given the integral of the spectral density of returns, the frequencies at which the spectrum is highest are those for which the bound is most binding. Moreover, an IMRS that satisfies the integral of the spectral bound (here, the HJ bound) need not satisfy the spectral bound frequency by frequency. Thus by considering the relationships among variances and covariances at all frequencies separately, departures from the Euler equation restrictions may more easily be detected.

To illustrate this, first note that the condition of the example—flat $\beta(e^{-i\omega})$—means that the first and second moments of the IMRS and return process are related via a relationship of the form

$$m_t = \lambda R_t + u_t$$

where $\{u_t\}$ has zero mean and is uncorrelated with $\{R_t\}$ contemporaneously and at all leads and lags. Restricting attention to univariate R_t, (7) implies that the HJ bound and the integral of the spectral bound are both equal to $(1\text{-}vER)^2/\sigma_R^2$. For the class of $\{m_t\}$ processes we are considering, v = $Em_t = \lambda ER$ and the bound will be $(1\text{-}\lambda(ER)^2)^2/\sigma_R^2$. The spectrum of $\{m_t\}$ is

$$S_m(e^{-i\omega}) = \lambda^2 S_R(e^{-i\omega}) + S_u(e^{-i\omega})$$

and of course

$$\sigma_m^2 = \lambda^2\sigma_R^2 + \sigma_u^2.$$

For this class of IMRS processes, the $\{m_t\}$ process that satisfies $ER_tm_t = 1$ has $\lambda = \lambda^* = 1/((ER)^2 + \sigma_R^2)$. When $\lambda = \lambda^*$, $\lambda^{*2}\sigma_R^2$ will equal the bound, so the HJ bound is satisfied because the variance of $\{u_t\}$ is nonnegative. As λ diminishes from λ^*, the variance (and spectrum) of $\{m_t\}$ decline, while the HJ bound rises.[2] Thus for a given $\lambda < \lambda^*$, the Euler equation condition will be violated but there will be a $\{u_t\}$ process with variance sufficiently large that the HJ bound is just satisfied.

For instance, suppose we further specialize the example to the situation in which the return is serially uncorrelated, so that the spectrum $S_R(e^{-i\omega})$ is flat, and equal to σ_R^2. Then there exists a $\{u_t\}$ process such that

$$\lambda^2\sigma_R^2 + \sigma_u^2 = \frac{(1-\lambda(ER)^2)^2}{\sigma_R^2}$$

or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[\lambda^2\sigma_R^2 + S_u(e^{-i\omega})]d\omega = \frac{(1-\lambda(ER)^2)^2}{\sigma_R^2}.$$

Clearly, if $S_u(e^{-i\omega})$ is not *flat*, there will be frequencies for which

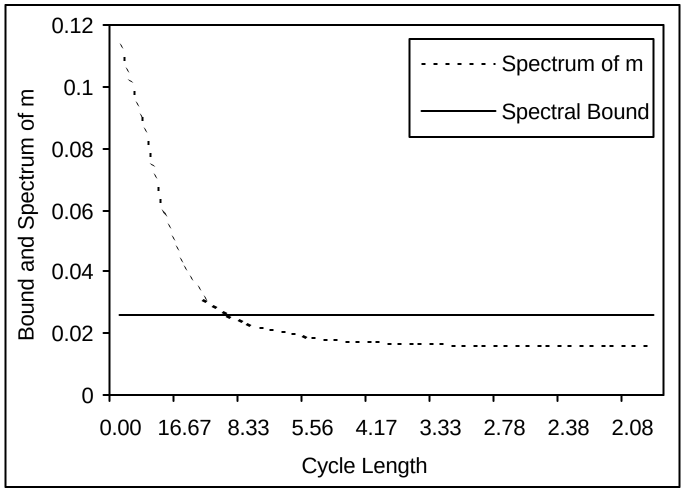
$$\lambda^2\sigma_R^2 + S_u(e^{-i\omega}) < (1-\lambda(ER)^2)^2/\sigma_R^2;$$

that is, for which the spectral bound will be violated.

As a numerical example, let $ER = 1.06$, $\sigma_R = 0.14$, $u_t = 0.0.8u_{t-1} + \eta_t$ with $E\eta_t = 0$ and $\sigma_\eta^2 = 0.004$. Then for $\lambda = 0.87$, the Euler equation is violated (the critical $\lambda^* = 0.875$), but the HJ bound is satisfied. The spectral bound, however, is violated at low frequencies, as depicted in Figure 4.

[2] As λ increases from λ^*, the variance of m_t increases and the HJ bound decreases. In fact, the bound will be satisfied for any λ between $1/((ER)^2 + \sigma_R^2)$ and $1/((ER)^2 - \sigma_R^2)$. Of course, all of the λ's inside this interval violate the Euler equation.

