Rationalizable Expectations

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1 Introduction

We study the implications of the assumptions of rationality and market clearing in economies with asymmetric information.

The starting point is the notion of rational expectations equilibrium (REE). REE extends the classical notion of a competitive equilibrium to economies with asymmetric information (i.e., economies in which different agents might have different information). When each agent has only partial information on the value of a commodity or an asset he can deduce additional information from the prices because prices reflect the information that other agents have. REE is a solution notion that is based on the assumption that agents make these inferences. However, the notion of REE is based on an additional strong assumption that agents know (and therefore agree on) the function that specifies the prices in each state. (A state specifies all the information that players have together.) This strong assumption leads to a strong result that in a general class of economies the only REE is a fully revealing equilibrium, i.e., an equilibrium in which each agent can infer from the prices all the information that any other agent has (Radner (1979).)

In the current research the assumption that players know the price function is relaxed, that is, we consider a situation where each agent has a different theory about how the vector of prices which is observed has materialized and about what would have happened in other states. However, the assumption is maintained that each agent makes inferences from the observed prices and furthermore assumes that other agents are doing likewise. More precisely we are interested in characterizing and analyzing properties of the set of outcomes that are consistent with *common knowledge of rationality and market clearing* (henceforth, outcomes that are $CKRMC$).

We view the contributions of this project as follows:

1. Defining and characterizing $CKRMC$.

It turns out that the definition of a solution notion that is consistent with common knowledge of rationality and market clearing is not obvious. In particular, the definition that we propose is different from the definitions that were suggested by MacAllister (1990), Dutta and Morris (1997), Desgranges and Guesnerie (1996) and Desgranges (2001). For example MacAllister and

Dutta and Morris propose a solution notion where in addition to $CKRMC$ there is common knowledge of the probability distribution of each player on the set of price functions. This additional assumption restricts in a significant way the set of possible outcomes.

2. We use the characterization result to compute the set of prices that are $CKRMC$ in some examples. In these examples $CKRMC$ captures intuitive properties which REE does not reflect well. In particular, for a general class of economies with two commodities we solve for the set of prices that are $CKRMC$ and obtain that :

(a) For a robust subset of these economies there is a whole range of prices that are consistent with all the possible states and therefore these prices do not reveal any information.

(b) Refining the knowledge of a positive measure of agents strictly shrinks the set of equilibrium prices.

Both these properties stand in contrast to the full revelation property of REE in economies with a finite number of states.

3. We study the relationship between $CKRMC$ and iterative deletion of weakly dominated strategies and obtain the following results:

Let F be a set of price functions and let z_i be a demand strategy for player i, that is, z_i is a function which assigns a bundle of commodities to each pair (l_i, p) where l_i is a private signal for player i and p a price vector. Say that z_i is weakly dominated w.r.t F if there exists another demand strategy z_i' which gives a higher expected payoff for every $f \in F$ and a strictly higher payoff for some $\overline{f} \in F$. We show that the set of price functions that are $CKRMC$ (henceforth $FCKRMC$) is the set of price functions that are generated by demand strategies that survive iterative deletion of weakly dominated strategies. (At each stage of the iteration the set of price functions is taken to be the functions that can be generated by demand strategies that have survived the deletion process up to the current stage.)

We then look at a market game in which each agent submits a demand function and an auctioneer selects a price which clears the market. This is a Bayesian game in which a strategy for player i is a demand strategy as was defined in the previous paragraph. Let $F(WS^\infty)$ denote the set of price functions that survive iterative deletion of weakly dominated strategies in this game. Our previous result would suggest that $F(WS^\infty)$ equals $FCKRMC$ and indeed this is "almost true". More precisely we show that $F(WS^\infty) \supseteq$

$FCKRMC$ and that for a general class of price functions $\overline{F}$ if $f \in \overline{F}$ then $f \in FCKRMC$ iff $f \in F(WS^{\infty})$. The reason why $F(WS^{\infty})$ can be bigger then $FCKRMC$ has to do with the existence of profiles of demand strategies that do not have a clearing price in each state.

The draft is organized as follows: In section 2 the definition of an exchange economy with asymmetric information is reviewed and a simple example which motivates and demonstrates the notion of $CKRMC$ is presented. We then define $CKRMC$ and present two preliminary results. In section 3 we present a result, theorem 3, which provides a characterization of outcomes that are $CKRMC$ under general conditions. This characterization simplifies the computation of the set of outcomes that are $CKRMC$. The result is also of some conceptual interest in that it establishes that two different types of assumptions on the uncertainty that players face determine, under general conditions, the same set of outcomes.

The current draft does not contain yet the solution of a general class of economies with two commodities (item (2) in the introduction) and the analysis of the relationship between $CKRMC$ and iterative deletion of weakly dominated strategies in a market game (item (3) in the introduction.)

2 The Model

In this section we review the definition of an exchange economy with asymmetric information and present a simple example which motivates the solution notion of $CKRMC$. We then define $CKRMC$ and present two preliminary results.

An economy with asymmetric information is defined by:

1. $I = [0, 1]-$ The set of players (consumers).
2. $X_1, ..., X_K-$ K commodities.
3. $S = \{s_1, ..., s_n\}-$ The set of states.
4. $\alpha \in \triangle(S)-$ α is a common prior on S .
5. P_i- A partition on S that describes the information of player i.
   $P_i(s) \subseteq S$ is the information that player i gets at the state s.
6. $u_i : R^K \times S \rightarrow R-$ A V.N.M utility function for player i.

$u_i(x,s)$ is the utility of player i from a bundle $x \in R^K$ in the state s.

7. $e_i : S \to R^K - e_i(s)$ is the initial bundle of player i at state s.
We assume that e_i is measurable w.r.t P_i and that $\forall s \in S \int_i e_i(s)-$ the aggregate supply in state $s-$exists.

A *price* p is a vector $p = (p_1, ...p_{K-1})$ where p_k is the price of X_k. The price of X_K is normalized to be 1.

A *price function* $f, f : S \to R^{K-1}$, assigns with every state s a price $f(s)$. We will sometimes think of a price function as a vector in R^{K-1}.

We let L_i denote the set of signals of agent i. So, $L_i \equiv \{P_i(s) : s \in S\}$.

A *demand strategy* for player i is a function z_i, $z_i : L_i \times R^{K-1} \to R^K$, such that $z_i(l_i, p)$ is in the budget set defined by the price p and the initial endowment $e_i(l_i)$. ($e_i(l_i)$ is well defined because e_i is measurable w.r.t P_i.)

The standard solution notion for economies with asymmetric information is Rational Expectations Equilibrium, REE. A REE is a price function f such that for each state s the price $f(s)$ clears the market when every agent i makes a demand which is optimal w.r.t the price $f(s)$ and the information that is revealed by his private signal $P_i(s)$ and the fact that the price is $f(s)$. Formally,

Definition: A price function f is a REE if there exists a profile of demand strategies, $\{z_i\}_{i \in I}$, that satisfies :

1. Rationality, $\forall s \in S$ $z_i(P_i(s), f(s))$ is optimal w.r.t the price $f(s)$ and the posterior $\alpha(\ |P_i(s) \cap f^{-1}(f(s)))$.
2. Market clearing, $\forall s \in S$ $\int_i z_i(P_i(s), f(s)) = \int_i e_i(s)$.

A price function f is a *fully revealing REE* ($FREE$) if $f(s) \neq f(s')$ when $s \neq s'$.

We turn now to a simple example which demonstrates the difference between REE and consistency with common knowledge of rationality and market clearing.

There are two commodities in the economy, X and M (money).

The set of states is $S = \{1, 3\}$.

