Learning and Disagreement in an Uncertain World*

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Abstract

Most economic analyses presume that there are limited differences in the prior beliefs of individuals, an assumption most often justified by the argument that sufficient common experiences and observations will eliminate disagreements. We investigate this claim using a simple model of Bayesian learning. Two individuals with different priors observe the same infinite sequence of signals about some underlying parameter. Existing results in the literature establish that when individuals are *certain* about the interpretation of signals, under very mild conditions there will be asymptotic agreement—their assessments will eventually agree. In contrast, we look at an environment in which individuals are *uncertain* about the interpretation of signals, meaning that they have non-degenerate probability distributions over the likelihood of signals given the underlying parameter. When priors on the parameter and the conditional distribution of signals have full support, we prove the following results: (1) Individuals will never agree, even after observing the same infinite sequence of signals. (2) Before observing the signals, they believe with probability 1 that their posteriors about the underlying parameter will fail to converge. (3) Observing the same sequence of signals may lead to a divergence of opinion rather than the typically-presumed convergence. We then characterize the conditions for asymptotic learning and agreement under "approximate certainty"—i.e., as we look at the limit where uncertainty about the interpretation of the signals disappears. When the family of probability distributions of signals given the parameter has "rapidly-varying tails" (such as the normal or the exponential distributions), approximate certainty restores asymptotic learning and agreement. However, when the family of probability distributions has "regularly-varying tails" (such as the Pareto, the log-normal, and the t-distributions), asymptotic learning and agreement do not result even in the limit as the amount of uncertainty disappears.

Lack of common priors has important implications for economic behavior in a range of circumstances. We illustrate how the type of learning outlined in this paper interacts with economic behavior in various different situations, including games of common interest, coordination, asset trading and bargaining.

Keywords: asymptotic disagreement, Bayesian learning, merging of opinions.

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1 Introduction

The common prior assumption is one of the cornerstones of modern economic analysis. Most models postulate that the players in a game have a common prior about the game form and payoff distributions—for example, they all agree that some payoff-relevant parameter vector θ is drawn from a known distribution G, even though each may also have additional information about some components of θ. A common justification for the common prior assumption comes from *learning*; individuals, through their own experiences and the communication of others, will have access to a history of events informative about the vector θ, and this process will lead to "agreement" among individuals about the distribution of the vector θ. A strong version of this view is expressed in Savage (1954, p. 48) as the statement that a Bayesian individual, who does not assign zero probability to "the truth," will learn it eventually as long as the signals are informative about the truth. A more sophisticated version of this conclusion also follows from Blackwell and Dubins' (1962) theorem about the "merging of opinions".[1]

Despite these powerful intuitions and theorems, disagreement is the rule rather than the exception in practice. Just to mention a few instances, there is typically considerable disagreement even among economists working on a certain topic. For example, economists routinely disagree about the role of monetary policy, the impact of subsidies on investment or the magnitude of the returns to schooling. Similarly, there are deep divides about religious beliefs within populations with shared experiences, and finally, there was recently considerable disagreement among experts with access to the same data about whether Iraq had weapons of mass destruction. In none of these cases, can the disagreements be traced to individuals having access to different histories of observations. Rather it is their *interpretations* that differ. In particular, it seems that an estimate showing that subsidies increase investment is interpreted very differently by two economists starting with different priors; for example, an economist believing that subsidies have no effect on investment appears more likely to judge the data or the methods leading to this estimate to be unreliable and thus to attach less importance to this evidence. Similarly, those who believed in the existence of weapons of mass destruction in Iraq presumably interpreted the evidence from inspectors and journalists indicating the opposite as

[1] Blackwell and Dubins' (1962) theorem shows that if two probability measures are absolutely continuous with respect to each other (meaning that they assign positive probability to the same events), then as the number of observations goes to infinity, their predictions about future frequencies will agree. This is also related to Doob's (1948) consistency theorem for Bayesian posteriors, which we discuss and use below.

biased rather than informative.

In this paper, we show that this type of behavior will be the outcome of learning by Bayesian individuals with different priors when they are *uncertain* about the informativeness of signals. In particular, we consider the following simple environment: one or two individuals with given priors observe a sequence of signals, $\{s_t\}_{t=0}^{n}$, and form their posteriors about some underlying state variable (parameter) θ. The only non-standard feature of the environment is that these individuals are uncertain about the distribution of signals conditional on the underlying state. In the simplest case where the state and the signal are binary, e.g., $\theta \in \{A, B\}$, and $s_t \in \{a, b\}$, this implies that $\Pr(s_t = \theta \mid \theta) = p_\theta$ is not a known number, but individuals may also have a prior over p_θ, say given by F_θ. We refer to this distribution F_θ as individuals' *subjective probability distribution* and to its density f_θ as *subjective (probability) density*. This distribution, which can differ among individuals, is a natural measure of their uncertainty about the informativeness of signals. When subjective probability distributions are non-degenerate, individuals will have some latitude in interpreting the sequence of signals they observe.

