The Ex Ante Incentive Compatible Core of the Assignment Game[*]

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Résumé

Nous considérons un marché bipartite dans lequel les agents disposent d'informations privées sur un état de la nature qui détermine leurs utilités d'appariement. Les transferts monétaires sont permis et les fonctions d'utilité sont quasi-linéaires. Le modèle étend donc les jeux d'allocation introduits par Shapley et Shubik. Nous démontrons que le coeur ex ante incitatif du jeu d'appariement est non-vide. Des exemples simples illustrent deux di¤érences avec l'information complète: tout d'abord, les mécanismes d'appariement aléatoires dé…nissent une fonction caractéristique (à utilité transférable) plus élevée que les mécanismes déterministes; de plus, les solutions du coeur ex ante incitatif ne coïncident pas nécessairement avec les résultats stables correspondants, et ce même si les valeurs sont privées et indépendantes. Notre approche s'étend au coeur brut (interim) incitatif, qui est lui aussi non-vide.

Abstract

We consider two-sided matching markets in which agents have private information on a state of nature which determines the agents' utilities of matching. Monetary transfers are allowed and utility functions are quasi-linear. The model thus extends the assignment game introduced by Shapley and Shubik. We prove that the ex ante incentive compatible core of the matching game is non-empty. Simple examples illustrate two di¤erences with complete information: …rst, random matching mechanisms de…ne a higher TU characteristic function than deterministic ones; furthermore, ex ante incentive compatible core solutions need not coincide with ex ante incentive compatible stable outcomes, even if values are independent and private. Our approach extends to the (interim) incentive compatible coarse core, which is also non-empty.

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1 Introduction

Two-sided matching markets have been extensively studied under the assumption of complete information (see, e.g., Roth and Sotomayor [23]). Roth [22] also investigated ex post stable mechanisms in marriage problems with incomplete information about others' preferences (i.e., private values).

We consider two-sided matching markets with arbitrary incomplete information (i.e., possible common values). The agents are divided into two disjoint sets (e.g., potential buyers and sellers, ..rms and workers, etc.). Every agent has private information on a state of nature which enters agents' (ex post) utility of being matched with a partner from the other side of the market. The (ex post) worth of a coalition is determined by pairwise combinations of agents (from di¤erent sides of the market) and arbitrary money transfers (e.g., in the auctions framework, we allow for entry fees, bidder rings, etc.). Utility functions are assumed to be linear in money. Our model is thus an extension of the (complete information) assignment game introduced by Shapley and Shubik [26]. Under incomplete information, a number of particular cases (bilateral trading, auctions, etc.) have been analyzed in great details (see, among others, [7], [8], [18], [21],[29]).

We extend Shapley and Shubik [26]'s results by proving that the ex ante incentive compatible core of the assignment game is non-empty. This solution concept has been mostly applied in di¤erential information exchange economies (see [13]). It is appropriate if coalitions can form before agents know their private information. The members of a coalition organize matchings and monetary transfers by means of (random) Bayesian incentive compatible mechanisms. This generates a well-de..ned TU characteristic function, namely the maximal sum of (ex ante) expected payo¤s that every coalition can guarantee by relying on an incentive compatible mechanism. The ex ante incentive compatible core is de..ned as the (standard) core of this characteristic function. To establish its non-emptiness, we apply the Bondareva-Shapley theorem.

As in most papers on auctions and bargaining under incomplete information, we assume that matching may result from a lottery. This procedure is natural, especially if one relies on the revelation principle (as will be the case here), but is not needed in Shapley and Shubik [26]'s original model (see also [23]). Random assignment procedures can nevertheless be useful under complete information, for instance to guarantee fairness when money is not available (see, e.g., [1], [5], [6], [16]). We shall show on an example that, when incentive constraints matter, the TU characteristic function associated with random mechanisms can take higher values than the one associated with deterministic mechanisms.

Under complete information, it is also well-known (see again [26]) that the core coincides with the outcomes which are stable, i.e., cannot be blocked by any single agent nor pair of agents (from di¤erent sides of the market). However, as we show on simple examples, no such property holds under incomplete information, and this even under private, independent values. More precisely, one can construct incentive compatible mechanisms which cannot be blocked by any single agent nor any pair of agents, but are blocked by three agents coalitions.

The non-emptiness of the ex ante incentive compatible core means that there exists an incentive compatible mechanism ^{1} for the grand coalition (which selects matchings and transfers) such that no coalition can propose an incentive compatible mechanism improving the expected payo¤ of all of its members. In particular, ^{1} is incentive ex ante e¢ cient (in the sense of Holmström and Myerson [15]) and ex ante individually rational. ^{1} is thus also incentive interim e¢ cient. But ^{1} is not necessarily interim individually rational.

Given the importance of the latter property in the mechanism design literature, a natural question concerns the existence of incentive compatible core solutions that would be interim individually rational, i.e., such that all types of agents would like to participate in the mechanism at the interim stage. A satisfactory solution concept in this respect is the incentive compatible coarse core, namely Vohra [27]'s incentive compatible version of Wilson [28]'s coarse core. In this approach, agents form coalitions after having learnt their types but can only exchange information inside coalitions, once these have formed. They thus use exactly the same incentive compatible mechanisms as above. However, blocking takes place at the interim stage so that, basically, every possible type of an agent should get a better payo¤. In particular, incentive compatible coarse core solutions are interim individually rational.

In a similar way as in Wilson [28], we construct an auxiliary (NTU) characteristic function such that the incentive compatible coarse core is the standard core of this characteristic function. Given the NTU structure of the game, we rely on on Scarf [24]'s theorem to prove that the incentive compatible coarse core is non-empty. This result guarantees that, in a large class of trading models with incomplete information, there exist incentive compatible mechanisms that are not only interim e¢ cient and interim individually rational (as established in a number of papers), but cannot even be improved by any coalition. In that sense, these mechanisms are "collusion-proof".

The positive results obtained here contrast with the ones that prevail in differential information exchange economies. As shown by Forges, Mertens and Vohra [11], in this model, both the ex ante incentive compatible core and the incentive compatible coarse core can be empty, even if random mechanisms and monetary transfers are allowed. The linear structure of the assignment game is certainly helpful but this explanation of the results should be used with care. The incentive compatible cores are not empty in exchange economies with linear utility functions (see [27]) but random mechanisms can be dispensed with in this model. On the contrary, here, they enable the agents to achieve higher expected payoffs. Hence, the result on linear exchange economies cannot be applied directly. It is also true that random mechanisms generate linearity in the sense that expected payoffs are linear in the mechanism probabilities. However, this does not suffice for the non-emptiness of the incentive compatible cores, as the counter-example in [11] shows.