The probability of each state is 0.5.

The set of agents is the interval $[0, 1]$. There are two types of agents I_1 and I_2. Agents in I_1 know the true state agents in I_2 do not know it.

$I_1 = [0, \delta]$ and $I_2 = (\delta, 1]$. All the agents have the same utility and the same initial bundle. The utility is:

(2.1) $$u(x, m, s) = s \times \log(x) + m$$

where x and m are the quantities of X and M respectively and s is the state.

The initial bundle consists of one unit of X and $\overline{m}$ units of M where $\overline{m} \geq 3$.

Let p be the price of a unit of X in units of M. It follows from the definition of the utility function in (2.1) that the demand for X of an agent who knows the true state is:

$$x = \frac{s}{p}$$

More generally, the demand of an agent i who assigns to the state s probability $\gamma(s)$ is

(2.2) $$x = \frac{\gamma(1)\times 1+\gamma(3)\times 3}{p}$$

In this example for every $\delta \succ 0$ there is only one REE, f^* ,where $f^*(s) = s$. To see that we, first, note that if f is a REE then $f(1) \neq f(3)$. This follows because if $f(1) = f(3) = p$ then agents in I_2 do not obtain any information about the true state and therefore their demand in both states is the same: $x = \frac{0.5\times 1+0.5\times 3}{p} = \frac{2}{p}$

However, the demand of agents from I_1 in state 1 is different than their demand in state 3 and therefore the aggregate demands are different as well. Since the aggregate amount of X is fixed this means that the market doesn't clear in at least one of the states and therefore f is not a REE. Thus, if f is a REE then $f(1) \neq f(3)$. In this case agents in I_2 infer the state from the price and it follows from (2.2) that $f(1) = 1$ and $f(3) = 3$. Thus, the only REE is a fully revealing equilibrium (henceforth, $FREE$) in which the price reveals the state. Indeed, Radner (1979) has shown that in a generic class of economies with a finite number of states the only REE is a $FREE$ in which the information that all the agents have together is revealed.

We now show that if we relax the assumption that players know the price function (and therefore agree on it) then there are other price functions which are consistent with common knowledge of rationality and market clearing. We call such price functions functions that are $CKRMC$.

Assume that $\delta = \frac{1}{6}$. We will show that the following price functions are $CKRMC$:

$$f(1) = 2 \qquad g(1) = 1$$
$$f(3) = 3 \qquad g(3) = 2$$

Suppose that a fraction β of the agents in I_2 assign probability $\frac{3}{4}$ to the event that f is the price function and a probability $\frac{1}{4}$ to the event that g is the price function, call this belief theory A. Assume that the other agents in I_2 think that g is more likely, they assign probability $\frac{1}{4}$ to the event that f is the price function and probability $\frac{3}{4}$ to the event that the price function is g, call this belief theory B.

What are the beliefs of different agents in I_2 about the true state when they observe the price 2 ?

Since the prior assigns probability 0.5 to each state it is easy to see that agents in I_2 who believe in theory A assign probability $\frac{3}{4}$ to the state 1 and probability $\frac{1}{4}$ to the state 3[3]. Similarly, agents who believe in theory B assign probability $\frac{1}{4}$ to the state 1 and probability $\frac{3}{4}$ to the state 3.

It follows from (2.2) that the demand for X at price 2 of agents who believe in theory A is $(\frac{3}{4} \times 1 + \frac{1}{4} \times 3)/2 = \frac{3}{4}$ while the demand of agents who believe in theory B is $(\frac{3}{4} \times 3 + \frac{1}{4} \times 1)/2 = \frac{5}{4}$.

Let $x(\beta, s, p)$ denote the aggregate demand for X in state s at price p when a proportion β of the agents in I_2 believe in theory A and the rest of I_2 believe in theory B. We have

$x(\beta, 1, 2) = (1-\delta) \times \beta \times \frac{3}{4} + (1-\delta) \times (1-\beta) \times \frac{5}{4} + \delta \times \frac{1}{2}$

$x(\beta, 3, 2) = (1-\delta) \times \beta \times \frac{3}{4} + (1-\delta) \times (1-\beta) \times \frac{5}{4} + \delta \times \frac{3}{2}$

Let β_f and β_g be the numbers which equate demand and supply at price 2 in the states 1 and 3 respectively, that is,

$x\left(\beta_f, 1, 2\right) = 1$ and $x\left(\beta_g, 3, 2\right) = 1$.For $\delta = \frac{1}{6}$ we obtain $\beta_f = 0.3$ and $\beta_g = 0.7$.

Now we observe that when β_f of the agents in I_2 believe in theory A and $1-\beta_f$ of them believe in B then the function f specifies prices which clear the market; We have just seen that the price 2 clears the market in $s=1$ and when the price is 3 everyone assigns probability 1 to the state 3 and therefore the price 3 clears the market. Similarly, when β_g of the agents

[3]Let $P_A(s\,|p=2)$ denote the posterior that an agent who believes in theory A assigns to the state s upon observing the price 2. Then $P_A(1\,|p=2) = \frac{0.75 \cdot \alpha(1)}{0.75 \cdot \alpha(1) + 0.25 \cdot \alpha(3)} = 0.75$

in I_2 believe in theory A (and the rest in B) the function g specifies prices which clear the market.

We have thus shown that when the assumption that players know the price function is relaxed then there is more than one price function that can be rationalized. Specifically, there exists a profile of beliefs (i.e., a belief for each player), $\overline{\beta_f}$, such that when each player makes a demand which is optimal w.r.t his beliefs the prices specified by f clear the market. Similarly, there is a profile of beliefs $\overline{\beta_g}$ which rationalizes g. Furthermore, since the beliefs of each player in the profiles $\overline{\beta_f}$ and $\overline{\beta_g}$ assign a positive probability only to f and g, which are functions that can be rationalized, f and g are consistent *not only* with rationality and market clearing but also with *common knowledge* of rationality and market clearing. Specifically, one can think of the theory A (B) not only as a theory which assigns probabilities to price functions, probability $\frac{3}{4}$ $(\frac{1}{4})$ to f and probability $\frac{1}{4}$ $(\frac{3}{4})$ to g, but as a richer theory which refers to the beliefs of agents as well. This extended theory A (B) assigns probability $\frac{3}{4}$ $(\frac{1}{4})$ to the event that the profile of beliefs in the population is $\overline{\beta_f}$ and probability $\frac{1}{4}$ $(\frac{3}{4})$ to the event that the profile of beliefs is $\overline{\beta_g}$.

We can now turn to the general definition of $CKRMC$.

We say that Borel set of functions $F, F \subseteq R^{K-1}$,is $CKRMC$ if every $f \in F$ defines prices which clear the market for demands that can be rationalized by beliefs on F.

To provide a completely formal description we need some preliminary definitions:

Definition: A *belief* μ_i for player i on a set of Borel price functions F is a finite lexicographic sequence of probability measures, $\mu_i = (\ \mu_i^1, ..., \mu_i^m)$, on F.