We identify conditions under which Bayesian updating leads to *asymptotic learning* (individuals learning, or believing that they are learning, the true value of θ with probability 1 after observing infinitely many signals) and *asymptotic agreement* (convergence between their assessments of the value of θ). When F_θ has a full support for each θ, we show that:

1. There will not be asymptotic learning. Instead each individual's posterior of θ continues to be a function of his prior.

2. There will not be asymptotic agreement; two individuals with different priors observing the *same* sequence of signals will reach different posterior beliefs even after observing infinitely many signals. Moreover, individuals attach *ex ante probability 1* that they will disagree after observing the sequence of signals.

3. Two individuals may *disagree more* after observing a common sequence of signals than they did so previously. In fact, for any model of learning under uncertainty that satisfies the full support assumption, there exists an open set of pairs of priors such that the disagreement between the two individuals will necessarily grow starting from these priors.

In contrast to these results, when each individual i is sure that $p_\theta = p^i$ for some known number $p^i > 1/2$ (with possibly $p^1 \neq p^2$), then asymptotic agreement is guaranteed. In fact, we show that similar asymptotic learning and agreement results hold even when there is some amount of uncertainty, but not full support.[2]

These results raise the question of whether the asymptotic learning and agreement results under certainty are robust to a small amount of uncertainty. We investigate this issue by studying learning under "approximate certainty," i.e., by considering a sequence of subjective density functions $\{f_m\}$ that become more and more concentrated around a single point—thus converging to full certainty. Interestingly, asymptotic learning and agreement under certainty may be a discontinuous limit point of a general model of learning under uncertainty. In particular, "approximate certainty" is not sufficient to ensure asymptotic agreement. We fully characterize the conditions under which approximate certainty will lead to asymptotic learning and agreement. Whether or not this is the case depends on the tail properties of the family of subjective density functions $\{f_m\}$. When this family has *regularly-varying tails* (such as the Pareto or the log-normal distributions), even under approximate certainty there will be asymptotic disagreement. When $\{f_m\}$ has rapidly-varying tails (such as the normal distribution), there will be asymptotic agreement under approximate certainty.

We also show that there may be substantial asymptotic disagreement even when the individuals' subjective probability distributions are approximately identical and there is approximate certainty. Nevertheless, when there is sufficient continuity of beliefs in the limit, we can link the extent of asymptotic disagreement to the differences in their interpretations of the signals. In this case, significant asymptotic disagreement under approximate certainty is possible only when their interpretations differ substantially.

Lack of asymptotic learning has important implications for a range of economic situations. We illustrate some of these by considering a number of simple environments where two individuals observe the same sequence of signals before or while playing a game. In particular, we discuss the implications of learning in uncertain environments for games of coordination, games of common interest, bargaining, games of communication and asset trading. Not surprisingly, given the above description of results, individuals will play these games differently than they would in environments with common priors—and also differently than in environments without common priors but where learning takes place under certainty. For example,

[2]For example, there will be asymptotic learning and agreement if both individuals attach probability 1 to the event that $p_\theta > 1/2$. See Theorem 2 below.

we establish that contrary to standard results, individuals may wish to play games of common interests before receiving more information about payoffs. Similarly, we show how the possibility of observing the same sequence of signals may lead individuals to trade *only after* they observe the public information. This result contrasts with both standard no-trade theorems (e.g., Milgrom and Stokey, 1982) and existing results on asset trading without common priors, which assume learning under certainty (Harrison and Kreps, 1978, and Morris, 1996). We also provide a simple example illustrating a potential reason why individuals may be uncertain about informativeness of signals—the strategic behavior of other agents trying to manipulate their beliefs.

Our results cast doubt on the idea that the common prior assumption may be justified by learning. In many environments, even when there is little uncertainty so that each individual believes that he will learn the true state, learning need not lead to similar beliefs about the relevant parameters, and the strategic outcome may be significantly different from that of the common-prior environment.[3] Whether this assumption is warranted will depend on the specific setting and what type of information individuals are trying to glean from the data.