The quasi-linear utility functions are useful in [11], as in many papers on mechanism design (see, e.g. [2], [3], [4], [7], [8] and [17]) to construct incentive compatible, ex post efficient mechanisms, which turn out to achieve expected payoffs in the ex ante incentive compatible core. Such results also apply here: under appropriate assumptions, the ex ante incentive compatible core contains ex post efficient allocations. However, no specific assumptions (e.g., on the agents' beliefs) are needed here to establish the non-emptiness of the ex ante incentive compatible core. Even more, this result (as well as the non-emptiness of the incentive compatible coarse core) still holds if there are no monetary transfers at all. The model covers then incomplete information versions of the marriage problem and the indivisible good economy of Shapley and Scarf [25]. Obviously, in this case, the underlying characteristic function is always NTU, even in the ex ante framework. The main reason to consider monetary transfers is to connect our analysis with the literature on trading with incomplete information (e.g., auctions), in which Bayesian incentive compatible mechanisms have become a standard tool.

The next section describes the model. Section 3 defines the ex ante incentive compatible core and establishes its non-emptiness. Section 4 is devoted to the incentive compatible coarse core. Examples illustrate our solution concepts in section 5. Section 6 concludes with further remarks on possible ex post properties.

2 Incentive compatible matching mechanisms

The economy consists of two, finite, disjoint sets of agents I and J: $I \cap J = \emptyset$; let $K = I \cup J$. Every agent $k \in K$ has private information, represented by his type $t_k \in T_k$, where T_k is a finite set[1]; let $T = \prod_{k \in K} T_k$ and let q be a probability distribution over T, the common prior of the agents, such that, without loss of generality, $q(t_k) > 0$ for every t_k. Every agent derives (von Neumann - Morgenstern) utility from being matched with an agent of the other side of the market and from monetary transfers. For every $i \in I$, $j \in J$ and $t \in T$, let $u_i(t; j)$ (resp., $u_j(t; i)$) denote agent i (resp., j)'s utility of being matched with agent j (resp., i) when the types are t. Let also $u_k(t; 0)$ denote agent k's utility of being unmatched ($k \in K$, $t \in T$). Monetary transfers are just added linearly to these utility numbers. Agents thus have additively separable utility functions for matching and transfers.

Except perhaps for the latter assumptions, the framework is quite general: any informational externality is allowed (the whole vector of types t enters all utility numbers, so that common values are possible). Furthermore, any matching externality is also allowed, since every agent cares about his partner.

As a typical example, extending Shapley and Shubik [26]'s housing market, the I-agents are sellers, each with a single item for sale, and the J-agents are potential buyers; $u_i(t; j)$ represents seller i's utility of selling his item to buyer j at the state of information t, while $u_j(t; i)$ represents buyer j's utility of buying an item from seller j at t. Standard assumptions are that the sellers are indifferent to sell to a buyer or another ($u_j(t; i) = u_j(t; i')$ for every $i, i' \in I$), or that the private information of traders coincides with their reservation prices (private values: $u_i(t; j) = u_i(t_i; j)$ represents buyer i's reservation price for seller j's item when his type is t_i, $u_j(t; i) = u_j(t; i') = u_j(t_j)$ represents seller j's reservation price for his item). This example covers most auction models. Monetary transfers are natural in this example; they basically correspond to sale prices, but may also involve compensations for not getting an item, entry fees, etc.

The restrictions imposed on the market will be reflected by the feasible matching mechanisms. The outcomes of such a mechanism consist of disjoint pairs $\{i, j\}$, $i \in I$, $j \in J$ of matched agents, a set of unmatched agents and

[1] This assumption (also made in, e.g., [7], [8], [11], [17], [27], etc.) is mainly for simplicity. Most of our results go through if the sets of types are real intervals and the beliefs have a density, as in [18], [19], [21], etc.

monetary transfers $m_k; k \in K$ such that $\sum_{k \in K} m_k \leq 0$. In particular, an agent can be matched to at most one agent of the other side of the market and arbitrary transfers are allowed, including between agents who are not matched together.

We assume that agents exchange information and match after having formed coalitions[2]. A coalition S is just a non-empty subset of K. The members of S can rely on any mechanism (in particular, any non-cooperative bargaining game) to organize matching and transfers. By the revelation principle (see, e.g., [18] for an application to auctions and [22] for an application to matching), all feasible collective decisions of S can be represented as the outcomes of a random, direct, incentive compatible, matching mechanism, which we define precisely below.

As in most papers on collective decisions under incomplete information (see [7], [8], [11], [12], [15], [18], [22], etc.), we allow that matching results from lotteries, which is usual in many auctions mechanisms. Lotteries also appear as a consequence of the revelation principle if agents are allowed to use mixed bargaining strategies. Under complete information, it is well-known that random matching mechanisms do not achieve more expected payoffs than the deterministic ones (see [26], [23]). This is no longer the case under asymmetric information, as we show in section 5.

Let S be a coalition; let us set $T_S = \prod_{k \in S} T_k$. For every $t = (t_k)_{k \in K} \in T$, let $t_S = (t_k)_{k \in S}$. A (random) matching mechanism[3] for S consists of mappings $\mu_S = (x_S; m_S) : T \to [0;1]^{I \times J} \times \mathbb{R}^K : t \to [(x_S^{ij}(t))_{(i;j) \in I \times J}; (m_S^k(t))_{k \in K}]$ such that

- μ_S is measurable w.r.t. T_S, namely $\mu_S(t) = \mu_S(t')$ for every $t; t' \in T$: $t_S = t'_S$, and for every $t \in T; x_S^{ij}(t) = 0$ if $i \notin S$ or $j \notin S$ and $m_S^k(t) = 0$ if $k \notin S$
- $\sum_{j \in J} x_S^{ij}(t) = \sum_{j \in J \cap S} x_S^{ij}(t) \leq 1$ for every $i \in I$ and $\sum_{i \in I} x_S^{ij}(t) = \sum_{i \in I \cap S} x_S^{ij}(t) \leq 1$ for every $j \in J$
- $\sum_{k \in K} m_S^k(t) = \sum_{k \in K \cap S} m_S^k(t) \leq 0$

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[2] Typically, coalitions form at the ex ante stage, i.e., before the agents know their types. In section 4, we will assume that coalitions form at the interim stage, but that agents do not exchange information until they are in a coalition.

[3] We will use the term "matching mechanism" to refer to a "feasible (matching) mechanism" satisfying the feasibility constraints.

The interpretation of these feasibility conditions is that every member of S is invited to report his private information to the mechanism μ_S, which, as a function of the reported types t, matches i and j with probability $x_S^{ij}(t)$, and distributes the (expected) transfers $m_S^k(t)$.[4] The feasibility conditions further assert that a coalition can only use the information of its members, that some agents may be left unmatched, and that transfers must balance. Observe that a matching mechanism μ_S, associated with an arbitrary coalition S, can be viewed as a matching mechanism for the grand coalition K, which leaves unmatched all agents in $K \setminus S$.