We assume that the selection of the state of nature is independent of the selection of the price function and therefore the beliefs of player i on the set $S \times F$ is a product of his prior probability distribution on S, α,and his beliefs on F, μ_i. Specifically, the belief of an agent i on $S \times F$ is the lexicographic sequence of probabilities $\alpha \times \mu_i = (\ \alpha \times \mu_i^1, ..., \alpha \times \mu_i^m)$ where for $S' \subseteq S$ and $F' \subseteq F$ $\alpha \times \mu_i^k(S' \times F') = \alpha(S') \cdot \mu_i^k(F')$. The information that player i has when he makes a demand is his private signal $l_i \in L_i$ and the fact that a given price p has materialized. Given a set of price functions F we let (l_i, p) denote the event in $S \times F$ which is consistent with l_i and p. That is,

$(l_i, p) = \{(s, f) : s \in S,\ f \in F\ \ P_i(s) = l_i \text{ and } f(s) = p\}$

We say that a belief $\mu_i, \mu_i = (\ \mu_i^1, ..., \mu_i^m)$, of player i is *consistent* with the event (l_i, p) if there exists k,$1 \leq\ k\ \leq\ m$, such that $\alpha \times \mu_i^k(l_i, p) \succ 0$. Given a belief μ_i and an event (l_i, p) which is consistent with it we allow for some abuse of notation and let $\mu_i(\ |(l_i, p))$ denote the marginal distribution of $\alpha \times \mu_i^k(\ |(l_i, p))$ on S, where k is the lowest index with the property that $\alpha \times \mu_i^k(l_i, p) \succ 0$. $\mu_i(\ |(l_i, p))$ is the posterior on S of a player i with a belief μ_i given the event (l_i, p).

We are now ready to give a formal definition of $CKRMC$.

Definition: A Borel set of price functions F is $CKRMC$ if $\forall f \in F$ there is a profile of demand strategies $\left\{z_i^f\right\}_{i\in I}$ and a profile of beliefs on F $\left\{\mu_i^f\right\}_{i\in I}$ that satisfy:

1. Rationality: for every $i \in I$ and every (l_i, p) that is consistent with μ_i^f $z_i^f(l_i, p)$ is an optimal bundle at the price p w.r.t $\mu_i^f(\ |(l_i, p))$. For every $s \in S$ and $i \in I$ $(P_i(s), f(s))$ is consistent with μ_i^f.

2. Market Clearing at the prices specified by f: for every $s \in S$ $\int_i z_i^f(P_i(s), f(s)) = \int_i e_i(s)$.

We will say that profiles of beliefs and demand strategies $\left\{\mu_i^f\right\}_{i\in I}$ and $\left\{z_i^f\right\}_{i\in I}$ support f w.r.t F if $\left\{\mu_i^f\right\}_{i\in I}$ and $\left\{z_i^f\right\}_{i\in I}$ satisfy conditions 1. and 2. in the definition of $CKRMC$.

Definition: A price function f is $CKRMC$ if there exists a set of price functions F such that $f \in F$ and F is $CKRMC$.

We let $FCKRMC$ denote the set of functions that are $CKRMC$.

Definition: An *outcome* (p, s), $p \in R^{K-1}$, $s \in S$,is a pair of a price and a state.

An outcome (p, s) is $CKRMC$ if there exists a price function $f \in FCKRMC$ such that $f(s) = p$.

To demonstrate and clarify the definitions we note that:

1. A price function f is an REE iff the set $F = \{f\}$ is $CKRMC$. In particular, a price function f that is an REE is also a function that is $CKRMC$.

2. In example 1 the set $F = \{f, g\}$ is $CKRMC$. In particular, any profile of beliefs $\{\mu_i\}_{i\in I}$, $\mu_i = (\mu_i^1)$ (i.e. the beliefs of each player consist of just

one probability distribution), where 0.3 of the agents in I_2 assign probability 0.75 to f and 0.25 to g and the rest assign 0.25 to f and 0.75 to g supports demands which clear the market in the prices specified by f. Similarly, as we have seen in the discussion of the example, there are profiles of beliefs that support g. It follows that (1,1), (2,1), (2,3) and (3,3) are outcomes that are $CKRMC$.

In the next section we will provide a characterization of outcomes that are $CKRMC$ and use this characterization to compute the whole set of outcomes that are $CKRMC$ in this example.

We conclude this section with two preliminary results.

1. $FCKRMC$ as the result of an iterative process.

Let F be set of price functions. We let $J(F)$ denote the set of price functions that can be supported w.r.t F. Theorem 1 below states that $FCKRMC$ is obtained by a process in which price functions that cannot be supported are iteratively deleted. Furthermore, there is only a finite number of iterations. Formally, define F^k, $k = 0, 1, 2, ...$ inductively as follows: $F^0 = R^{n\times(K-1)}$ and $F^{k+1} = J(F^k)$. Define $F^\infty = \cap_{k=0}^{\infty} F^k$.

Theorem 1:
$(a.)$ $F^\infty = FCKRMC$.
$(b.)$ There exists a number M which depends on n ($n = |S|$) such that $F^\infty = F^M$
$(c.)$ Let $f \in FCKRMC$. There exists a finite set of functions $F(f)$ such that $f \in F(f)$ and $F(f)$ is $CKRMC$.

The proof of the theorem is omitted from the current draft.

2. Common Knowledge of Rationality and Market clearing.

Our solution notion does not include a description of what one player knows, or believes, about another player so the reader might ask in what sense is a function that is $CKRMC$ indeed consistent with common knowledge of rationality and market clearing. On an intuitive level if a function f can be supported by beliefs on functions that are consistent with rational behavior and if each one of these functions, in turn, can be supported by such beliefs and so forth then f is consistent with common knowledge of rationality and market clearing. This argument can be made precise by embedding our model in a richer model which includes beliefs of players about the beliefs of other players. This is done as follows:

Say that an abstract measurable space (Ω, β) is a *model* for a given economy E if each state $\omega \in \Omega$ specifies :

1. A price function f.

2. A profile of demand strategies $\{z_i\}_{i \in I}$.

3. A profile of beliefs $\{\mu_i\}_{i \in I}$ where $\mu_i = (\mu_i^1, ..., \mu_i^m)$ is a finite lexicographic sequence of probabilities on Ω such that for every $i \in I$ and for every index $k, 1 \leq k \leq m$, μ_i^k assigns probability 1 to states in which the demand strategy and the belief of player i are z_i and μ_i respectively. (This requirement reflects the assumption that player i knows his demand and belief.)

In addition the transformation T_f which associates with each state $\varpi \in \Omega$ a price function $T_f(\varpi)$ is measurable and $F(\Omega) \equiv \{T_f(\varpi) : \varpi \in \Omega\}$ is a Borel set.

Say that model (Ω, β) is *consistent with common knowledge of rationality and market clearing*, henceforth $CKRMC$ if for each state $\varpi \in \Omega$ the following two conditions are satisfied:

1.Rationality: for every $i \in I$ and (l_i, p) that is consistent with μ_i $z_i(l_i, p)$ is an optimal bundle at the price p w.r.t $\mu_i(\ |(l_i, p))$. For every $i \in I$ and for every $s \in S$ $(P_i(s), f(s))$ is consistent with μ_i. (just like in the definition of $CKRMC$.)

2. Market Clearing, for every $s \in S$ $\int_i z_i(P_i(s), f(s)) = \int_i e_i(s)$.

Conditions 1. and 2. are similar to the conditions in the definition of $CKRMC$. Here these conditions say that in every state $\varpi \in \Omega$ every player is making a rational choice and markets clear .Now the point is that here the belief of a player i, μ_i, is on the space Ω, (that is, $\mu_i^k \in \triangle(\Omega), k = 1, ..., m$.) So each state ϖ describes the beliefs of each player i on the set of states in the model, which in turn describe the beliefs of each other player on the set of states and so forth. In particular, each state ϖ describes what each player i believes about the beliefs of any other player j about the beliefs of any other player k and so forth. Since rationality and market clearing are satisfied in every state ϖ every proposition of the type, player i knows that player j knows that .... player k knows that everyone is rational and markets clear, is true in every state $\widehat{\varpi}$ and therefore there is common knowledge of rationality and market clearing in every state $\widehat{\varpi} \in \Omega$.