Relating our results to the famous Blackwell-Dubins (1962) theorem may help clarify their essence. As briefly mentioned in Footnote 1, this theorem shows that when two agents agree on zero-probability events (i.e., their priors are absolutely continuous with respect to each other), asymptotically, they will make the same predictions about future frequencies of signals. Our results do not contradict this theorem, since we impose absolute continuity throughout. Instead, our results rely on the fact that agreeing about future frequencies is not the same as agreeing about the underlying state (or the underlying payoff relevant parameters).[4] Put differently, under uncertainty, there is an "identification problem" making it impossible for individuals to infer the underlying state from limiting frequencies, and this leads to different interpretations of the same signal sequence by individuals with different priors. In most economic situations, what is important is not the future frequencies of signals, but some payoff-relevant parameter. For example, what was essential for the debate on the weapons of mass destruction was not the frequency of news about such weapons but whether or not they existed. What is relevant for economists trying to evaluate a policy is not the frequency of estimates on the effect of

[3] For the previous arguments about whether game-theoretic models should be formulated with all individuals having a common prior, see, for example, Aumann (1986, 1998) and Gul (1998).

[4] In this respect, our paper is also related to Kurz (1994, 1996), who considers a situation in which agents agree about long-run frequencies, but their beliefs fail to merge because of the non-stationarity of the world.

similar policies from other researchers, but the impact of this specific policy when (and if) implemented. Similarly, what may be relevant in trading assets is not the frequency of information about the dividend process, but the actual dividend that the asset will pay. Thus, many situations in which individuals need to learn about a parameter or state that will determine their ultimate payoff as a function of their action falls within the realm of the analysis here.

In this respect, our work differs from papers, such as Freedman (1964) and Miller and Sanchirico (1999), that question the applicability of the absolute continuity assumption in the Blackwell-Dubins theorem in statistical and economic settings. Similarly, a number of important theorems in statistics, for example, Berk (1966), show that under certain conditions, limiting posteriors will have their support on the set of all identifiable values (though they may fail to converge to a limiting distribution). Our results are different from those of Berk both because in our model individuals always place positive probability on the truth and also because we provide a tight characterization of the conditions for lack of asymptotic learning and agreement.

Finally, our paper is also related to models of media bias, for example, Baron (2004), Besley and Prat (2006) and Gentzkow and Shapiro (2006), which investigate the causes or consequences of manipulation of information by media outlets. We show in Section 4 how reporting by a biased media outlet can lead to a special case of the learning problem studied in this paper.

The rest of the paper is organized as follows. Section 2 provides all our main results in the context of a two-state two-signal setup. Section 3 provides generalizations of these results to an environment with K states and $L \geq K$ signals. Section 4 considers a variety of applications of our results, and Section 5 concludes.

2 The Two-State Model

2.1 Environment

We start with a two-state model with binary signals. This model is sufficient to establish all our main results in the simplest possible setting. These results are later generalized to arbitrary number of states and signal values.

There are two individuals, denoted by $i = 1$ and $i = 2$, who observe a sequence of signals $\{s_t\}_{t=0}^{n}$ where $s_t \in \{a, b\}$. The underlying state is $\theta \in \{A, B\}$, and agent i assigns ex ante probability $\pi^i \in (0,1)$ to $\theta = A$. The individuals believe that, given θ, the signals are exchangeable, i.e., they are independently and identically distributed with an unknown distribution.[5] That is, the probability of $s_t = a$ given $\theta = A$ is an unknown number p_A; likewise, the probability of $s_t = b$ given $\theta = B$ is an unknown number p_B—as shown in the following table:

	A	B
a	p_A	$1 - p_B$
b	$1 - p_A$	p_B

Our main departure from the standard models is that we allow the individuals to be uncertain about p_A and p_B. We denote the cumulative distribution function of p_θ according to individual i—i.e., his *subjective probability distribution*—by F_θ^i. In the standard models, F_θ^i is degenerate, putting probability 1 at some $\hat{p}_\theta^i$. In contrast, we will assume:

Assumption 1 *For each i and θ, F_θ^i has a continuous, non-zero and finite density f_θ^i over* $[0,1]$.