In order to define incentive compatibility of μ_S, consider $i \in I \cap S$; let t_i (resp., t_i') denote agent i's true (resp., reported) type; agent i's expected utility is

$$U_i(\mu_S|t_i; t_i') = \sum_{t_{-i}} q(t_{-i}|t_i) \left[\sum_{j \in J \cup \{0\}} x_S^{ij}(t_i'; t_{-i}) u_i((t_i; t_{-i}); j) + m_S^i(t_i'; t_{-i}) \right]$$

where $t_{-i} = t_{K \setminus \{i\}}$ and for every $t \in T$, $x_S^{i0}(t) = 1 - \sum_{j \in J} x_S^{ij}(t)$ is the probability that agent i is left unmatched by μ_S at t. Let us denote as $U_i(\mu_S|t_i)$ the (interim) expected utility of agent i when he truthfully reports his type to μ_S, namely

$$U_i(\mu_S|t_i) = U_i(\mu_S|t_i; t_i)$$

μ_S is incentive compatible if

$$U_i(\mu_S|t_i) \geq U_i(\mu_S|t_i; t_i') \text{ for every } i \in I \cap S; t_i; t_i' \in T_i \tag{1}$$

and similarly for the agents $j \in J \cap S$ on the other side of the market.

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[4] Every $x_S(t) \in [0;1]^{I \times J}$ satisfying the above conditions is a substochastic matrix and thus a convex combination of deterministic matchings in $\{0;1\}^{I \times J}$ (this is a variant of the well-known Birkhoff-von Neumann theorem, see, e.g., [26] or [6] for a recent application). The description of random mechanisms by matching probabilities (as opposed to probability distributions over deterministic matchings) suffices here, thanks to the properties of the utility functions (such a simplification is not always possible in exchange economies, see [11], [12] and the discussion following proposition 1). In the same way, transfers are deterministic without loss of generality.

We denote as $U_k(\mu_S)$ the (ex ante) expected utility of agent $k \in S$ from the matching mechanism μ_S, namely

$$U_k(\mu_S) = \sum_{t_k} q(t_k) U_k(\mu_S | t_k)$$

All the notions introduced in this section are illustrated in section 5.

3 The ex ante incentive compatible core

We define a cooperative game, namely a characteristic function[5] V_A^*, from the previous set up. A coalition S can achieve a payoff vector $v = (v_k)_{k \in S} \in R^S$, i.e., $v \in V_A^*(S)$, iff there exists an incentive compatible mechanism μ_S such that $v_k \leq U_k(\mu_S)$ for every $k \in S$. The characteristic function V_A^* is consistent with the following scenario:

- Every coalition S chooses a matching mechanism μ_S, coalitions possibly form
- A vector of types $t \in T$ is selected according to q; every agent $k \in K$ is informed of his own type $t_k \in T_k$
- If S has formed, μ_S is implemented, members of S report their types to μ_S, matchings and tranfers are realized according to μ_S.

One immediately checks that if $v = (v_k)_{k \in S} \in V_A^*(S)$ and $w = (w_k)_{k \in S} \in R^S$ is such that $\sum_{k \in S} w_k = \sum_{k \in S} v_k$, then $w \in V_A^*(S)$. Indeed, w can be achieved by modifying the transfers used for v independently of types, i.e., in an incentive compatible way. The cooperative game is thus TU and one can describe the characteristic function as

$$v_A^*(S) = \max_{\mu_S \ IC} \left[\sum_{k \in S} U_k(\mu_S) \right] \tag{2}$$

where the maximum[6] is over all incentive compatible (IC) matching mechanisms μ_S. This defines the ex ante assignment game. Observe that the expected sum of transfers, namely $\sum_t q(t) \sum_{k \in S} m_S^k(t) \leq 0$, is part of the above

[5] As in [15], we put a "*" for the incentive compatible concepts.

[6] (2) defines a "general linear problem" (see, e.g., Gale [14]). It is easily checked that the primal and the dual problems are feasible and thus have a solution.

objective function, and that, implicitly, $v_A^*(\{k\}) = \sum_t q(t)u_k(t;0)$ for any single agent k.

As a benchmark, we consider the first best characteristic function v_A in which agents are not submitted to incentive compatibility constraints. In this case, at an optimum, the sum of monetary transfers is exactly 0:

$$\begin{aligned} v_A(S) &= \max_{\mu_S}\Big[\sum_{k\in S} U_k(\mu_S)\Big] \\ &= \max_{x_S} \sum_t q(t)\left[\sum_{i\in I}\sum_{j\in J\cup\{0\}} x_S^{ij}(t)u_i(t;j) + \sum_{j\in J}\sum_{i\in I\cup\{0\}} x_S^{ij}(t)u_j(t;i)\right] \end{aligned}$$

Obviously,

$$v_A^*(S) \leq v_A(S)$$

Shapley and Shubik [26] (see also [23]) have shown that under complete information, one can restrict oneself on deterministic mechanisms without loss of generality. Hence, in the above expression of $v_A(S)$, the maximum is in fact over all deterministic mechanisms x_S, with values in $\{0;1\}$. As we already mentioned, this property is no longer true under incomplete information. In example 3 (section 5), the best expected payoff a coalition S can get by relying on deterministic incentive compatible matching mechanisms is strictly less than $v_A^*(S)$.

Another interesting property of the assignment game with complete information is that all gains are generated by pairs of agents from the different sides of the market (see [23]). More precisely, the worth of any coalition S can then be computed directly from the worths of all singletons and pairs contained in S and, as a consequence, transfers are only needed between pairs of matched agents. This structure does not survive under incomplete information, even in the simplest sellers-buyers models with independent private values (see example 1 section 5). The intuitive reason for this is that in our model, the players exchange information after coalitions have formed. If they partition into pairs at the coalition formation stage, they may loose the interaction possibilities which are offered by larger coalitions' mechanisms.

Recall that the core of a TU game v over K is the set of all vector payoffs $w = (w_k)_{k\in K}$ which can be achieved by the grand coalition, i.e., $\sum_{k\in K} w_k \leq v(K)$, and cannot be improved upon by any coalition, i.e., $\sum_{k\in S} w_k \geq v(S)$ for every S. We refer to $C(v_A^*)$ as to the "ex ante incentive compatible core" of the assignment game. Under complete information, Shapley and Shubik [26] have proved that the core of the assignment game is not empty. A straighforward application of the Bondareva-Shapley theorem will enable us to show that Shapley and Shubik's result extends, namely $C(v_A^*) \neq \emptyset$. However, as we pointed out above, many nice properties of the characteristic function disappear. In particular, the ex ante incentive compatible core does not coincide with the (payoffs of) stable matchings (which are individually rational and cannot be blocked by any pair $(i;j)$ of agents in $I \times J$; see examples 1 and 2 in section 5).

Proposition 1 The (TU) assignment game defined by v_A^* is balanced, so that the ex ante incentive compatible core $C(v_A^*)$ is non-empty.

The result is a direct consequence of the following lemma:

Lemma 1 Let $\mathcal{S}$ be a balanced family of coalitions with associated weights λ_S, $S \in \mathcal{S}$, and let $\mu_S = (x_S; m_S)$ be a mechanism for S, $S \in \mathcal{S}$. Then $\mu = \sum_{S \in \mathcal{S}} \lambda_S \mu_S$ is a matching mechanism for K and satisfies

$$U_k(\mu|t_k; t_k') = \sum_{S \in \mathcal{S}: S \ni k} \lambda_S U_k(\mu_S|t_k; t_k') \quad \forall k; t_k; t_k' \tag{3}$$

In particular, if every μ_S is incentive compatible, so is μ.