In the appendix we prove the following proposition.

Theorem 2: A price function f is $CKRMC$ iff there exists a model (Ω, β) that is consistent with common knowledge of rationality and market clearing and a state $\varpi \in \Omega$ such that f is the price function that is specified in ϖ.

Theorem 2 makes precise the sense in which $FCKRMC$ is the set of functions that are consistent with common knowledge of rationality and market clearing.

3 A Characterization

In this section we present a result which provides a characterization of outcomes that are $CKRMC$ under general conditions. This characterization simplifies the computation of the set of outcomes that are $CKRMC$. The result is also of some conceptual interest in that it shows that two different types of assumptions about the uncertainty that players face determine, under mild restrictions, the same set of outcomes.

The following solution notion which we call Ex-Post Rationalizable was first proposed by Desgranges (2001) (Desgranges called it Common Knowledge Equilibrium.)

Definition: A price p is Ex-Post Rationalizable (henceforth $EXPR$) w.r.t to a set of states $\widehat{S} \subseteq S$ if for every $s \in \widehat{S}$ there exists a profile of probabilities on $\widehat{S}$ $\{\gamma_i^s\}_{i \in I}$, $\gamma_i^s \in \triangle(\widehat{S} \cap P_i(s))$, and a profile of demands $\{x_i^s\}_{i \in I}$, $x_i^s \in R^K$, such that:

1. For every $i \in I$ x_i^s is an optimal at the price p w.r.t γ_i^s.
2. Markets clear, that is, $\int_i x_i^s = \int_i e_i$.

The idea is that if p is $EXPR$ w.r.t $\widehat{S}$ then $\widehat{S}$ is a set of states in which p could be a clearing price because for every $s \in \widehat{S}$ there is a profile of beliefs on $\widehat{S}$, $\{\gamma_i^s\}_{i \in I}$, which is consistent with the private information of the players and which rationalizes demands that clear the markets at p. (The belief γ_i^s,in turn, is possible for player i because p can be a clearing price in every $s \in \widehat{S}$.)

Definition: An outcome (p, s) is $EXPR$ (alternatively, p is $EXPR$ in s) if there exists a set of states $\widehat{S}$ such that p is $EXPR$ w.r.t $\widehat{S}$.

Let $S(p)$ be the set of all states such that (p,s) is $EXPR$. It is easy to see that p is $EXPR$ w.r.t $S(p)$ and that $S(p)$ is the maximal set w.r.t which p is $EXPR$. It is not difficult to show that $S(p)$ is the set of all states that are consistent with common knowledge of rationality and market clearing when there are no (further) restrictions on the beliefs of the players on the states.[4]

It is important to observe that $EXPR$ is a weak (i.e. permissive) notion in two ways:

1. A price p in a state $s \in S(p)$ can be supported by any profile of subjective probabilities $\{\gamma_i^s\}_{i \in I}$ on $S(p) \cap P_i(s)$.

2.The beliefs of the players can be correlated with the state. That is, γ_i^s can be different from $\gamma_i^{\widetilde{s}}$ even when s and $\widetilde{s}$ belong to the same information set of player i (i.e. $P_i(s) = P_i(\widetilde{s})$.)

We can now state the main result in this section.

Theorem 3:

a. If (p,s) is $CKRMC$ then (p,s) is $EXPR$.

b. Let E be an economy in which there is a fully revealing REE, f. Let p be a price such that $\forall s \in S$, $f(s) \neq p$,then (p,s) is $CKRMC$ iff (p,s) is $EXPR$.

Before demonstrating the result and proving it we make a few further comments on the relationship between $CKRMC$ and $EXPR$. There are two main differences between the two notions. First, $CKRMC$ assumes that agents have a common prior on the set of states S. Second, $CKRMC$ assumes that each player has a complete theory about what price might materialize in each state. Then, given a price p, each player updates his probability distribution on S. By contrast, $EXPR$ assumes that given a price p each player i may have any probability distribution on $S(p)$ that is consistent with his private signal. Furthermore, the belief of player i may be correlated with the state. There is no common prior, in fact, there are no priors and no updating at all. Player i does not assess the likelihood of a state s given the price p by asking himself what is the prior probability distribution on S and how likely is p in different states, rather, given p, he forms some probability

[4] A formal statement and proof of this proposition is done by defining a richer model in which a state of the world defines not only the preferences of the players but also their beliefs, their beliefs on the beliefs of other players and so forth. We have defined such a model in the proof of theorem 2 in section 2 so we do not repeat this here.

on the states in which p can be a clearing price. In this sense he does not have a complete theory and his reasoning is Ex-Post.

We now use the result to solve the set of outcomes (p, s) that are $CKRMC$ in example 1. Let P_s , $s = 1, 3$, denote the set of prices that are $EXPR$ in s. We will compute P_s and conclude, using theorem 3, that P_s is also the set of prices that are $CKRMC$ in the state s. Let $P(\widehat{S})$ denote the set of prices that are $EXPR$ w.r.t the set of states $\widehat{S}$, $\widehat{S} \subseteq S$. It follows from the definitions that: $P_s = \cup_{s \in \widehat{S}} P(\widehat{S})$. In our example:

(3.1) $P_1 = P(\{1\}) \cup P(\{1, 3\})$

(3.2) $P_3 = P(\{3\}) \cup P(\{1, 3\})$

$P(\{1\}) = 1$ and $P(\{3\}) = 3$ because 1 and 3 are prices which clear the markets in the states 1 and 3 respectively when everyone knows the state. We now compute $P(\{1, 3\})$. Let $P_s(\{1, 3\})$ denote the set of prices that can clear the markets in state s, $s = 1, 3$, when players in I_2 may have any profile of beliefs on $\{1, 3\}$. It follows from the definition of $P(\{1, 3\})$ that

(3.3) $P(\{1, 3\}) = P_1(\{1, 3\}) \cap P_3(\{1, 3\})$.

We claim that $P_1(\{1, 3\}) = [1, 3 - 2 \cdot \delta]$. This follows because the price 1 clears the market when every agent in I_2 assigns probability 1 to the state 1(every agent in I_1 <u>knows</u> that the state is 1.) Clearly, the aggregate demand for X and therefore it's price are minimal when everyone assigns the state 1 probability1. Similarly, the price 3-2$\cdot\delta$ clears the market when every agent in I_2 assigns probability 1 to the state 3 and therefore the maximal point in $P_1(\{1, 3\})$ is $3 - 2 \cdot \delta$ It is easy to see that for every 1$\leq p \leq 3 - 2 \cdot \delta$ there is a probability $\gamma(p)$ such that if every agent in I_2 assigns probability $\gamma(p)$ to the state 3 then p clears the market. The set $P_3(\{1, 3\})$ is computed in a similar way. When each agent in I_2 assigns the state 1 probability 1 the clearing price is $1 + 2 \cdot \delta$. When agents in I_2 assign the state 3 probability 1 the clearing price is 3. It follows that $P_3(\{1, 3\}) = [1 + 2 \cdot \delta, 3]$.

From (3.3) we obtain that for $\delta \leq 0.5$ $P(\{1, 3\}) = [1 + 2 \cdot \delta, 3 - 2 \cdot \delta]$. For $\delta \succ 0.5$ $P(\{1, 3\}) = \emptyset$. From (3.1) and (3.2) we have that for $\delta \leq 0.5$ $P_1 = \{1\} \cup [1 + 2 \cdot \delta, 3 - 2 \cdot \delta]$ and $P_3 = \{3\} \cup [1 + 2 \cdot \delta, 3 - 2 \cdot \delta]$ and for $\delta \succ 0.5$ $P_1 = \{1\}$ and $P_3 = \{3\}$. It follows from theorem 3 that the difference between the set P_s and the set of prices that are $CKRMC$ in s, $s = 1, 3$, is at most the price s. Now, s is the REE price in the state s and therefore s is a price that is $CKRMC$ in the state s. It follows that the sets P_s $s = 1, 3$ that we have computed are the sets of prices that are $CKRMC$ in the respective states.