Figure 4: Spectral Bound Violations



Recall that while Hansen and Jagannathan projected the IMRS on current returns only, they neither specified nor exploited the cross-covariances between returns and the IMRS other than the contemporaneous one. Our example *does* aim to exploit such information, but in this special case, the joint stochastic process for m and R is such that lagged and future returns carry no information for current m. In such a case, integrating our spectral bound yields the static HJ bound. Even so, the spectral bound will detect departures from the Euler equation restrictions that the HJ bound will not. Of course, in general, the two projections of the IMRS—one on current R alone and the other on current, past, and future R—will differ and thus, the integral of our spectral bound will be strictly tighter than the HJ bound.

The next two examples help illustrate how serial correlation in returns or intertemporal covariance between returns and the IMRS affect the spectral bound. To facilitate understanding of this, analogous to (8), notice from (13) that

$$\begin{aligned} S_{Rg}(e^{-iw}) &= S_{Rm}(e^{-iw}) - \text{cov}(R,m) + i - vER \\ &= S_{Rm}(e^{-iw}) - \Psi_0 + i - vER \end{aligned}.$$

Using this notation, the spectral bound can be written as

$$S_m(e^{-iw}) \geq \left(S_{Rm}(e^{iw}) - \Psi_0 + i - vER\right)' S_R(e^{-iw})^{-1}\left(S_{Rm}(e^{-iw}) - \Psi_0 + i - vER\right) \quad . \tag{17}$$

We now turn to examples 2 and 3, involving, in turn, a flat cross-spectrum between returns and the IMRS, and a flat spectrum of returns.

Example 2: Flat S_{Rm}*(e^{-iw})*. Suppose that in contrast to example 1, the cross-spectrum S_{Rg}, rather than being proportional to returns, is *flat*. In this case, S_{Rm} is also flat, and equal to cov(R,m) = Ψ_0, so the inequality in (12) becomes

$$(18) \qquad S_m(e^{-iw}) \geq \left(i - vER\right)' S_R(e^{-iw})^{-1} \left(i - vER\right) \quad .$$

Notice that *given* ω, this bound has the same form as the HJ bound, with the spectral density matrix $S_R(e^{-i\omega})$ in place of the variance-covariance matrix Ω. Similarly, analogous to the HJ bound, but in contrast to example 1, the spectral bound in this case is tightest at frequencies at which returns are least volatile. The analogue to Figure 1 for this case, for example, is nearly the mirror image of Figure 1, with the "U" shape most pronounced at high frequencies rather than the low ones. In addition, from lemma 1, in this example it may even the case that an IMRS that satisfies the HJ bound could violate the integral of the spectral bound. To see this, integrate both sides of (18) to get

$$\begin{aligned} \frac{1}{2p}\int_{-p}^{p} S_m(e^{-iw})dw &\geq (i - vER)'[\frac{1}{2p}\int_{-p}^{p} S_R(e^{-iw})^{-1}dw](i - vER) \\ &\geq (i - vER)'[\frac{1}{2p}\int_{-p}^{p} S_R(e^{-iw})dw]^{-1}(i - vER) \\ &= (i - vER)'\Omega^{-1}(i - vER). \end{aligned}$$

The last expression, of course, is the HJ bound. In this case it is persistence in returns (positive or negative serial correlation) that provides extra information that may permit the detection of violations of the Euler equation conditions.

Example 3: Flat S_R*(e^{-iw})*. Suppose now that the spectrum of returns, $S_R(e^{-i\omega})$, is flat. In this case, $S_R(e^{-i\omega}) = \Omega$, and the bound becomes

$$(19) \qquad S_m(e^{-iw}) \geq \left(S_{Rm}(e^{iw}) - \Psi_0 + i - vER\right)' \Omega^{-1} \left(S_{Rm}(e^{-iw}) - \Psi_0 + i - vER\right) \quad .$$

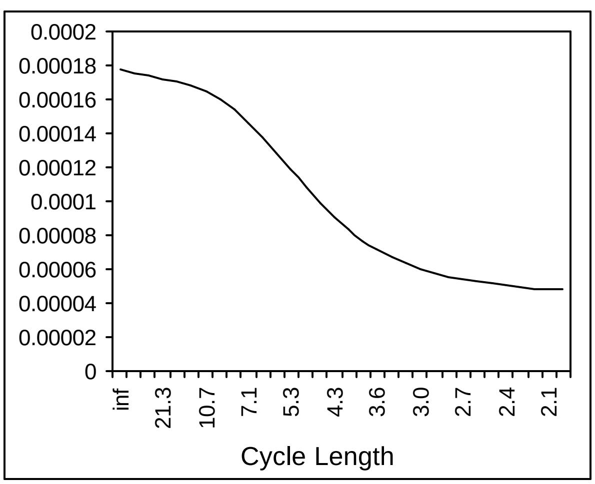
Now the bound varies by frequency because returns and the IMRS covary at leads and lags, making S_{Rm} vary by frequency. As with the HJ bound, the spectral bound is tight when returns are not very volatile, but now the bound also tightens at frequencies for which the covariance between returns and the IMRS is large (positive or negative). In this case, covariation between returns and the IMRS at leads and lags carries information beyond that in the HJ bound that may be used to detect violations of the first order conditions.