The assumption implies that F_θ^i has *full support* over $[0,1]$. This assumption ensures that there is absolute continuity of priors as in the Blackwell-Dubins theorem and will also play an important but different role in our analysis. It is worth noting that while this assumption allows $F_\theta^1(p)$ and $F_\theta^2(p)$ to differ, for many of our results it is not important whether or not this is so (i.e., whether or not the two individuals have a common prior about the distribution of p_θ). Throughout, we assume that π^1, π^2, F_θ^1 and F_θ^2 are known to both individuals.[6]

We consider infinite sequences $s \equiv \{s_t\}_{t=1}^\infty$ of signals and write S for the set of all such sequences. The posterior belief of individual i about θ after observing the first n signals $\{s_t\}_{t=1}^n$ is

$$\phi_n^i(s) \equiv \Pr^i\left(\theta = A \mid \{s_t\}_{t=1}^n\right),$$

[5] See, for example, Billingsley (1995). If there were only one state, then our model would be identical to De Finetti's canonical model (see, for example, Savage, 1954). In the context of this model, De Finetti's theorem provides a Bayesian foundation for classical probability theory by showing that exchangeability (i.e., invariance under permutations of the order of signals) is equivalent to having an independent identical unknown distribution and implies that posteriors converge to long-run frequencies. De Finetti's decomposition of probability distributions is extended by Jackson, Kalai and Smorodinsky (1999) to cover cases without exchangeability.

[6] The assumption that player 1 knows the prior and probability assessment of player 2 regarding the distribution of signals given the state is used in the "asymptotic agreement" results and in applications. Since our purpose is to understand whether learning justifies the common prior assumption, we depart from Aumann's (1976) approach and assume that agents do not change their views because the beliefs of others differ from theirs.

where $\Pr^i(\theta = A \mid \{s_t\}_{t=1}^n)$ denotes the posterior probability that $\theta = A$ given a sequence of signals $\{s_t\}_{t=1}^n$, prior π^i and subjective probability distribution F_θ^i (see footnote 7 for a formal definition).

Throughout, without loss of generality, we suppose that in reality $\theta = A$. The two questions of interest for us are:

1. **Asymptotic learning:** whether $\Pr^i\left(\lim_{n\to\infty} \phi_n^i(s) = 1 | \theta = A\right) = 1$ for $i = 1, 2$.

2. **Asymptotic agreement:** whether $\Pr^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| = 0\right)$ for $i = 1, 2$.

Notice that both asymptotic learning and agreement are defined in terms of the ex ante probability assessments of the two individuals. Therefore, asymptotic learning implies that an individual believes that he or she will ultimately learn the truth, while asymptotic agreement implies that both individuals believe that their assessments will eventually converge.

2.2 Asymptotic Learning and Disagreement

The following theorem gives the well-known result, which applies when Assumption 1 does *not* hold. A version of this result is stated in Savage (1954) and also follows from Blackwell and Dubins' (1962) more general theorem applied to this case. Since the proof of this theorem uses different arguments than those presented below and is tangential to our focus here, it is relegated to the Appendix.

Theorem 1 *Assume that for some* $\hat{p}^1, \hat{p}^2 \in (1/2, 1]$*, each* F_θ^i *puts probability 1 on* $\hat{p}^i$*, i.e.,* $F_\theta^i(\hat{p}^i) = 1$ *and* $F_\theta^i(p) = 0$ *for each* $p < \hat{p}^i$*. Then, for each* $i = 1{,}2$*,*

1. $\Pr^i\left(\lim_{n\to\infty} \phi_n^i(s) = 1 | \theta = A\right) = 1$.

2. $\Pr^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| = 0\right) = 1$.

Theorem 1 is a slightly generalized version of the standard theorem where the individual will learn the truth with experience (almost surely as $n \to \infty$) and two individuals observing the same sequence will necessarily agree. The generalization arises from the fact that learning and agreement take place even though $\hat{p}^1$ may differ from $\hat{p}^2$ (while Savage, 1954, assumes that $\hat{p}^1 = \hat{p}^2$). Even if the two individuals have different expectations about the probability of $s_t = a$ conditional on $\theta = A$, the fact that $\hat{p}^i > 1/2$ and that they hold these beliefs with *certainty* is sufficient for asymptotic learning and agreement. Intuitively, this is because both individuals will, with certainty, interpret one of the signals as evidence that the state is $\theta = A$, and also believe that when the state is $\theta = A$ the majority of the signals in the limiting distribution will be $s_t = a$. Based on this idea, we generalize Theorem 1 to the case where the individuals are not necessarily certain about the signal distribution but their subjective distributions do not satisfy the full support feature of Assumption 1.