It is interesting to observe that that the set P_s depends on δ - the fraction of agents who the true state. As δ increases the set P_s shrinks and when more than 0.5 of the population is informed ($\delta \succ 0.5$) the only price function that is $CKRMC$ is the REE.

We now turn to the proof of theorem 3.

Proof: Start with part 1. If $(p, \widehat{s})$ is $CKRMC$ then there exists a set of price functions F that is $CKRMC$ and a function $\widehat{f} \in F$ such that $\widehat{f}(\widehat{s}) = p$. Define

$\widehat{S} \equiv \{s : \exists f \in F \text{ s.t. } f(s) = p\}$. Clearly, $\widehat{s} \in \widehat{S}$. We now show that p is $EXPR$ w.r.t to $\widehat{S}$. So let $\overline{s} \in \widehat{S}$ we have to show that there exists a profile of probabilities $\{\gamma_i^{\overline{s}}\}_{i \in I}$, $\gamma_i^{\overline{s}} \in \triangle(\widehat{S} \cap P_i(\overline{s}))$, and a profile of demands $\{x_i^{\overline{s}}\}_{i \in I}$ s.t $x_i^{\overline{s}}$ is optimal for player i w.r.t $\gamma_i^{\overline{s}}$ and $\int_i x_i^{\overline{s}} = \int_i e_i(\overline{s})$. Let $f \in F$ be a function such that $f(\overline{s}) = p$. Since F is $CKRMC$ there exists a profile of beliefs on F, $\left\{\mu_i^f\right\}_{i \in I}$, and a profile demands $\left\{z_i^f(P_i(s), f(s)) : s \in S\right\}_{i \in I}$ such that in every state s the aggregate demand equals the aggregate supply and $z_i^f(P_i(s), f(s))$ is optimal w.r.t $\mu_i^f(\ |P_i(s), f(s))$. In particular, these properties are satisfied in the state $\overline{s}$. Now, $\mu_i^f(\ |P_i(\overline{s}), f(\overline{s}))$=$\mu_i^f(\ |P_i(\overline{s}), p) \in \triangle(\widehat{S} \cap P_i(\overline{s}))$ and therefore by defining $\gamma_i^{\overline{s}} = \mu_i^f(\ |P_i(\overline{s}), p)$ and $x_i^{\overline{s}} = z_i^f(P_i(\overline{s}), p)$ we have defined probabilities and demands which satisfy the requirements in the definition of $EXPR$.

We turn now to the proof of the second part of the theorem. Let $\overline{f}$ be a fully revealing REE and let $\overline{p}_s \equiv \overline{f}(s)$. Let $(p, \widehat{s})$ be an outcome which is $EXPR$ where $p \neq \overline{p}_s$ for every $s \in S$. We want to show that there exists a price function $f_{\widehat{s}}$that is $CKRMC$ such that $f_{\widehat{s}}(\widehat{s}) = p$. To prove this we now define a price function f_s for every $s \in S(p)$ as follows:

$$(3.4) \quad f_s(s') = \begin{cases} p & s' = s \\ \overline{p}_{s'} & s' \neq s \end{cases}$$

We will show that the set $F = \{f_s : \ s \in S(p)\}$ is $CKRMC$. Since $f_{\widehat{s}} \in F$ this will complete the proof. So let $s \in S(p)$ we need to show that there exists a profile of beliefs $\left\{\mu_i^{f_s}\right\}_{i \in I}$ on F and demands $\left\{z_i^{f_s}(P_i(s'), f(s')) : s' \in S\right\}_{i \in I}$ such that the demands clear the market and are optimal w.r.t the beliefs. First, we observe that for any probability distribution μ_i on F and for any state $s' \in S$ $\mu_i(\ |P_i(s'), \overline{p}_{s'})$ assigns probability 1 to the state s' (because for every $f \in F$ $f(\widetilde{s}) = \overline{p}_{s'}$ implies $\widetilde{s} = s'$.) Therefore, if for every $s' \neq s$ we define the demand $z_i^{f_s}(P_i(s'), f(s')) = z_i^{f_s}(P_i(s'), \overline{p}_{s'})$ to be the optimal bundle for player i at the price $\overline{p}_{s'}$ in the state s' then we have satisfied the requirements for rationality and market clearing in state s' for any belief μ_i on F.(Market clearing follows because $\overline{p}_{s'}$ is the clearing price in state s' in the fully revealing REE $\overline{f}$.) So the only question is how to define the demands $z_i^{f_s}(P_i(s), f(s))$ (which is $z_i^{f_s}(P_i(s), p)$) and the beliefs $\mu_i^{f_s}$ so that the requirements of market clearing and rational choice are satisfied in the state s. Since (p, s) is $EXPR$ there exists a profile of probabilities, $\{\gamma_i^s\}_{i \in I}$, $\{\gamma_i^s\} \in \triangle(S(p) \cap P_i(s))$, and a profile of demands $\{x_i^s\}_{i \in I}$ such that x_i^s is an optimal choice for player i w.r.t γ_i^s at the price p and such that the aggregate demand equals the aggregate supply. We now define $z_i^{f_s}(P_i(s), f(s)) = x_i^s$ and establish the result by showing that we can define probabilities $\left\{\mu_i^{f_s}\right\}_{i \in I}$ on F so that

(3.5)

$$\mu_i^{f_s}(\ |P_i(s), p) = \gamma_i^s$$

To show this we rely on the following lemma which is proved in the appendix.

Lemma 1.1: Let $\alpha_1, ..., \alpha_m$ be m positive numbers and let $\gamma = (\gamma_1, ..., \gamma_m)$ be a probability vector. There exists a probability vector $\delta = (\delta_1, ..., \delta_m)$ which solves the following system of equations:

$$\gamma_k = \frac{\alpha_k \cdot \delta_k}{\sum_{j=1}^m \alpha_j \cdot \delta_j} \quad k = 1, ..., m$$

We map the lemma to our proof as follows: Suppose that $S(p) \cap P_i(s)$ is the set $\{1, .., m\}$. Define $\alpha_j \equiv \alpha(j)$, the prior probability of the state j and $\gamma_k \equiv \gamma_i^s(k)$, the probability of the state k according to γ_i^s. The lemma says that if we define $\mu_i^{f_s}(f_k)$, the probability of the price function f_k to be δ_k (and the probability of a price function different from $f_1, ..., f_m$ to be zero) then the equation (3.5) is satisfied. This follows because Bayesian updating implies that for $k = 1, ..., m$

$$\mu_i^{f_s}(k\ |P_i(s), p) = \frac{\alpha(k) \cdot \mu_i^{f_s}(f_k)}{\sum_{j=1}^m \alpha(j) \cdot \mu_i^{f_s}(f_j)}$$

and therefore $\mu_i^{f_s}(k \mid P_i(s), p) = \gamma_i^s(k)$.
This completes the proof of the theorem.

We now present an example which shows two things: First, the possibility of non-existence of a price function that is $CKRMC$ (and therefore non-existence of an outcome (p, s) which is $CKRMC$.) Second, the possibility of a difference between the set of outcomes that are $CKRMC$ and the set of outcomes that are $EXPR$. The example is similar to examples of non-existence of REE that were given by Kreps(1977) and Allen(1986). However, as we will demonstrate later on non-existence of REE does not imply non-existence of price functions that are $CKRMC$.