III. An Application to Asset Pricing Models

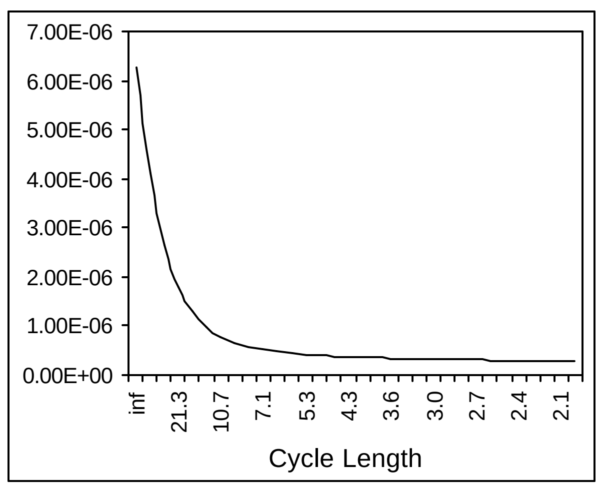
Several studies in the last 15 years have documented the time series properties of asset returns. For instance, Fama and French (1989) found that asset returns could be predicted by variables, such as default spread, that exhibit business cycle variation; Ferson and Harvey (1991) found that "the expected risk premium (associated with the stock market index) increases during economic contractions and peaks near business cycle troughs." (p.402). Figure 5 displays the spectra of real equity and Treasury bill returns using the *ex-post* real returns on the S&P 500 and 90-day T-bills for 1947:1-1997:4. The real return on the S&P 500 and the T-bills have substantial volatility concentrated over the long-run and business cycle frequencies. For the quarterly data from 1947:1 to 1997:4, following Watson's (1993) definition of a business cycle (6 to 32 quarters), the business cycle volatility is 48% of the total volatility in the equity return and 38% of total volatility in the Treasury bill return. Defining the long-run to be cycles longer than 32 quarters, the fraction of the volatility in the equity return is 11%; the corresponding fraction in the Treasury bill return is 39%. The figure indicates that the return data display the "typical spectral shape of an economic variable" (Granger, 1966)—substantial spectral power at low frequencies that declines rapidly as frequency increases (as cycle length decreases).

Figure 5: Return Spectra

a: Equity



b: Treasury bill



An intertemporal asset-pricing model carries implications for return spectra. While the HJ bound helps evaluate an asset-pricing model via restrictions based on mean and variance in asset returns, our spectral bound exploits the spectral shape of returns (as in Figure 5) and imposes additional restrictions on the model. For instance, (16) implies that the spectral bound will be tighter at the long-run and business cycle frequencies relative to high frequencies, so an asset-pricing model's IMRS faces a more stringent restriction at the long-run and business cycle frequencies. We illustrate this by evaluating 4 asset-pricing models.

We examine time-separable-, Epstein-Zin-, internal-habit-formation-, and external-habit-formation-preferences. The consumption data are given by per capita consumption of nondurables and services. We use the two return time series—quarterly real S&P 500 returns and real Treasury bill returns—used to calculate the spectra in Figure 5.[3] We calculate the spectrum of the IMRS, the spectral density matrix of returns, and the cross-spectrum of returns and IMRS by computing the necessary auto and cross-covariances, weighting using Bartlett weights, and Fourier transforming the result. We choose preference parameters for each model IMRS so that the volatility of the IMRS just exceeds the integral of the spectral bound.

III.1 Time-Separable Preferences

Time-separable constant relative risk aversion preferences (e.g., Mehra and Prescott 1985) are described by

$$U_o = E_o \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0$$

where E_0 denotes conditional expectation given information at time 0, C_t denotes consumption at time t and $\beta \in (0,1)$ is the discount factor. (The $\sigma = 1$ case will be interpreted as logarithmic.) The agent's intertemporal marginal rate of substitution is:

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} .$$

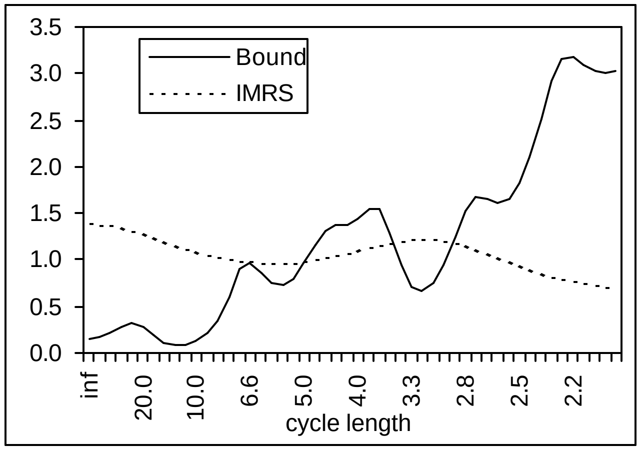
For these preferences, we find that the volatility of the IMRS exceeds the integral of the spectral volatility bound with $\beta = 0.99$ and $\sigma = 264.7$ ($\sigma = 263$ is the smallest σ for which the IMRS volatility exceeds the standard HJ bound). For this parameter setting the volatility of the IMRS (variance) is 8.5, the integral of the spectral bound is 7.8, and the HJ bound is 1.7.