Theorem 2 *Assume that each F_θ^i has a density f_θ^i and satisfies $F_\theta^i(1/2) = 0$. Then, for each $i = 1,2$,*

1. $\Pr^i\left(\lim_{n\to\infty} \phi_n^i(s) = 1 | \theta = A\right) = 1.$
2. $\Pr^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| = 0\right) = 1.$

This theorem will be proved together with the next one, Theorem 3, below. It is evident that the assumption $F_\theta^i(1/2) = 0$ implies that $p_\theta > 1/2$, contradicting the full support assumption imposed in Assumption 1. The intuition for this result is similar to that of Theorem 1: when both individuals attach probability 1 to the event that $p_\theta > 1/2$, they will believe that the majority of the signals in the limiting distribution will be $s_t = a$ when $\theta = A$,. Thus, each believes that both he and the other individual will learn the underlying state with probability 1—even though they may both be uncertain about the exact distribution of signals conditional on the underlying state.

In contrast to the previous two theorems, which establish asymptotic learning and agreement results, our next result is a negative one and shows that when F_θ^i has full support as specified in Assumption 1, there will be neither asymptotic learning nor asymptotic agreement.

Theorem 3 *Suppose Assumption 1 holds for $i = 1,2$. Then,*

1. $\Pr^i\left(\lim_{n\to\infty} \phi_n^i(s) \neq 1 | \theta = A\right) = 1$ *for $i = 1,2$;*
2. $\Pr^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| \neq 0\right) = 1$ *whenever $\pi^1 \neq \pi^2$ and $F_\theta^1 = F_\theta^2$ for each $\theta \in \{A, B\}$.*

This theorem therefore contrasts with Theorems 1 and 2 and implies that the individual in question will fail to learn the true state with probability 1. The second part of the theorem states that if the individuals' prior beliefs about the state differs (but they interpret the signals in the same way), then their posteriors will eventually disagree, and moreover, they will both attach probability 1 to the event that their beliefs will eventually diverge. Put differently, this implies that there is "agreement to eventually disagree" between the two individuals, in the sense that they both believe ex ante that after observing the signals they will fail to agree. This feature will play an important role in the applications in Section 4 below.

Towards proving the above theorems, we now introduce some notation, which will be used throughout the paper. Recall that the sequence of signals, s, is exchangeable, so that the order of the signals does not matter for the posterior. Let

$$r_n(s) \equiv \# \{t \leq n | s_t = a\}$$

be the number of times $s_t = a$ out of first n signals.[7] By the strong law of large numbers, $r_n(s)/n$ converges to some $\rho(s) \in [0,1]$ almost surely according to both individuals. Defining the set

$$\bar{S} \equiv \{s \in S : \lim_{n\to\infty} r_n(s)/n \text{ exists}\}, \tag{1}$$

this observation implies that $\Pr^i\left(s \in \bar{S}\right) = 1$ for $i = 1, 2$. We will often state our results for all sample paths s in $\bar{S}$, which equivalently implies that these statements are true almost surely or with probability 1. Now, a straightforward application of the Bayes rule gives

$$\phi_n^i(s) = \frac{1}{1 + \frac{1-\pi^i}{\pi^i} \frac{\Pr^i(r_n|\theta=B)}{\Pr^i(r_n|\theta=A)}}, \tag{2}$$

where $\Pr^i(r_n|\theta)$ is the probability of observing the signal $s_t = a$ exactly r_n times out of n signals with respect to the distribution F_θ^i. The next lemma provides a very useful formula for $\phi_\infty^i(s) \equiv \lim_{n\to\infty} \phi_n^i(s)$ for all sample paths s in $\bar{S}$.

Lemma 1 *Suppose Assumption 1 holds. Then for all $s \in \bar{S}$,*

$$\phi_\infty^i(\rho(s)) \equiv \lim_{n\to\infty} \phi_n^i(s) = \frac{1}{1 + \frac{1-\pi^i}{\pi^i} R^i(\rho(s))}, \tag{3}$$

[7] Given the definition of $r_n(s)$, the probability distribution $\Pr^i$ (on $\{A, B\} \times S$ with respect to the product topology) can be formally defined as

$$\Pr^i\left(E^{A,s,n}\right) \equiv \pi^i \int_0^1 p^{r_n(s)} (1-p)^{n-r_n(s)} f_A^i(p)\, dp, \text{ and}$$
$$\Pr^i\left(E^{B,s,n}\right) \equiv \left(1-\pi^i\right) \int_0^1 (1-p)^{r_n(s)} p^{n-r_n(s)} f_B^i(p)\, dp$$

at each event $E^{\theta,s,n} = \{(\theta, s') | s'_t = s_t \text{ for each } t \leq n\}$, where $s \equiv \{s_t\}_{t=1}^\infty$ and $s' \equiv \{s'_t\}_{t=1}^\infty$.