Example 2: The example is a simple variation on example 1.
There are two states, $S = \{1, 2\}$. The probability of each state is 0.5. The set of agents is $I = [0, 1]$ where agents in $I_1 = [0, \delta]$ know the true state and agent in $I_2 = (\delta, 1]$ don't know it. The utility of an agent in I_1 is $u_1(x, m, s) = a_s \cdot \log(x) + m$. The utility of an agent in I_2 is $u_2(x, m, s) = b_s \cdot \log(x) + m$. The aggregate amount of X is 1 and the number of units of M that each agent has exceeds $\text{Max}\{a_s, b_s : s = 1, 2\}$. All this implies that if p is the price of X in units of M then the demand for X of an agent in I_1 in state s is $\frac{a_s}{p}$ and the demand of an agent in I_2 who assigns probability $\gamma(s)$ to the state s is $\frac{\gamma(1) \cdot b_1 + \gamma(2) \cdot b_2}{p}$.
We make the following assumption:

(3.6) $a_1 \succ a_2$ and $b_1 \prec b_2$

(3.7) There exists a number $\widehat{p}$ such that

$$a_1 \cdot \delta + b_1(1 - \delta) = a_2 \cdot \delta + b_2(1 - \delta) = \widehat{p}$$

We claim that under these assumptions $FCKRMC = \emptyset$.
To prove this we compute, first, the set of outcomes that are $EXPR$. Let $\gamma = \{\gamma_i\}_{i \in I_2}$ be a profile of probabilities on S, (agents in I_1 assign probability 1 to the true state), and let $x_s^p(\gamma)$ denote the aggregate demand for X in the state s at the price p when the profile is γ. Since $b_1 \prec b_2$ the demand of each agent in I_2 is increasing in the probability which he assigns to the state 2. It follows that for every profile γ $x_1^p(\gamma) \geq \frac{a_1 \cdot \delta + b_1(1 - \delta)}{p}$ and $x_2^p(\gamma) \leq \frac{a_2 \cdot \delta + b_2(1 - \delta)}{p}$

. These two inequalities (plus (3.7) and the fact that the aggregate supply of X is 1) imply that the only outcomes that are $EXPR$ are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$. To see that we, first, observe that $\widehat{p}$ is the clearing price in state s when every agent in I_2 assigns the state s probability 1and therefore $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$ are $EXPR$. Now, assume by contradiction that $(p, 1)$ is some other outcome that is $EXPR$. It follows from the first inequality that $p \succ \widehat{p}$ but because of the second inequality p cannot be $EXPR$ w.r.t S (because for any profile of probabilities γ the aggregate demand in state 2, $x_2^p(\gamma)$,is smaller than 1.) Clearly, p cannot be $EXPR$ w.r.t $\{1\}$ and therefore we have obtained a contradiction. A similar argument establishes that $\widehat{p}$ is the only price that is $EXPR$ in the state 2. It follows from this and from part 1 of theorem 1 that the only price function that could possibly be $CKRMC$ is the function $\widehat{f}$ where $\widehat{f}(1) = \widehat{f}(2) = \widehat{p}$,but it is impossible to support $\widehat{f}$, because when each agent in I_2 assigns $\widehat{f}$ probability 1(which he must because there is no other function that is $CKRMC$) his posterior on the states (upon observing the price $\widehat{p}$) is the prior, probability 0.5 for each state, but with such a posterior the aggregate demand does not equal the aggregate supply. It follows that $\widehat{f}$ is not $CKRMC$ and therefore $FCKRMC = \emptyset$.

We note that the reason for the discrepancy between $EXPR$ and $CKRMC$ is that $EXPR$ allows for any probability distribution on S. $CKRMC$ on the other hand requires from an agent to have a complete theory about what price will occur in each state. In particular, if a price $\widehat{p}$ is the only price that could be $CKRMC$ in a state s then upon observing $\widehat{p}$ each agent must assign the state s a probability which is at least the prior of s.

Example 2 is an example where a REE does not exist and where the set of outcomes that are $EXPR$ is different from the set of outcomes that is $CKRMC$. Neither one of these properties implies the other. In the appendix we present two examples, examples 3 and 4, which demonstrate this point. In example 3 there is a fully revealing REE with a price $\widehat{p}_1$ in state 1. The price $\widehat{p}_1$ is $EXPR$ in the two other states- states 2 and 3-but it is not $CKRMC$ in these states. The reason for this discrepancy is similar to the one in example 2: The price $\widehat{p}_1$ is the only price that is $CKRMC$ in the state 1. Therefore, the conditional on S given $\widehat{p}_1$ of a probability distribution on $FCKRMC$ assigns the state 1 a probability which is greater or equal to the prior probability of state1. However, such a probability distribution on S cannot support $\widehat{p}_1$ as a clearing price in states 2 and 3. (In the example $\widehat{p}_1$ is supported in states 2 and 3 by probabilities which assign a probability zero to the state 1.) On the other hand, $EXPR$ allows an agent to assign any probability to the state 1 and therefore $\widehat{p}_1$ is $EXPR$ in states 2 and 3.

Example 4 is an example of an economy with two states in which there is no REE and yet there is a segment of prices that are $CKRMC$ in both states. Furthermore, the set of outcomes that are $CKRMC$ equals the set of outcomes that are $EXPR$. In the simple examples that we have considered where there are two types of agents, agents who know the true state and agents who don't know anything, non-existence of REE occurs (generically) whenever there are two states in which there is the same unique equilibrium price in the two respective economies where the state is known. Non-existence of $CKRMC$ requires more. In particular, we show in the appendix that in economies with two states existence of at least three different prices that are $EXPR$ in both states implies that the set of outcomes that are $EXPR$ equals the set of outcomes that are $CKRMC$. (In particular, $FCKRMC$ is not empty.)

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Appendix

Section 2:

Proof of Theorem 2:

One direction is immediate: Let $\widehat{f}$ be a price function such that there exists a model (Ω, β) that is $CKRMC$ and a state $\widehat{\varpi}$ such that $T_f(\widehat{\varpi}) = \widehat{f}$. We have to show that $\widehat{f}$ is $CKRMC$. We will do that by showing that the set $F = \digamma(\Omega)$ is $CKRMC$. So let $f \in F$ and let $\varpi \in \Omega$ be a state such that $T_f(\varpi) = f$. Let $\{\mu_i\}_{i \in I}$ and $\{z_i\}_{i \in I}$ be the profiles of beliefs and demands in ϖ. Define a profile of beliefs $\left\{\mu_i^f\right\}_{i \in I}$, $\mu_i^f = (\mu_i^{f,1}, ..., \mu_i^{f,m})$, on F as follows: For a Borel set of functions $\overline{F} \subseteq F$ define $\mu_i^{f,k}(\overline{F}) \equiv \mu_i^k(T_f^{-1}(\overline{F}))$ for $1 \leq k \leq m$. Since T_f is a measurable transformation $\mu_i^{f,k}$ is well defined. It is easy to see that the profile of beliefs $\left\{\mu_i^f\right\}_{i \in I}$ and the profile of demand strategies $\{z_i\}_{i \in I}$ support the function f w.r.t to the set F.