Proof of the lemma: Recall that a mechanism μ_S for an arbitrary coalition S can be viewed as a mechanism for the grand coalition. The mechanism μ defined in the statement is T-measurable as a linear combination of T-measurable mechanisms. By definition of balancedness, $\sum_{S \in \mathcal{S}: S \ni k} \lambda_S = 1$ for every $k \in K$; $\mu = (x; m)$ is not, as such, a convex combination of the μ_S's but satisfies

$$x^{ij}(t) = \sum_{S \in \mathcal{S}: S \supseteq \{i;j\}} \lambda_S x_S^{ij}(t) \quad \forall i \in I; j \in J; t \in T \tag{4}$$

$$m^k(t) = \sum_{S \in \mathcal{S}: S \ni k} \lambda_S m_S^k(t) \quad \forall k \in K; t \in T \tag{5}$$

so that μ is feasible. Indeed, let $j \in J$ and $t \in T$:

$$\begin{aligned}
\sum_{i \in I} x^{ij}(t) &= \sum_{i \in I} \sum_{S \in \mathcal{S}: S \supseteq \{i;j\}} \lambda_S x_S^{ij}(t) \\
&= \sum_{S \in \mathcal{S}: S \ni j} \lambda_S \sum_{i \in I \cap S} x_S^{ij}(t) \leq 1
\end{aligned}$$

and similarly on the other side of the market. The transfers satisfy

$$\sum_{k\in K} m^k(t) = \sum_{k\in K}\sum_{S\in\mathcal{S}:k\in S}\lambda_S m^k_S(t) = \sum_{S\in\mathcal{S}}\lambda_S\sum_{k\in S} m^k_S(t) \le 0$$

The equalities (3) can be checked in a similar way:

$$\begin{aligned}
&\sum_{j\in J} x^{ij}(t_i'; t_{-i})u_i((t_i; t_{-i}); j) \\
&= \sum_{j\in J}\Big[\sum_{S\in\mathcal{S}:S\subseteq\{i,j\}}\lambda_S x^{ij}_S(t_i'; t_{-i})\Big]u_i((t_i; t_{-i}); j) \\
&= \sum_{S\in\mathcal{S}:S\ni i}\lambda_S\sum_{j\in J\cap S} x^{ij}_S(t_i'; t_{-i})u_i((t_i; t_{-i}); j)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&x^{i0}(t_i'; t_{-i})u_i((t_i; t_{-i}); 0) \\
&= \Big[1 - \sum_{j\in J}\sum_{S\in\mathcal{S}:S\subseteq\{i,j\}}\lambda_S x^{ij}_S(t_i'; t_{-i})\Big]u_i((t_i; t_{-i}); 0) \\
&= \sum_{S\in\mathcal{S}:S\ni i}\lambda_S\Big[1 - \sum_{j\in J\cap S} x^{ij}_S(t_i'; t_{-i})\Big]u_i((t_i; t_{-i}); 0)
\end{aligned}$$

and the transfers satisfy (5).

Q.E.D.

Proof of the proposition: Let $\mathcal{S}$ be a balanced family of coalitions with associated weights λ_S, $S \in \mathcal{S}$. We must show that $v^*_A(K) \ge \sum_{S\in\mathcal{S}}\lambda_S v^*_A(S)$. Let, for every $S \in \mathcal{S}$, μ_S be a mechanism achieving the maximum in (2), namely $v^*_A(S) = \sum_{k\in S} U_k(\mu_S)$ and define μ as in the lemma. μ is incentive compatible. By the linearity of utility functions,

$$\begin{aligned}
v^*_A(K) &\ge \sum_{k\in K} U_k(\mu) = \sum_{k\in K}\sum_{S\in\mathcal{S}:S\ni k}\lambda_S U_k(\mu_S) \\
&= \sum_{S\in\mathcal{S}}\lambda_S\sum_{k\in S} U_k(\mu_S) = \sum_{S\in\mathcal{S}}\lambda_S v^*_A(S)
\end{aligned}$$

Q.E.D.

Let us compare the previous proposition with the results obtained for the ex ante incentive compatible core of an exchange economy in Vohra [27], Forges and Minelli [12] and Forges, Mertens and Vohra [11]. The sellers-buyers example (without matching externalities for the sellers) is basically a particular case of the economies considered in [11], where arbitrary monetary transfers are allowed and utility functions are quasi-linear. As shown in [11], these conditions do not guarantee the non-emptiness of the ex ante incentive compatible core. Here, the utility functions are in fact linear, and this even if one restricts oneself to deterministic mechanisms (in which all probabilities of matching are 0 or 1).

The ex ante incentive compatible core of an exchange economy with linear utility functions is known to be non-empty; random mechanisms are clearly useless in that framework (see, e.g., [27], [13]). Let us show that this result cannot be applied directly here. It is tempting to view the matching probabilities of section 2 as quantities of indivisible goods, e.g., by appealing to a time-sharing interpretation. However, the standard resource constraints do not account for the feasibility constraints stating that each buyer is matched with (total) probability less than one at every state of nature. This should result from an assumption on the buyers' initial utility functions, namely that, at every state of nature, a buyer's utility for several houses is just his utility for his favorite house. Such utility functions are obviously not linear. To restrict on time-shares (as opposed to probability distributions over deterministic allocations) and utility functions that are linear in these, as in section 2 (recall footnote 4), a construction much more tedious than above is needed. This approach generates core mechanisms which are feasible in expectation. These are always feasible here (see again footnote 4), unlike in general exchange economies with asymmetric information.

Let us further illustrate the differences between the assignment game and an exchange economy by considering a simple example. Let $I = \{1;2\}$ and $J = \{3;4\}$; consider the balanced family $S = \{\{1;2;4\};\{1;3\};\{2;3\};\{4\}\}$ with all weights equal to $\frac{1}{2}$ and the following deterministic matchings for some state of nature t: 2 and 4 in $\{1;2;4\}$, 1 and 3 in $\{1;3\}$, 2 and 3 in $\{2;3\}$ (and null transfers). The mechanism constructed in the lemma is, at t,

$$x(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which is feasible as expected. Let us now view the example as an exchange economy, in which agent 1 and agent 2, the sellers, both initially have 1 unit of good. The following deterministic mechanisms are now feasible: agent 2

gets 2 units in $\{1;2;4\}$; agent 3 gets 1 unit in $\{1;3\}$ and 1 unit as well in $\{2;3\}$. Proceeding as in the lemma yields the mechanism which allocates 0 or 2 units, each with probability $\frac{1}{2}$, to agent 2 and 1 unit with probability 1 to agent 3. This mechanism cannot be feasible since it distributes 3 units with positive probability. However, the average mechanism, which gives 1 to agent 2 and 1 to agent 3 is feasible[7]. If the utility functions are linear, the average mechanism is obviously equivalent to the random one, but then, by contrast with the assignment game, random mechanisms are not useful at all.