where $\rho(s) = \lim_{n\to\infty} r_n(s)/n$, *and* $\forall \rho \in [0,1]$,

$$R^i(\rho) \equiv \frac{f_B^i(1-\rho)}{f_A^i(\rho)}. \tag{4}$$

Proof. Write

$$\begin{aligned}\frac{\Pr^i(r_n|\theta=B)}{\Pr^i(r_n|\theta=A)} &= \frac{\int_0^1 p^{r_n}(1-p)^{n-r_n} f_B(1-p)dp}{\int_0^1 p^{r_n}(1-p)^{n-r_n} f_A(p)dp} \\ &= \frac{\frac{\int_0^1 p^{r_n}(1-p)^{n-r_n} f_B(1-p)dp}{\int_0^1 p^{r_n}(1-p)^{n-r_n}dp}}{\frac{\int_0^1 p^{r_n}(1-p)^{n-r_n} f_A(p)dp}{\int_0^1 p^{r_n}(1-p)^{n-r_n}dp}} \\ &= \frac{\mathbb{E}^\lambda[f_B(1-p)|r_n]}{\mathbb{E}^\lambda[f_A(p)|r_n]}\end{aligned}$$

where the first equality is obtained by dividing the numerator and the denominator by the same term, and the second uses the fact that these expressions correspond to the expectation of f_B and f_A given r_n under the flat (Lebesgue) prior, denoted by $\mathbb{E}^\lambda[f_\theta(p)|r_n]$. By Doob's consistency theorem for Bayesian posterior expectation of the parameter as $r_n \to \rho$, we have that $\mathbb{E}^\lambda[f_B(1-p)|r_n] \to f_B(1-\rho)$ and $\mathbb{E}^\lambda[f_A(p)|r_n] \to f_A(\rho)$ (see, e.g., Doob, 1949, Ghosh and Ramamoorthi, 2003, Theorem 1.3.2). This establishes

$$\frac{\Pr^i(r_n|\theta=B)}{\Pr^i(r_n|\theta=A)} \to R^i(\rho),$$

as defined in (4). Equation (3) then follows from (2). ■

In equation (4), $R^i(\rho)$ is the *asymptotic likelihood ratio* of observing frequency ρ of a when the true state is B versus when it is A. Lemma 1 states that, asymptotically, individual i uses this likelihood ratio and Bayes rule to compute his posterior beliefs about θ.

An immediate implication of Lemma 1 is that given any $s \in \bar{S}$,

$$\phi_\infty^1(\rho(s)) = \phi_\infty^2(\rho(s)) \text{ if and only if } \frac{1-\pi^1}{\pi^1}R^1(\rho(s)) = \frac{1-\pi^2}{\pi^2}R^2(\rho(s)) \ . \tag{5}$$

The proofs of Theorems 2 and 3 now follow from Lemma 1 and equation (5).

Proof of Theorem 2. Under the assumption that $F_\theta^i(1/2) = 0$ in the theorem, the argument in Lemma 1 still applies, and we have $R^i(\rho(s)) = 0$ when $\rho(s) > 1/2$ and $R^i(\rho(s)) = \infty$ when $\rho(s) < 1/2$. Given $\theta = A$, then $r_n(s)/n$ converges to some $\rho(s) > 1/2$ almost surely according to both $i = 1$ and 2. Hence, $\Pr^i\left(\phi_\infty^1(\rho(s)) = 1|\theta = A\right) = \Pr^i\left(\phi_\infty^2(\rho(s)) = 1|\theta = A\right) =$

1 for $i = 1, 2$. Similarly, $\Pr^i\left(\phi^1_\infty(\rho(s)) = 0|\theta = B\right) = \Pr^i\left(\phi^2_\infty(\rho(s)) = 0|\theta = B\right) = 1$ for $i = 1, 2$, establishing the second part. ■

Proof of Theorem 3. Since $f^i_B(1-\rho(s)) > 0$ and $f_A(\rho(s))$ is finite, $R^i(\rho(s)) > 0$. Hence, by Lemma 1, $\phi^i_\infty(\rho(s)) \neq 1$ for each s, establishing the first part. The second part follows from equation (5), since $\pi^1 \neq \pi^2$ and $F^1_\theta = F^2_\theta$ implies that for each $s \in \bar{S}$, $\phi^1_\infty(s) \neq \phi^2_\infty(s)$, and thus $\Pr^i\left(\left|\phi^1_\infty(s) - \phi^2_\infty(s)\right| \neq 0\right) = 1$ for $i = 1, 2$. ■