Figure 6 illustrates the spectral bound (solid curve) and the spectrum of the IMRS (dotted curve). The figure is very striking: the model exceeds the bound at low frequencies, yet misses at high frequencies. Of course the amount of risk aversion used here is excessive, and we would reject the time-

[3] All data are quarterly, for 1947:1-1997:4. Real (and nominal) consumption of non-durables and services data and the non-institutional population over age 16 were obtained from the BEA website. A consumption deflator was calculated from the ratio of nominal to real consumption of non-durables and services, and this deflator was used to deflate nominal dividends and the S&P 500 index obtained from the Citibase dataset. Nominal Treasury bill returns were obtained from the FRED database at the St. Louis Fed website and converted to real returns by subtracting the inflation rate implied by the implicit deflator for consumption of non-durables and services. The mean quarterly

separable model on these grounds alone. We turn to more popular asset pricing models in the next subsection.

Figure 6: Time -Separable Model



III.2 State Non-Separable Preferences

Epstein and Zin (1991) and Weil (1989) generalized the time-separable preferences to allow for an independent parameterization of attitudes towards risk and intertemporal substitution. Following Weil (1989), we assume that the preferences are given by:

$$V_t = U[c_t, E_t V_{t+1}]$$

where

$$U[c,V] = \frac{\left\{(1-b)c^{1-r} + b[1+(1-b)(1-s)V]^{\left(\frac{1-r}{1-s}\right)}\right\}^{\left(\frac{1-s}{1-r}\right)} - 1}{(1-b)(1-s)}.$$

The elasticity of intertemporal substitution is 1/ρ and σ is the coefficient of relative risk aversion. The IMRS for these preferences is:

$$m_{t+1} = \left[b\left(\frac{c_{t+1}}{c_t}\right)^{-r}\right]^{\frac{1-s}{1-r}} [R_{t+1}]^{\left(\frac{1-s}{1-r}\right)-1}$$

where R_{t+1} is the return on wealth.[4] For our dataset with β = 0.987, ρ = 3.85 and σ = 32.4 the volatility in the IMRS exceeds the integral of the spectral bound. (with β = 0.9895, ρ = 3.1 and σ = 16.35 the model

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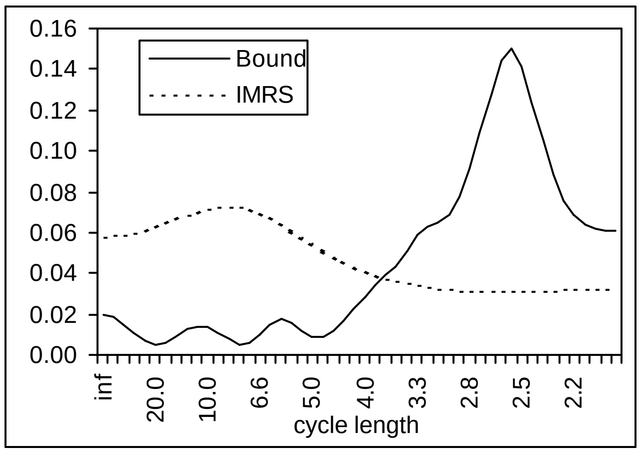
equity return is 2.06% and its standard deviation is 5.80%. The mean quarterly Treasury bill return is 0.28% with standard deviation 0.64%.

[4] We follow Cochrane and Hansen (1992) and use the stock market return to proxy for the return on wealth.

satisfies the HJ bound). For this parameter setting the volatility of the IMRS (variance) is 0.292, the integral of the spectral bound is 0.290, and the HJ bound is 0.094.

Figure 7 graphs the spectral bound and IMRS. The model misses dramatically at high frequencies and exceeds the bound at business cycle and lower frequencies. Note that the shape of the spectral bound as well as the integral of the spectral bound in Figure 7 is different from those in Figure 6. This is due to two reasons: (i) as already noted in Section II our spectral bound is model dependent, and (ii) the mean IMRS for state non-separable preferences is different from that of time-separable preferences.