4 The incentive compatible coarse core

Up to now, we have assumed that coalitions form at the ex ante stage, before agents know their types. Such a scenario is obviously not always feasible (see, e.g.,[15]): if types represent intrinsic characteristics of the agents, the matching procedure cannot start before the interim stage, in which every agent only knows his own type. If we maintain the assumption that agents do not exchange information until they are in a coalition, the incentive compatible version of Wilson [29]'s coarse core proposed by Vohra [27] is then an appropriate notion of the core. In particular, solutions in the incentive compatible coarse core are interim individually rational in the standard sense (see [18], [19], [21] etc.) and incentive interim efficient (see [15]). Agents will exchange information only after having formed coalitions, by means of the same mechanisms as in section 2, but will possibly block proposals at the interim stage. As a consequence, they will base their objections on events that are common knowledge inside the coalition at the interim stage.

In order to make the formal definition of common knowledge events (and thus of the incentive compatible coarse core) as simple as possible, we shall make, throughout this section, the assumption that all vector of types occur with positive probability: $q(t) > 0$ for every $t \in T$, e.g., that types are independent.[8] In this case, the set of all types T is the only event that can be common knowledge in a coalition S of two agents or more at the interim

[7] The present example is simplistic since there is only one good. [11] fully defines a TU exchange economy in which the ex ante incentive compatible core is empty. Hence, the corresponding TU game cannot be balanced (which is obviously much stronger than verifying that the most naive construction to establish balancedness does not work, as we did above).

[8] All our results go through without this restriction, by defining the incentive compatible

stage (since the information partition of agent k of type t_k^0 is the set of all type vectors t such that $t_k = t_k^0$). Obviously, if $S = \{k\}$, "common knowledge" is synonymous with "knowledge", and at the interim stage, agent k knows his own type t_k.

Let μ be a matching mechanism for the grand coalition and let ν_S be a matching mechanism for coalition S, $|S| \geq 2$, as defined in section 2. ν_S is a coarse objection to μ iff

$$U_k(\nu_S|t_k) > U_k(\mu|t_k) \quad \forall k \in S; t_k \in T_k \tag{6}$$

In a similar way, agent k of type t_k has a coarse objection to μ iff

$$\sum_{t_{-k}} q(t_{-k}|t_k) u_k(t;0) > U_k(\mu|t_k) \tag{7}$$

so that agent k blocks μ at the interim stage as soon as there exists $t_k \in T_k$ satisfying (7).

μ is an incentive compatible coarse core mechanism iff μ is incentive compatible and no coalition S has an incentive compatible coarse objection to μ. In particular, μ is incentive interim efficient, namely there is no incentive compatible mechanism $\nu = \nu_K$ such that (6) is satisfied for $S = K$, and μ is interim individually rational, namely

$$U_k(\mu|t_k) \geq \sum_{t_{-k}} q(t_{-k}|t_k) u_k(t;0) \quad \forall k \in K; t_k \in T_k$$

In a similar way as in section 3, we define the incentive compatible coarse core as the set of (interim) payoffs (of the form $[(w_k(t_k))_{t_k \in T_k}]_{k \in K} \in \mathbb{R}^N$, $N = \sum_k |T_k|$) to incentive compatible coarse core mechanisms. As pointed out in [27] and [13], there is no inclusion relationship between the ex ante incentive compatible core and the incentive compatible coarse core; indeed, interim individual rationality obviously implies ex ante individual rationality, but the implication goes the other way round for efficiency (see [15]).

Wilson [29] showed that, in the absence of incentive constraints, the coarse core of a well-behaved exchange economy is non-empty, as the standard core of an appropriately defined balanced NTU cooperative game with N players $(k; t_k)$, $k \in K$, $t_k \in T_k$ (see also [13]). Vohra [27] extended his argument to

coarse core as in [27] and [13]. From a conceptual point of view, the concept of the coarse core seems more attractive when no state of nature has zero probability.

exchange economies with linear utility functions. We proceed in the same way here and establish that the incentive compatible coarse core is the core $C(V_I^*)$ of a characteristic function V_I^*. The players in this game are, as in Wilson [29], the types of the original model; we call them "auxiliary players"; the only viable coalitions are of the form $\{(k; t_k)\}$ for some $k \in K$ and some type $t_k \in T_k$, or of the form $\{(k; t_k) : k \in S; t_k \in T_k\}$ for some original coalition $S \subseteq K$, $|S| \geq 2$. The interpretation is clear: at the interim stage, a coalition consists either of a single agent who knows his type or of several agents who have trivial common knowledge (as a consequence of our assumptions) and thus consider all their types as possible. The viable coalitions of two agents or more will thus be identified with the original coalitions $S \subseteq K$, $|S| \geq 2$ with a little abuse of language.

The characteristic function V_I^* is now easily defined on viable coalitions. For every $(k; t_k)$, $V_I^*(\{(k; t_k)\})$ is the set of vector payoffs $w \in R^N$ in which agent k of type t_k gets at most his individually rational level $\sum_{t_{-k}} q(t_{-k}|t_k) u_k(t; 0)$. For every coalition $S \subseteq K$, $|S| \geq 2$, $V_I^*(S)$ is the set of vector payoffs $w \in R^N$ for which there exists an incentive compatible mechanism μ_S for coalition S such that $w_k(t_k) \leq U_k(\mu|t_k)$ for every $k \in S$ and $t_k \in T_k$. Finally, as usual, let us define V_I^* on arbitrary coalitions by taking the superadditive cover. This amounts to allowing the auxiliary players $(k; t_k)$ of a coalition to use an arbitrary incentive compatible matching mechanism if all $(k; t_k')$, $t_k' \in T_k$, are also in the coalition and to leaving unmatched the other auxiliary players. V_I^* is a well-defined NTU game, but clearly, the argument showing that V_A^* defines a TU game does not extend here (see [11] and [13] for related comments).

The core of the interim assignment game V_I^* is the set of all vector payoffs $w = [(w_k(t_k))_{t_k \in T_k}]_{k \in K} \in V_I^*(K)$ which cannot be improved upon by any (viable) coalition. We deduce the non-emptiness of $C(V_I^*)$ from Scarf's [24] theorem.

Proposition 2 The (NTU) assignment game defined by V_I^* is balanced, so that the incentive compatible coarse core $C(V_I^*)$ is non-empty.

Proof: Let $\mathcal{B}$ be a balanced family of viable coalitions in the auxiliary N player game defined above, with associated weights λ_B, $B \in \mathcal{B}$. We must show that $\cap_{B \in \mathcal{B}} V_I^*(B) \subseteq V_I^*(K)$. Assume that, for some k and t_k, the singleton $\{(k; t_k)\}$ belongs to $\mathcal{B}$; then by balancedness, all singletons $\{(k; t_k')\}$, $t_k' \in T_k$, must also be in $\mathcal{B}$, all with same weight, say λ_k. We can thus associate with $\mathcal{B}$ a balanced family $\mathcal{S}$ of coalitions of original players $k \in K$

by merging all B's elements $\{(k,t_k)\}$, $t_k \in T_k$, into a single coalition, with weight λ_k (and keeping unchanged the coalitions of at least two players and their weights, since these already correspond to original players). Since by definition of V_I^*, for every $k \in K$, $V_I^*(\{(k,t_k) : t_k \in T_k\}) = \cap_{t_k} V_I^*(\{(k,t_k)\})$, we also have $\cap_{B \in \mathcal{B}} V_I^*(B) = \cap_{S \in \mathcal{S}} V_I^*(S)$. We can thus pursue the reasoning in terms of the original players, but considering vector payoffs indexed by the types.