Intuitively, when Assumption 1 (in particular, the full support feature) holds, an individual is never sure about the exact interpretation of the sequence of signals he observes and will update his views about p_θ (the informativeness of the signals) as well as his views about the underlying state. For example, even when signal a is more likely in state A than in state B, a very high frequency of a will not necessarily convince him that the true state is A, because he may infer that the signals are not as reliable as he initially believed, and they may instead be biased towards a. Therefore, the individual never becomes certain about the state, which is captured by the fact that $R^i(\rho)$ defined in (4) never takes the value zero or infinity. Consequently, as shown in (3), his posterior beliefs will be determined by his prior beliefs about the state and also by R^i, which tells us how the individual updates his beliefs about the informativeness of the signals as he observes the signals. When two individuals interpret the informativeness of the signals in the same way (i.e., $R^1 = R^2$), the differences in their priors will always be reflected in their posteriors.

In contrast, if an individual were sure about the informativeness of the signals (i.e., if i were sure that $p_A = p_B = p^i$ for some $p^i > 1/2$) as in Theorem 1, then he would never question the informativeness of the signals—even when the limiting frequency of a converges to a value different from p^i or $1 - p^i$. Consequently, in this case, for each sample path with $\rho(s) \neq 1/2$ both individuals would learn the true state and their posterior beliefs would agree asymptotically.

As noted above, an important implication of Theorem 3 is that there will typically be "agreement to eventually disagree" between the individuals. In other words, given their priors, both individuals will agree that after seeing the same infinite sequence of signals they will still disagree (with probability 1). This implication is interesting in part because the common prior assumption, typically justified by learning, leads to the celebrated "no agreement to disagree" result (Aumann, 1976, 1998), which states that if the individuals' posterior beliefs are common knowledge, then they must be equal.[8] In contrast, in the limit of the learning process here, the individuals' beliefs are common knowledge (as there is no private information), but they are different with probability 1. This is because in the presence of uncertainty, as defined by Assumption 1, both individuals understand that their priors will have an effect on their beliefs even asymptotically; thus they expect to disagree. Many of the applications we discuss in Section 4 exploit this feature.

We have established that the differences in priors are reflected in the posteriors even in the limit $n \to \infty$ when the individuals interpret the informativeness of the signals similarly. This raises the question of whether two individuals that observe the same sequence of signals may have diverging posteriors, i.e., whether common information can turn agreement into disagreement. The next theorem shows this can be the case as long as individuals start with relatively similar priors.

Theorem 4 *Suppose that Assumption 1 holds and that there exists $\epsilon > 0$ such that $\left|R^1(\rho) - R^2(\rho)\right| > \epsilon$ for each $\rho \in [0,1]$. Then, there exists an open set of priors π^1 and π^2, such that for all $s \in \bar{S}$,*

$$\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| > \left|\pi^1 - \pi^2\right|;$$

in particular,

$$\Pr^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| > \left|\pi^1 - \pi^2\right|\right) = 1.$$

Proof. Fix $\pi^1 = \pi^2 = 1/2$. By Lemma 1 and the hypothesis that $\left|R^1(\rho) - R^2(\rho)\right| > \epsilon$ for each $\rho \in [0,1]$, $\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| > \epsilon'$ for some $\epsilon' > 0$, while $\left|\pi^1 - \pi^2\right| = 0$. Since both expressions are continuous in π^1 and π^2, there is an open neighborhood of $1/2$ such that the above inequality uniformly holds for each ρ whenever π^1 and π^2 are in this neighborhood. The last statement follows from the fact that $\Pr^i\left(s \in \bar{S}\right) = 1$. ■

Intuitively, even a small difference in priors ensures that individuals will interpret signals differently, and if the original disagreement was relatively small, after almost all sequences of signals, the disagreement between the two individuals grows. Consequently, the observation of a common sequence of signals causes an initial difference of opinion between individuals to widen (instead of the standard merging of opinions under certainty). Theorem 4 also shows that both individuals are certain ex ante that their posteriors will diverge after observing

[8] Note, however, that the "no agreement to disagree" result derives from individuals updating their beliefs because those of others differ from their own, whereas here individuals only update their beliefs by learning.

the same sequence of signals, because they understand that they will interpret the signals differently. This strengthens our results further and shows that for some priors individuals will "agree to eventually disagree even more".