Figure 7: State Non-Separable Model



III.3 Habit Formation Preferences

Sundaresan (1989), Constantinides (1990), and others model consumers who are habitual, in that levels of consumption in adjacent periods are complementary. That is, the preferences of consumers (in a discrete-time version of Constantinides, 1990) are given by:

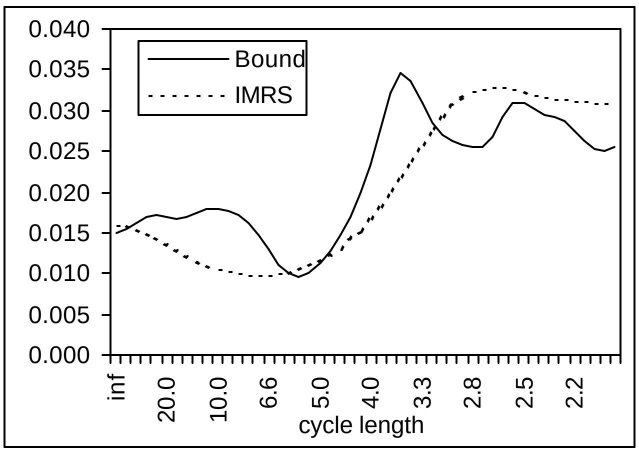
$$U_o = E_o \sum_{t=0}^{\infty} b^t \frac{[(1+d(L))c_t]^{1-s}}{1-s},$$

where δ(L) is a polynomial in the lag operator L. When the lag coefficients are all negative, the preferences exhibit habit-persistence. When the coefficients in δ(L) are zero, the preferences are time-separable. Here we work with the popular “one-lag habit” case with δ(L) = δL. The representative agent's IMRS is given by:

$$m_{t+1} = b\,\frac{(c_{t+1} + dc_t)^{-s} + bdE_{t+1}(c_{t+2} + dc_{t+1})^{-s}}{(c_t + dc_{t-1})^{-s} + bdE_t(c_{t+1} + dc_t)^{-s}}.$$

With β = 0.95, δ = -0.69 and σ = 4.949 the volatility in the IMRS exceeds the integral of the spectral bound (with β = 0.95, δ = -0.77 and σ = 2.41 the model satisfies the standard HJ bound). For this parameter setting the volatility of the IMRS is 0.1369, the integral of the spectral bound is 0.1367, and the HJ bound is 0.095. Figure 8 indicates that the model is doing reasonably well at most frequencies except the business cycle frequencies. Alternative parameterizations of the δ(L) function will alter the spectral nature of the IMRS and the spectral bound. In the next subsection we investigate a related habit model in which the consumers' habit adjusts slowly in response to changes in consumption.

Figure 8: Internal Habit Model



III.4 External Habit Formation

Abel (1990) introduced a form for preferences in which habit formation is external, that is, people care about their consumption relative to the consumption of others. Campbell and Cochrane (1999) developed a more elaborate external habit formation model that mimics many features of the asset return data. They specify preferences as:

$$U(c_t, X_t) = E_0 \sum_{t=0}^{\infty} d^t \frac{(c_t - X_t)^{1-g} - 1}{1-g}.$$

The IMRS is given by:

$$m_{t+1} = b\left(\frac{S_{t+1}C_{t+1}}{S_t C_t}\right)^{-s},$$

where $c_t = C_t$ in equilibrium and S_t is the surplus consumption ratio is given by $S_t = \dfrac{C_t - X_t}{C_t}$. The evolution of the Habit stock is defined in terms of $\ln(S_t)$:

$$\ln(S_{t+1}) = (1-f)\ln(\bar{S}) + f\ \ln(S_t) + l[\ln(S_t)](\ln(C_{t+1}) - \ln(C_t) - m) \text{ where}$$

$$l[\ln(S_t)] = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2(\ln(S_t) - \ln(\bar{S}))} - 1 & \ln(S_t) \le S_{\max} \\ 0 & \ln(S_t) > S_{\max} \end{cases} \quad \text{and}$$

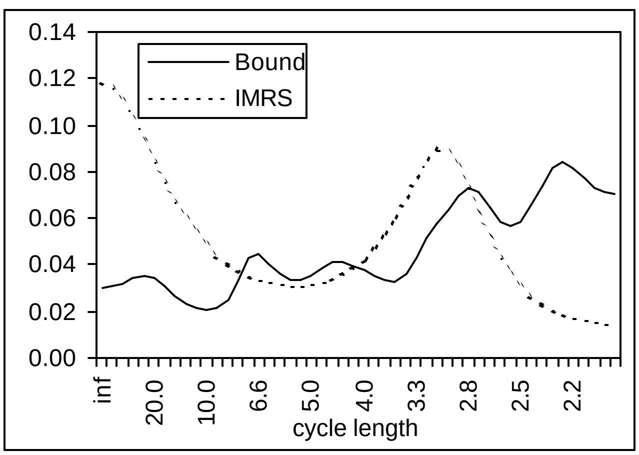
$$\bar{S} = k\sqrt{\frac{s}{1-f}}.$$

The parameter κ is the standard deviation of consumption growth, while μ is its mean.

With β = 0.9, φ = 0.9 and σ = 1.275 the volatility in the IMRS exceeds the integral of the spectral bound (with β = 0.89, φ = 0.87 and σ = 1.05 the model satisfies the standard HJ bound). For this parameter setting the volatility of the IMRS is 0.317, the integral of the spectral bound is 0.296, and the standard HJ bound is 0.13.