Let $w = [(w_k(t_k))_{t_k \in T_k}]_{k \in K} \in \cap_{S \in \mathcal{S}} V_I^*(S)$: for every $S \in \mathcal{S}$, there exists an incentive compatible mechanism μ_S achieving w. Let us define μ as in lemma 1. μ is an incentive compatible mechanism for the grand coalition, such that $U_k(\mu|t_k) \geq w_k(t_k)$ for every k, t_k. Hence $w \in V_I^*(K)$.

Q.E.D.

Remarks:

(i) The previous analysis shows that under the assumption that all type vectors have positive probability, the relevant characteristic function at the interim stage can be defined over the original coalitions $S \subseteq K$, by considering as achievable all vector payoffs $[(w_k(t_k))_{t_k \in T_k}]_{k \in S}$ that result from an incentive compatible mechanism. Coalition S blocks a mechanism μ if there is an incentive compatible mechanism ν_S such that (6) holds. Of course, if S is a singleton, the incentive compatibility conditions are vacuous so that a single agent blocks any mechanism that is not interim individually rational. The construction of the auxiliary game is only necessary to use Scarf's theorem, which is not formulated for vector payoffs.

(ii) Unlike the core of a matching game with complete information, and as the ex ante incentive compatible core, the incentive compatible coarse core is smaller than the set of interim payoffs which cannot be blocked (in the sense of (6) and (7)) by any single agent nor by any pair $(i,j) \in I \times J$ (see example 1 in section 5).

5 Examples

In this section, we motivate the basic model and the solution concept of sections 2 and 3 on simple housing markets and we illustrate two differences between matching games with complete and incomplete information. Example 1 consists of a simple auction (one seller) with independent private values, in which objections by coalitions of more than two agents matter.

The latter property is also true in the second example, but is less surprising there, because values are common and correlated. The goal of this example is to show that the grand coalition deals better with incentive compatibility than small coalitions. Finally, in example 3, random mechanisms improve the characteristic function under incomplete information.

In the sequel, for $i \in I$ and $j \in J$, $-u_i(t;j) = \pi_i(t;j)$ is interpreted as the minimum price at which seller i is willing to sell his house to buyer j when the state of information is t and $u_j(t;i)$ is interpreted as the maximum price that buyer j is willing to pay for seller i's house. The utility of being unmatched is 0 for every trader k.

Example 1

Agent 1, the seller, has no private information and a null reservation price; agents 2 and 3, the potential buyers, have independent private values uniformly distributed over $\{0;1\}$.[9] The basic parameters of the assignment game are thus: $I = \{1\}$, $J = \{2;3\}$, $T_j = \{0;1\}$, $j = 2;3$, t_2 and t_3 are i.i.d. uniformly over $\{0;1\}$, $u_1(t;2) = u_1(t;3) = 0$, $u_j(t;1) = u_j(t_j;1) = t_j$, $j = 2;3$.

We shall show that in this example,

$$v^*_A(\{1;2\}) = v^*_A(\{1;3\}) = \frac{1}{2}$$

while

$$v^*_A(\{1;2;3\}) = \frac{3}{4}$$

Hence a feasible vector payoff like $(\frac{1}{2};0;0)$ cannot be blocked by any two agent coalition but can be blocked by the grand coalition.

Since the seller is not submitted to incentive constraints, one can restrict on transfers summing up to 0 without loss of generality. In particular, in a seller-buyer coalition $\{1;j\}$, $j = 2$ or 3, a mechanism $(x;m)$ can be defined by the probability of trade $x(t_j)$ and the expected transfer $m(t_j)$ from buyer j to the seller, $t_j = 0;1$. The total expected gains from mechanism $(x;m)$ are $\frac{1}{2}x(1) \le \frac{1}{2}$. Hence $v^*_A(\{1;2\}) = v^*_A(\{1;3\}) \le \frac{1}{2}$. On the other hand, $\frac{1}{2}$ can be

[9] The example corresponds to the simplest possible discrete model. Everything goes through if for instance, the pair $\{0;1\}$ is replaced by the interval $[0;1]$. In this case, $v^*_A(\{1;2;3\}) \ge \frac{2}{3}$.

achieved with a (constant) mechanism selling always the object at the price $\frac{1}{2}$.

Let us turn to the grand coalition and consider the feasible, incentive compatible mechanism induced by a second price auction, namely

	0	1
0	no sale	sell to 3 at price 0
1	sell to 2 at price 0	sell to 2 or 3 with probability $\frac{1}{2}$ at price 1

where the rows (resp., columns) correspond to the reported types of agent 2 (resp., 3). The total expected gains from trade are $\frac{3}{4}$, which is the maximum expected value that can be achieved, even in the absence of incentive constraints.

One can also check that no seller-buyer coalition can block the previous mechanism at the interim stage. Indeed, this mechanism is classically (i.e., without incentive constraints) ex ante efficient and interim individually rational.[10]

Example 2

The sellers, agents 1 and 2, both know, at the interim stage, whether the quality of their own house is high or low: $T_1 = \{h_1, l_1\}$, $T_2 = \{h_2, l_2\}$. The probability distribution over the sellers' types is $q(h_1, h_2) = q(l_1, l_2) = \frac{3}{8}$, $q(h_1, l_2) = q(l_1, h_2) = \frac{1}{8}$. The potential buyers, agents 3 and 4, have no private information. The reservation price of a seller for a high (resp., low) quality house is θ_h (resp., θ_l), while for a buyer, the reservation prices are u_h and u_l. We assume that $u_l < \theta_l < \theta_h < u_h$ and that $\frac{1}{2}(u_l + u_h) < \frac{1}{2}(\theta_l + \theta_h)$ (e.g., 0, 9, 12, 20).[11]

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[10] If the buyers' types are uniformly distributed over $[0, 1]$, coalition $\{1, j\}$, $j = 2, 3$, can achieve the vector payoff $(\frac{1}{4}, (t_j - \frac{1}{2}) 1(t_j \geq \frac{1}{2}))$ with an incentive compatible, interim individually rational, mechanism. Furthermore, one can check that $\frac{1}{4}$ is the maximum the seller can expect from a mechanism with these properties in a two-agent coalition. In the grand coalition, the second price auction mechanism yields the expected payoffs $(\frac{1}{3}, \frac{t_2^2}{2}, \frac{t_3^2}{2})$ and thus stricly improves all agents' payoffs.

[11] For a concrete example, think of the termites invading some regions of France. The qualities of houses in the same neighborhood are highly correlated. Many owners do not know whether their house is infected or not, but will go through a test in case of potential sale. Hence, there is an ex ante stage, before the sellers know the quality of their houses.One might argue that the quality of a house is verifiable in this example but, given the risk of false certificates, an incentive mechanism looks safer.

Since the buyers are not submitted to incentive constraints, we will, without loss of generality, focus on transfers summing up to 0 throughout the example.