We turn now to the second direction. Let $\widehat{f}$ be a function that is $CKRMC$ and let $\widehat{F}$ be a Borel set of functions such that $\widehat{f} \in \widehat{F}$ and $\widehat{F}$ is $CKRMC$. It follows from part (3) of Theorem 1 that we can assume w.l.o.g that $\widehat{F}$ is finite. We will construct a model (Ω, β) that is $CKRMC$ such that $\digamma(\Omega) = \widehat{F}$. Define now (Ω, β) as follows:

(A.2.1) $\quad \Omega \equiv \widehat{F} \times \widehat{F} \times I$

and β is the product of the Borel sets in $\widehat{F} \times \widehat{F}$ and I.

To understand the idea behind this definition of Ω it would be useful to point out why a simpler definition would <u>not</u> work. So suppose we would have defined Ω to be F where we associate with the state f the price function f and the profiles of demands and beliefs $\left\{z_i^f\right\}_{i \in I}$ and $\left\{\mu_i^f\right\}_{i \in I}$ (the beliefs are now interpreted as beliefs on states.) Let i be a specific player and suppose that $\mu_i^{f,1}(\overline{f}) \succ 0$ for some $\overline{f} \in F$. This means that player i in the state f assigns a positive probability to the state $\overline{f}$ which implies that he is assigning a positive probability to the event where his beliefs are $\mu_i^{\overline{f}}$, but these beliefs are different than his beliefs in the state f, (μ_i^f), so this construction contradicts the assumption that a player knows his own beliefs.

To avoid this contradiction we need a richer set of states, in particular we need a state where the function that is materialized is $\overline{f}$ but player i has the belief μ_i^f and the demand z_i^f. The definition of Ω in (A.2.1) implements this requirement in the following way: The state $\widehat{\varpi} = (\overline{f}, f, i)$ is a state in which $\overline{f}$ is materialized and player i has the belief μ_i^f and the demand z_i^f. The complete and formal definition of $\widehat{\varpi}$ is as follows: Let $z_j^{\widehat{\varpi}}$ and $\mu_j^{\widehat{\varpi}}$ denote respectively the demand and belief on Ω of player j in $\widehat{\varpi}$. Define $z_j^{\widehat{\varpi}} \equiv z_j^{\overline{f}}$ for every $j \neq i$ and $z_i^{\widehat{\varpi}} \equiv z_i^f$. So all the players different from i have demand strategies that support $\overline{f}$ while player i has a demand strategy that supports the function f. (We note that since there is a continuum of players the fact that a single player i has a demand that is different than $z_i^{\overline{f}}$ does not change the fact that $\overline{f}$ specifies prices that clear the market.) The beliefs are defined according to the correspondence between states and functions that are materialized in them. Start with player i :

$$\mu_i^{\widehat{\varpi}}(\varpi) = \begin{cases} \mu_i^f(g) & \varpi = (g, f, i) \\ 0 & \text{otherwise} \end{cases}$$ [5]

and for a player $j \neq i$ define:

$$\mu_j^{\widehat{\varpi}}(\varpi) = \begin{cases} \mu_j^{\overline{f}}(g) & \varpi = (g, \overline{f}, j) \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to check that these definitions satisfy requirements 1. and 2. in the definition of a $CKRMC$ model.

Section 3

Proof of Lemma 1.1:

First, we assume w.l.o.g that $\gamma_k \succ 0$ for every k because if this is not the case we define $\delta_j = 0$ if $\gamma_j = 0$ and proceed to prove the lemma for the set $\{k : \gamma_k \succ 0\}$.

[5] The belief $\mu_i^{\widehat{\varpi}}$ is a finite sequence of probabilities-$\mu_i^{\widehat{\varpi}} = (\mu_i^{\widehat{\varpi},1}, ...\mu_i^{\widehat{\varpi},m})$. So the equation $\mu_i^{\widehat{\varpi}}(\varpi) = \mu_i^f(g)$ should be read as follows: for any $1 \leq k \leq m$ $\mu_i^{\widehat{\varpi},k}(\varpi) = \mu_i^{f,k}(g)$.
The definition of the belief of a player j different from i should be read in a similar way.

Second, multiplying the equations by the denominator and subtracting the RHS from the LHS gives a system of m homogeneous linear equations in $\delta_1,..,\delta_m$ and therefore there exists a solution to this system, $\overline{\delta} = (\overline{\delta_1},....,\overline{\delta_m})$.

Third, if $\overline{\delta}$ is a solution and c is a constant then $c \cdot \overline{\delta}$ is also a solution.

Finally, since $\gamma_k \succ 0$ for all $k = 1,...,m$ then if $\overline{\delta}$ is a solution then $\overline{\delta_1},....,\overline{\delta_m}$ all have the same sign which is the sign of the denominator.

It follows from all this that there is a solution $\widehat{\delta}$ to the system that is a probability vector because if $\overline{\delta}$ is some solution there is a constant c such that $c \cdot \overline{\delta}$ is a probability vector.

Example 3 :

There are three states in the economy, $S = \{1,2,3\}$ and two commodities X and M. There are three sets of agents $I_1 = [0,\delta]$, $I_2 = \left(\delta, \frac{1+\delta}{2}\right]$, $I_3 = \left(\frac{1+\delta}{2}, 1\right]$. Agents in I_1 know the true state. The others don't know anything. The prior on S, α,could be any probability distribution with full support. The utility of agent $i, u_i(x,m,s)$, is $a_s \log(x) + m$ if $i \in I_1$ it is $b_s \log(x) + m$ if $i \in I_2$ and it is $c_s \log(x) + m$ if $i \in I_3$. We assume that the aggregate amount of $X, \overline{X}$, is 1 and that each agent i has enough money, that is, $\overline{m_i} \geq \max\{a_s, b_s, c_s : s \in S\}$ (where $\overline{m_i}$ is the initial amount of money of agent i.)

We assume:

A.3.1 $a_1 \succ a_2 = a_3$; $b_1 \prec b_2 \prec b_3$; $c_1 \prec c_3 \prec c_2$

A.3.2 $\widehat{p_1} \equiv a_1 \cdot \delta + (b_1 + c_1) \cdot \frac{(1-\delta)}{2} \succ \widehat{p_2} \equiv a_2 \cdot \delta + (b_2 + c_2) \cdot \frac{(1-\delta)}{2} \succ \widehat{p_3} \equiv a_3 \cdot \delta + (b_3 + c_3) \cdot \frac{(1-\delta)}{2}$

A.3.3 $\widehat{p_1} = a_2 \cdot \delta + (b_3 + c_2) \cdot \frac{(1-\delta)}{2} = a_3 \cdot \delta + (b_3 + c_2) \cdot \frac{(1-\delta)}{2}$

The inequalities in A.3.2 imply that the price function $\widehat{f}(s) = \widehat{p_s}$ is a fully revealing REE. (This follows because the equations in A.3.2 imply that the aggregate demand for X in the state s at the price $\widehat{p_s}$ when everyone assigns probability 1 to s is equal to the aggregate supply, $\widehat{X} = 1$.) The equalities in A.3.3 imply that the price $\widehat{p_1}$ is $EXPR$ w.r.t S. In both states 2 and 3 $\widehat{p_1}$ is supported by the profile of probabilities $\widehat{\gamma} = \{\widehat{\gamma}_i\}_{i\in[0,1]}$ in which each agent in I_2 assigns probability 1 to the state 3 while each agent in I_3 assigns probability 1to the state 2. (Obviously, each agent in I_1 assigns probability 1 to the true state.)