Figure 9 and illustrates the spectral bound and spectrum of the IMRS. The model fairs very well at the lowest frequencies, but misses at the higher frequencies.

Figure 9: External Habit Model



The spectral bounds presented in Figures 6 to 9 allow us to observe the set of frequencies at which the model does or does not do well. It is natural to focus on the set of frequencies that the model is designed to explain and ask how well the model does at those frequencies. Here we examine how well the models do at business cycle frequencies in order to assess whether the various asset-pricing models are consistent with the business cycle properties of returns alluded to at the outset of this section. This exercise is also useful in thinking about the literature that tries to explain the joint behavior of asset prices and business cycle quantities. Examples in this literature include Rouwenhorst (1995), Jermann (1998), Lettau and Uhlig (2000), Tallarini (2000) and Boldrin, Christiano and Fisher (2001). To evaluate the models at business cycle frequencies we integrate the spectral bound over cycles of length 6-32 quarters. We also integrate the spectrum of the model IMRS over the same set of frequencies. Table 1 reports the results of this exercise. Of the four models only the Epstein-Zin model reaches the spectral bound over business cycle frequencies. Evidently, the other models require modifications if they are to approach the bound at the business cycle frequencies; the next section begins exploring one such modification.

Table 1. IMRS and Spectral Bound for Asset-Pricing Models

Model	IMRS Mean	IMRS Volatility	Spectral Bound	IMRS Volatility business cycles	Spectral Bound business cycles
Time-Separable	0.989	8.490	7.810	1.970	0.690
Epstein-Zin	0.997	0.292	0.291	0.119	0.016
Internal Habit	0.998	0.137	0.137	0.020	0.029
External Habit	0.996	0.317	0.296	0.099	0.054

IV. Spectral Bound with Frictions

He and Modest (1995) and Luttmer (1996) find that frictions such as short sales constraints, borrowing constraints, transactions costs and solvency constraints can explain the apparent failure of the standard consumption-based asset-pricing model without resorting to non-separabilities in preferences.[5] They show how to construct HJ bounds for economies that allow for such frictions. In this section we show how to extend the spectral bound from Section II.1 to economies with frictions. For illustrative purposes we study only economies with short-sale constraints.[6]

Our derivation of the HJ bound with frictions follows closely the approach in He and Modest. They show that in the presence of short sales constraints the Euler equation (1) becomes an inequality:

$$ERm \leq \iota.$$

For assets with no short sale constraints, the restriction holds with equality. They then modify the inequality to

[5] Aiyagari (1993), based on analysis in Aiyagari and Gertler (1991), argues that "… deviating from the complete frictionless markets framework is … both necessary and fruitful for solving the return puzzles …" (p. 18).

[6] The derivation of the bound with short sale constraints is also applicable to some models with incomplete markets. For instance, Constantinides and Duffie (1996) show that some heterogeneous agent economies with uninsurable idiosyncratic shocks yield Euler equation inequalities that are identical to those that arise in representative agent models with frictions.

$ERm = \chi, \ \chi \leq \iota,$

where χ is a vector of unknown parameters. Given χ and v, He and Modest show that the HJ bound for this economy is

(20) $\mathrm{var}(m_v) = (\chi - vER)'\Omega^{-1}(\chi - vER).$

They restrict all the Lagrange multipliers on the short-sale constraints to be the same. Since they are interested in constructing a lower bound they choose the unknown χ to minimize var(m_v) in (31).

The spectral bound corresponding to (20) follows the derivation in Section II.2. We begin with equation (12)

$$ER_{t-j}g_{v,t} = \Phi_j \text{ , j = 0, } \pm 1, \pm 2, \ldots$$

where Φ_0 is now equal to χ. Following the same steps as in Section II.2, we find the spectral bound applicable in the presence of frictions:

$$S_g(e^{-iw}) = S_{Rg}(e^{iw})' S_R(e^{-iw})^{-1} S_{Rg}(e^{-iw}) \text{ .}$$

where

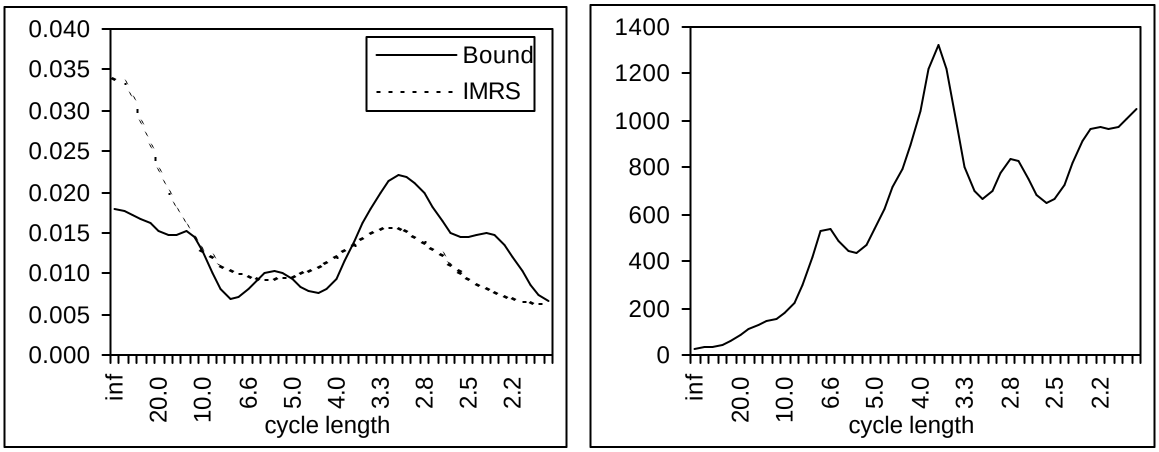
$$S_{Rg}(e^{-iw}) = \sum_{j=-\infty}^{-1} \Psi_j e^{-iwj} + c - vER + \sum_{j=1}^{\infty} \Psi_j e^{-iwj}$$

IV.1.1 Time-Separable Preferences

For these preferences, we find that the volatility of the IMRS exceeds the integral of the spectral volatility bound with β = 0.99 and σ = 65. For this parameter setting the volatility of the IMRS is 0.085, the integral of the spectral bound is 0.084, and the HJ bound is 0.059. Panel a of Figure 10 illustrates the spectral bound (solid curve) and the spectrum of the IMRS (dotted curve). Panel b is the spectral bound with the above parameters but without frictions. This figure illustrates the effect of frictions on the bound itself. Perhaps counterintuitively, the imposition of frictions is not just a high frequency phenomenon—such frictions affect the equilibrium relationship between consumption and asset returns at all frequencies.

Figure 10: Time Separable Model

a: Spectral Bound with Frictions **b: Spectral Bound without Frictions**

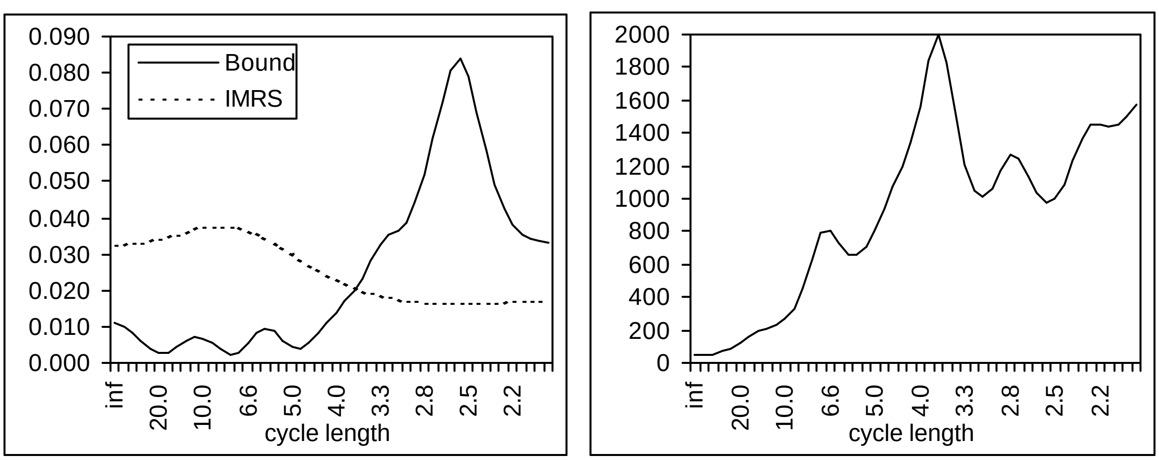


IV.1.2 State Non-Separable Preferences

For our dataset with β = 0.96, ρ = 3.65 and σ = 30 the volatility in the IMRS exceeds the integral of the spectral bound. For this parameter setting the volatility of the IMRS is 0.16, the integral of the spectral bound is 0.15, and the HJ bound is 0.052. In the figure below we can see that the model does well at business cycle frequencies, but not at higher frequencies.