Let us first consider a seller-buyer coalition $\{i, j\}$, $i \in \{1, 2\}$, $j \in \{3, 4\}$. The seller has then two equiprobable types, which we denote as h and l. We face a simple, discrete version of Myerson [19]'s "lemon problem". A mechanism for a seller-buyer coalition consists of the probability of trade x_h (resp., x_l) when the seller reports type h (resp., l) and the corresponding expected transfers m_h, m_l from the buyer to the seller. By eliminating the transfers from the incentive constraints (see, e.g., [17] or [19]), the optimization problem of a seller-buyer coalition is

$$v_A^*(\{i; j\}) = \max[\frac{1}{2}x_h(u_h - \pi_h) + \frac{1}{2}x_l(u_l - \pi_l)] \quad s.t. \quad 0 \le x_h \le x_l \le 1$$

so that, under our assumptions,

$$v_A^*(\{i; j\}) = 0 \quad i \in \{1; 2\}; j \in \{3; 4\}$$

Let us set $g_h = u_h - \pi_h > 0$. Observe that, in absence of incentive constraints, $v_A(\{i; j\}) = \frac{1}{2}g_h$. Trade is indeed beneficial in state h, but the incentive compatibility conditions prevent revelation of information from the seller.

Let us turn to the grand coalition. First best efficiency requires to sell the high quality houses, and only those, at every state of nature. Hence,

$$v_A(K) = g_h$$

We will construct an incentive compatible mechanism achieving g_h as sum of expected payoffs, so that

$$v_A^*(K) = g_h$$

Since $v_A(S) \ge v_A^*(S)$ for every S, this will also show that $C(v_A) \subseteq C(v_A^*)$ and provide a simple procedure to construct expected payoffs in the ex ante incentive compatible core.[12]

[12] The procedure can be applied to a large class of assignment problems (see, e.g., [2], [3], [4], [7], [8], [17]).

Consider a matching mechanism in which only high quality houses are sold, e.g., $x \in \{0;1\}^{I \times J}$ described by

$$x(h_1;h_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad x(h_1;l_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$x(l_1;h_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad x(l_1;l_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The sum of expected payoffs from x is g_h. Obviously, x is not incentive compatible, but one can construct transfers m such that $(x;m)$ is incentive compatible. For instance, the transfers m_1 to the first seller must satisfy

$$-\pi_h + \frac{3}{4}m_1(h_1;h_2) + \frac{1}{4}m_1(h_1;l_2) \geq \frac{3}{4}m_1(l_1;h_2) + \frac{1}{4}m_1(l_1;l_2)$$
$$\frac{1}{4}m_1(l_1;h_2) + \frac{3}{4}m_1(l_1;l_2) \geq -\pi_l + \frac{1}{4}m_1(h_1;h_2) + \frac{3}{4}m_1(h_1;l_2)$$

A possible solution is

$$m_1(h_1;h_2) = \frac{3\pi_h - \pi_l}{2} + \frac{g_h}{4} \qquad m_1(h_1;l_2) = \frac{3\pi_l - \pi_h}{2} + \frac{g_h}{4}$$
$$m_1(l_1;h_2) = \frac{g_h}{4} \qquad m_1(l_1;l_2) = \frac{g_h}{4}$$

The transfers m_2 to the second seller can be chosen in a similar way. In order to balance the transfers, one can simply set $m_3 = -m_1$, $m_4 = -m_2$. The mechanism $(x;m)$ thus associates buyer 3 (resp., 4) with seller 1 (resp., 2) but sale only takes place if the seller's house is of high quality. $(x;m)$ yields the expected payoff $\frac{g_h}{4}$ to each trader. The mechanism reflects that sale prices are influenced by the presence of low quality items; the transfers in the low state should be interpreted as a fee that the potential buyers pay to get information and avoid a bad decision. Many other mechanisms achieving ex post efficiency can be constructed. In particular, as in [8], it is possible to design the transfers in such a way that the mechanism is interim individually rational for the sellers, who fully extract the surplus[13].

[13] Adding (resp., subtracting) $\frac{g_h}{4}$ to (resp., from) all previous transfers gives the surplus to the sellers (resp., buyers). All these mechanisms, including the latter, are interim individually rational for the sellers. Another mechanism with the same properties is

$$m_1(h_1;h_2) = \frac{3u_h - \pi_l}{2} \qquad m_1(h_1;l_2) = \frac{3\pi_l - u_h}{2}$$
$$m_1(l_1;h_2) = \frac{3g_h}{2} \qquad m_1(l_1;l_2) = \frac{-g_h}{2}$$

Let us end the analysis of the example by showing that the expected payoff from (x, m), namely, $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4})$, belongs to $C(v^*_A)$. We have evaluated $v_A(\{i, j\})$, $i \in \{1, 2\}$, $j \in \{3, 4\}$ and $v_A(K)$. To complete the description of v_A, we compute that

$$\begin{aligned} v_A(\{i, 3, 4\}) &= \frac{g_h}{2} \quad i = 1, 2 \\ v_A(\{1, 2, j\}) &= \frac{5g_h}{8} \quad j = 3, 4 \end{aligned}$$

Hence, $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}) \in C(v_A)$. It follows from our previous remarks that $(\frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}, \frac{g_h}{4}) \in C(v^*_A)$. The same reasoning applies to $(\frac{g_h}{2}, \frac{g_h}{2}, 0, 0)$.

As announced above, this example shows that it may be better for the agents (in the sense of generating a higher sum of expected payoffs) to stay together at the ex ante (or interim) stage in order to exchange information within the grand coalition. For instance, this enables the agents to exploit the possible correlation between types and to achieve first best efficiency through full revelation.

Example 3

This example will confirm the advantage of random mechanisms, by illustrating that $v^*_A(K)$ can be strictly larger than the maximum sum of payoffs that can be achieved with an incentive compatible deterministic matching mechanism (to which we will refer as $v^*_{AD}(K)$). The framework will consist of one seller (agent 1) and one buyer (agent 2), with independent types t_1 and t_2, but common values: at the state of information (t_1, t_2), the reservation price of trader 1 (resp., 2) is $\pi_1(t_1, t_2)$ (resp., $u_2(t_1, t_2)$). We will further assume that each trader only has two equiprobable types ($T_k = \{t_{k1}, t_{k2}\}$, $q(t_{k1}) = q(t_{k2}) = \frac{1}{2}$, $k = 1, 2$).

This seems one of the simplest models in which one can expect to have $v^*_{AD}(K) < v^*_A(K)$. For instance, under independent private values, first best efficiency can be achieved in an incentive compatible way, so that $v^*_A(K) = v_A(K)$ (see, e.g., [3] and [4] and the concluding remarks below) and thus $v^*_{AD}(K) = v^*_A(K)$ in this case. It is a remarkable fact that $v^*_{AD}(K)$ and $v^*_A(K)$ also coincide in seller-buyer models with common values in which only one trader has private information, as it can be checked by adapting Myerson [19]'s results on the "lemon problem" of to our discrete framework.