We now show that $(\widehat{p_1}, 2)$ and $(\widehat{p_1}, 3)$ are not $CKRMC$ outcomes. First, we observe that the profile $\widehat{\gamma}$ which supports $\widehat{p_1}$ in states 2 and 3 generates the maximal aggregate demand for X in these states. Any other profile of probabilities will lead to a smaller demand and therefore to a lower price. It follows that any price that is $EXPR$ in state 2 or state 3 is smaller or equal to $\widehat{p_1}$. We now check which prices are $EXPR$ in state 1. Because b_1 is smaller than b_2 and b_3 and because c_1 is smaller than c_2 and c_3 the demand of an agent in $I_2 \cup I_3$ is minimal when he assigns the state 1 probability 1. It follows that if p_1 is a price that is $EXPR$ in state 1 then $p_1 \geq \widehat{p_1}$. Now since a price p_1 that is higher than $\widehat{p_1}$ is not $EXPR$ in states 2 or 3 we obtain that the only price that is $EXPR$ in state 1 is $\widehat{p_1}$. It follows (Theorem 1, part 1) that $\widehat{p_1}$ is also the only price that is $CKRMC$ in state 1. This means that every price function that is $CKRMC$ receives the value $\widehat{p_1}$ in state 1. Therefore, an agent who has a probability distribution on functions that are $CKRMC$ and who observes the price $\widehat{p_1}$ will assign state 1 a (conditional) probability which is at least the prior probability of this state. However, as we have seen, a profile of probabilities that assign the state 1 a positive probability cannot support $\widehat{p_1}$ in the states 2 and 3 and therefore $(\widehat{p_1}, 2)$ and $(\widehat{p_1}, 3)$ are not $CKRMC$ outcomes.

Example 4:

There are two states, $S = \{1, 2\}$, and each one of them has a probability 0.5. There are three sets of agents : $I_1 = [0, \delta]$, $I_2 = \left(\delta, \frac{1+\delta}{2}\right]$, $I_3 = \left(\frac{1+\delta}{2}, 1\right]$. Agents in I_1 know the true state. The others don't know it. The utilities of the agents are similar to those defined in the previous example so $u_i(x, m, s)$ is $a_s \log(x) + m$ if $i \in I_1$ it is $b_s \log(x) + m$ if $i \in I_2$ and it is $c_s \log(x) + m$ if $i \in I_3$. Also, the aggregate amount of X is 1 and each agent has enough money.

We assume:

A.3.4 $a_1 \succ a_2$; $b_1 \prec b_2$; $c_1 \succ c_2$.

A.3.5 $\widehat{p} \equiv a_1 \cdot \delta + (b_1 + c_1) \cdot \frac{(1-\delta)}{2} = a_2 \cdot \delta + (b_2 + c_2) \cdot \frac{(1-\delta)}{2}$

The equality in A.3.5 implies non-existence of a REE. The argument is familiar: Full revelation would imply that the price which clears the market is $\widehat{p}$ in both states. However, if that is the case then $\widehat{p}$ does not reveal the true state. On the other hand there cannot be a non-revealing REE $f, f(1) = f(2)$ because the demands of agents in I_1 for X in states 1and 2 are different so the same price cannot clear the market in both states.

We now compute the set of prices that are $EXPR$ w.r.t S and we will then show that these prices are also $CKRMC$ in both states. The computation is similar to the one in example 1 in the main text. Let $P_s, s = 1, 2$, be the set of prices that clear the market in state s when agents in $I_2 \cup I_3$ may have any profile of probabilities $\widehat{\gamma} = \{\widehat{\gamma}_i\}_{i \in I_2 \cup I_3}$ on S (agents in I_1 assign , of course, probability 1 to the true state.) We claim that :

$P_1 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_1 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

$P_2 = \left[a_2 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

To see this we note that the extreme points in each set are clearly the lowest and highest prices in the respective states (for example, the demand of agents in $I_2 \cup I_3$ is minimal when agents in I_2 assign probability 1 to the state 1and agents in I_3 assign probability 1to the state 2. When these are the beliefs the clearing prices in states 1 and 2 are the respective minimal points in P_1 and P_2.) Any price p between these points can be obtained as a clearing price by having a fraction $\beta = \beta(p)$ of the agents in I_2 and I_3 assign probability 1 to the states 1 and 2 respectively and a fraction $1 - \beta(p)$ assign probability 1 to the states 2 and 1 respectively. The set of prices that are $EXPR$ w.r.t S is:

$P \equiv P_1 \cap P_2 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

It follows from A.3.4 and A.3.5 that this is a non-empty segment. We now show that P is also the set of prices that are $CKRMC$ in S. We will do that by proving a more general claim.

Claim: Let E be an economy in which there are two states, $S = \{1, 2\}$. Let P denote the set of prices that are $EXPR$ w.r.t S. If $|P| \geq 3$ then the set outcomes that are $EXPR$ equals the set of outcomes that are $CKRMC$.

Proof: Let p_1, p_2 and p_3 be three different prices that are $EXPR$ w.r.t S. Define:

$F \equiv \{f, \ f(s) \in \{p_1, p_2, p_3\} \text{ and } f(1) \neq f(2)\}$

We will show that F is $CKRMC$. The proof is similar to the proof of theorem 1. Let $f \in F$ we will show that there exists profile of probabilities $\left\{\mu_i^f\right\}_{i \in I}$ on F ($I \equiv [0, 1]$ is the set of agents) and a profile of demands

$\left\{z_i^f(P_i(s), f(s)), s \in S\right\}_{i \in I}$ such that the demands are optimal w.r.t the beliefs and the aggregate demand equals the aggregate supply. Assume w.l.o.g that $f = (p_1, p_2)$, that is, $f(j) = p_j, j = 1, 2$. It will be useful to denote p_3 by $\overline{p}$. Since $p_1$and p_2 are $EXPR$ w.r.t S there exist profiles of probabilities on S $\gamma^{j,s} = \left\{\gamma_i^{j,s}\right\}_{i \in I}, j = 1, 2, s = 1, 2,$ $\gamma_i^{j,s}(s) = 1$ for $i \in I_1$(agents in I_1 know the true state) and profiles of demands $x^{j,s} = \left\{x_i^{j,s}\right\}_{i \in I}$ such that $x_i^{j,s}$ is an optimal demand for agent i at the price p_j w.r.t the probability $\gamma_i^{j,s}$ and the aggregate demand equals the aggregate supply. Applying lemma 1.1 we obtain that for $j \in \{1, 2\}$ and $s \in \{1, 2\}$ there exists a probability distribution $\delta_i^{j,s}$ on the pair of functions ($p_j, \overline{p}$), ($\overline{p}$, p_j) such that the conditional probability of agent i on S upon observing p_j is $\gamma_i^{j,s}$.

Thus, we have two probability distributions on F, $\delta_i^{1,1}$ and $\delta_i^{2,2}$, such that the conditional of $\delta_i^{j,j}$ on S given p_j is $\gamma_i^{j,j}$. (Recall that $\gamma^{j,j} = \left\{\gamma_i^{j,j}\right\}_{i \in I}$ is the profile of probabilities that rationalizes the demands $\left\{x_i^{j,j}\right\}_{i \in I}$ at the price p_j.) However, to show that (p_1, p_2) is $CKRMC$ we have to show the existence of one probability distribution δ_i on F with the property that it's conditional on S given p_1 is $\gamma_i^{1,1}$ and it's conditional given p_2 is $\gamma_i^{2,2}$. It is easy to see that we can obtain this by defining $\delta_i \equiv 0.5 \cdot \delta_i^{1,1} + 0.5 \cdot \delta_i^{2,2}$ (So, for example, the probability that δ_i assigns to ($\overline{p}$,p_1) is half the probability that $\delta_i^{1,1}$ assigns it.) Thus, defining $\mu_i^f \equiv \delta_i$ and $z_i^f(P_i(s), f(s)) \equiv x_i^{s,s}$ we obtain profiles of beliefs and demands that support f w.r.t F.