Figure 11: State Non-Separable Model

a: Spectral Bound with Frictions **b: Spectral Bound without Frictions**

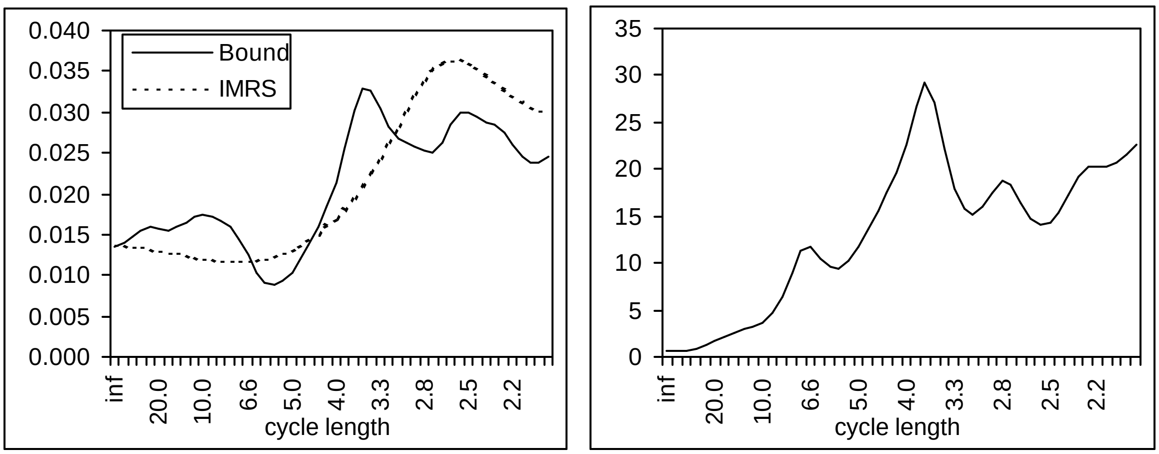


IV.1.3 Habit Formation Preferences

With β = 0.91, δ = -0.75 and σ = 3.6 the volatility in the IMRS exceeds the integral of the spectral bound. For this parameter setting the volatility of the IMRS is 0.133, the integral of the spectral bound is 0.130, and the HJ bound is 0.090. The model appears to do well at high frequencies, but not at lower frequencies.

Figure 12: Internal Habit Model

a: Spectral Bound with Frictions **b: Spectral Bound without Frictions**



IV.1.4 External Habit Formation

With β = 0.9, ϕ = 0.94 and σ = 0.9 the volatility in the IMRS exceeds the integral of the spectral bound For this parameter setting the volatility of the IMRS is 0.115, the integral of the spectral bound is 0.113, and the HJ bound is 0.085.

Figure 13: External Habit Model

a: Spectral Bound with Frictions **b: Spectral Bound without Frictions**

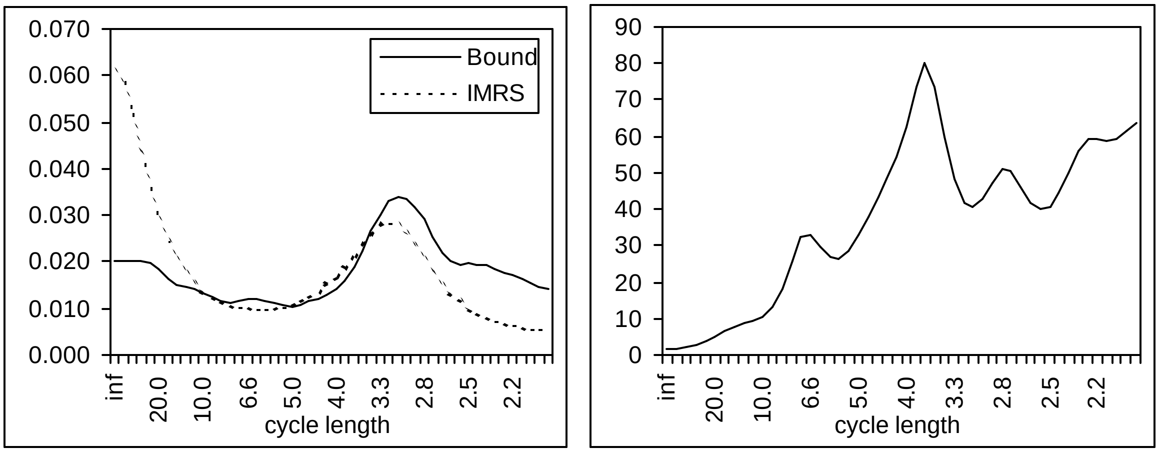


Table 2: IMRS and Spectral Bound for Economies with Frictions

Model	IMRS Mean	IMRS Volatility	Spectral Bound	IMRS Volatility business cycles	Spectral Bound business cycles
Time-Separable	0.788	0.085	0.084	0.028	0.022
Epstein-Zin	0.742	0.161	0.159	0.067	0.009
Internal Habit	0.966	0.133	0.131	0.023	0.028
External Habit	0.945	0.115	0.113	0.035	0.026

V. Conclusion

In this paper we proposed a volatility bound for the evaluation of asset pricing and business cycle models. The bound is a generalization of the volatility bound proposed by Hansen and Jagannathn (1991) that incorporates serial correlation properties of returns. The spectral bound that we develop allows for model evaluation by frequency; it thus allows the researcher to determine the frequencies at which the model is not doing well. A business cycle model that violates the bound at business cycle frequencies would be cause for concern, though one might not care that a consumption-based asset pricing model is inconsistent with the movement of asset prices and consumption at the higher frequencies. Under the assumption that the business cycle frequencies *are* of interest, our results suggest that asset-pricing models that incorporate state non-separable preferences perform best.

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