Let us complete our example by assuming the following possible reservation prices $\pi_1(t_1, t_2)$, $u_2(t_1, t_2)$:

	t_{21}	t_{22}
t_{11}	$6,2$	$2,3$
t_{12}	$7,6$	$0,4$

Since the traders' types are independent, one can restrict on mechanisms in which the transfers sum up to 0 without loss of generality[14]. Furthermore, by eliminating the transfers from the incentive compatibility conditions (see e.g., Johnson, Pratt and Zeckhauser [17]), we can write the optimization problem of the seller-buyer coalition as

$$v_A^*(\{1,2\}) = \max\{-4\alpha_{11} + \alpha_{12} - \alpha_{21} + 4\alpha_{22}\}$$

$$\text{s.t.} \quad 0 \leq \alpha_{rs} \leq 1 \quad r,s = 1,2 \quad \text{and}$$

$$\alpha_{11} - 2\alpha_{12} - \alpha_{21} + 2\alpha_{22} \geq 0 \tag{8}$$

$$-\alpha_{11} + \alpha_{12} + 2\alpha_{21} - 2\alpha_{22} \geq 0 \tag{9}$$

where α_{rs} is the probability of trade when the seller (resp., buyer) reports his r^{th} (resp., s^{th} type) $r,s = 1,2$. (8) is the seller's incentive compatibility constraint (after elimination of the transfers) which expresses that the sum of his expected payoffs when he always tells the truth is larger than the sum of his expected payoffs when he always lies. (9) has a similar interpretation for the buyer.

The only ex post efficient mechanism in this example, $\alpha = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$, is not incentive compatible for the buyer.

The mechanism $\alpha = \begin{array}{|c|c|}\hline 0 & \frac{2}{3} \\ \hline \frac{2}{3} & 1 \\ \hline \end{array}$ satisfies (8) and (9) - with an equality - so that $v_A^*(\{1,2\}) \geq 4$.[15]

[14] As shown in, e.g., [17], if types are independent and $\mu = (x,m)$ is incentive compatible, so are $(x,\overline{m})$, $\overline{m}_k(t_k) = \sum_{t_{-k}} q(t_{-k}) m_k(t)$, and $(x,\tilde{m})$, $\tilde{m}_k(t) = \overline{m}_k(t_k) - \frac{1}{|K|-1}\sum_{l \neq k} \overline{m}_l(t_l)$. $\tilde{m}(t)$ is exactly balanced for every t.

[15] In fact, one can check that $v_A^*(\{1,2\}) = 4$. Under the feasibility constraints, the objective is $\leq \alpha_{12} - \alpha_{21} + 4$ while the sum of the incentive compatibility constraints yields $\alpha_{12} - \alpha_{21} \leq 0$.

We now check that $v^*_{AD}(\{1;2\}) < 4$. Let α be a deterministic mechanism; if $\alpha_{11} = 1$, the objective $-4\alpha_{11} + \alpha_{12} - \alpha_{21} + 4\alpha_{22}$ is ≤ 1; and similarly if $\alpha_{22} = 0$. $v^*_{AD}(\{1;2\})$ is the maximum of $\alpha_{12} - \alpha_{21} + 4$ s.t. α_{12}, $\alpha_{21} \in \{0;1\}$, (8) and (9) (with $\alpha_{11} = 1$ and $\alpha_{22} = 0$) , which is 3.

6 Concluding remarks

In this paper, we have focused on the ex ante assignment game (which is defined without any ambiguity) and a possible version of the interim assignment game, in which agents do not communicate until they are in a coalition. We have established that the associated cores are non-empty. Other core concepts have been proposed at the interim stage, starting with Wilson [28]'s fine core (defined in absence of incentive constraints). The definition of an interim core concept turns out to be quite delicate, especially if some communication is allowed at the coalition formation stage (see, e.g., [9] [10], [15], [13], [20]) and we will not investigate this problem further here. But the previous analysis suggests that two-sided matching games with incomplete information could be an appropriate framework in which to apply or develop interim core concepts. Indeed, the model has a simple, well-behaved structure, a number of particular cases are well-understood (e.g., seller-buyer bargaining, auctions,...) and the benchmark incentive compatible coarse core is not empty.

Let us turn to possible ex post properties of ex ante incentive compatible core solutions. A number of papers (see, e.g., [2], [3], [4], [7], [8], [17]) identify assumptions (on beliefs and utility functions) which ensure the existence of mechanisms (involving exactly balanced transfers) which are both incentive compatible and ex post efficient. Given the TU aspect, it is not difficult to see that the expected payoffs achieved by such mechanisms are in the ex ante incentive compatible core (this is illustrated on example 2 in the previous section). In other words, one can easily produce assumptions under which the ex ante incentive compatible core contains ex post efficient mechanisms. Such results are used in [11] (where an analog of proposition 1 does not hold) to establish the non-emptiness of the ex ante incentive compatible core.

Transfers are crucial in the previous approach, which guarantees ex post efficiency but not ex post individual rationality.[16] Roth [22] considers a much stronger requirement: ex post stability in the absence of transfers. He analyzes "marriage problems" (typically without transfers) in which agents privately know their own utilities (in our terminology, "private values"). As in the present paper, he considers general mechanisms (allowing in particular for lotteries over matchings) and applies the revelation principle in order to focus on direct revelation mechanisms. He calls such a mechanism "stable" if it selects a stable matching for any stated utilities. To express a similar stability property in our framework, let us define, for every $t \in T$, v_t as the (TU) matching game (with complete information) when the types are t. As shown in [26], v_t is the superadditive cover of $v_t(\{k\}) = u_k(t; 0)$, $k \in K$, $v_t(\{i; j\}) = u_i(t; j) + u_j(t; i)$, $i \in I$, $j \in J$. A matching mechanism μ is "stable" in the sense of Roth [22] if for every $t \in T$ (interpreted as a vector of reported types), $\mu(t)$ selects a solution in $C(v_t)$, i.e., if μ is ex post stable. Roth argues that this requirement is not "excessively strong" in a model with private values. However, he shows that incentive compatible, ex post stable, matching mechanisms do not exist in general. As suggested above, in our framework (with transfers), it is not difficult to construct examples in which incentive compatible ex post efficient mechanisms do exist, but none of them is ex post individually rational.

Roth [22] also observes that several properties involving strategy-proofness in marriage problems with complete information have an immediate counterpart in marriage problems with incomplete information but private values.[17] In particular, there exist matching mechanisms such that revealing one's true preferences is a dominant strategy for one side of the market. Such results are also valid here, but do not seem to be helpful in analyzing the incentive compatible core unless one side of the market contains only one agent (as in standard auctions).

The previous comments raise the question of the validity of our results in matching markets where monetary transfers are not possible (like in Shapley and Scarf [25]). Our solution concepts can be used in this framework. Except for the TU aspect in section 3, all our results hold when all monetary transfers are imposed to be null. The (NTU) ex ante incentive compatible core is then non-empty as a consequence of Scarf [24]'s theorem.

[16] Ex ante and, in some cases like [7] and [8], interim individual rationality can be guaranteed.

[17] A strategy-proof mechanism induces a dominant-strategy incentive compatible mechanism in matching problems with private values.

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