An interesting implication of Theorem 4 is also worth noting. As demonstrated by Theorems 1 and 2, when there is learning under certainty individuals initially disagree, but each individual also believes that they will eventually agree (and in fact, that they will converge to his or her beliefs). This implies that each individual expects the other to "learn more". More specifically, let $\mathbf{I}_{\theta=A}$ be the indicator function for $\theta = A$ and $\Lambda^i = \left(\pi^i - \mathbf{I}_{\theta=A}\right)^2 - \left(\phi^i_\infty - \mathbf{I}_{\theta=A}\right)^2$ be a measure of learning for individual i,and let $\mathbb{E}^i$ be the expectation of individual i (under the probability measure $\Pr^i$). Under certainty, Theorem 1 implies that $\phi^i_\infty = \phi^j_\infty = \mathbf{I}_{\theta=A}$, so that $\mathbb{E}^i\left[\Lambda^i - \Lambda^j\right] = -\left(\pi^i - \pi^j\right)^2 < 0$ and thus $\mathbb{E}^i\left[\Lambda^i\right] < \mathbb{E}^i\left[\Lambda^j\right]$. Under uncertainty, this is not necessarily true. In particular, Theorem 4 implies that, under the assumptions of the theorem, there exists an open subset of the interval $[0,1]$ such that whenever π^1 and π^2 are in this subset, we have $\mathbb{E}^i\left[\Lambda^i\right] > \mathbb{E}^i\left[\Lambda^j\right]$, so that individual i would expect to learn more than individual j. The reason is that individual i is not only confident about his initial guess π^i, but also expects to *learn more* from the sequence of signals than individual j, because he believes that individual j has the "wrong model of the world." The fact that an individual may expect to learn more than others will play an important role in some of the applications in Section 4.

2.3 Nonmonotonicity of the Likelihood Ratio

We next illustrate that the asymptotic likelihood ratio, $R^i(\rho)$, may be non-monotone, meaning that when an individual observes a high frequency of signals taking the value a, he may conclude that the signals are biased towards a and may put lower probability on state A than he would have done with a lower frequency of a among the signals. This feature not only illustrates the types of behavior that are possible when individuals are learning under uncertainty but is also important for the applications we discuss in Section 4.

Inspection of expression (3) establishes the following:

Lemma 2 *For any $s \in \bar{S}$, $\phi^i_\infty(s)$ is decreasing at $\rho(s)$ if and only if R^i is increasing at $\rho(s)$.*

Proof. This follows immediately from equation (3) above. ■

When R^i is non-monotone, even a small amount of uncertainty about the informativeness may lead to significant differences in limit posteriors. The next example illustrates this point, while the second example shows that there can be "reversals" in individuals' assessments, meaning that after observing a sequence "favorable" to state A, the individual may have a lower posterior about this state than his prior. The impact of small uncertainty on asymptotic learning and agreement will be more systematically studied in the next subsection.

Example 1 ***(Nonmonotonicity)*** Each individual i thinks that with probability $1-\epsilon$, p_A and p_B are in a δ-neighborhood of some $\hat{p}^i > (1+\delta)/2$, but with probability $\epsilon > 0$, the signals are not informative. More precisely, for $\hat{p}^i > (1+\delta)/2$, $\epsilon > 0$ and $\delta < \left|\hat{p}^1 - \hat{p}^2\right|$, we have

$$f_\theta^i(p) = \begin{cases} \epsilon + (1-\epsilon)/\delta & \textit{if } p \in \left(\hat{p}^i - \delta/2, \hat{p}^i + \delta/2\right) \\ \epsilon & \textit{otherwise} \end{cases} \tag{6}$$

for each θ and i. Now, by (4), the asymptotic likelihood ratio is

$$R^i(\rho(s)) = \begin{cases} \frac{\epsilon\delta}{1-\epsilon(1-\delta)} & \textit{if } \rho(s) \in \left(\hat{p}^i - \delta/2, \hat{p}^i + \delta/2\right) \\ \\ \frac{1-\epsilon(1-\delta)}{\epsilon\delta} & \textit{if } \rho(s) \in \left(1-\hat{p}^i - \delta/2, 1-\hat{p}^i + \delta/2\right) \\ 1 & \textit{otherwise.} \end{cases}$$

This and other relevant functions are plotted in Figure 1 for $\epsilon \to 0$. The likelihood ratio $R^i(\rho(s))$ is 1 when $\rho(s)$ is small, takes a very high value at $1-\hat{p}^i$, goes down to 1 afterwards, becomes nearly zero around $\hat{p}^i$, and then jumps back to 1. By Lemmas 1 and 2, $\phi_\infty^i(s)$ will also be non-monotone: when $\rho(s)$ is small, the signals are not informative, thus $\phi_\infty^i(s)$ is the same as the prior, π^i. In contrast, around $1-\hat{p}^i$, the signals become very informative suggesting that the state is B, thus $\phi_\infty^i(s) \cong 0$. After this point, the signals become uninformative again and $\phi_\infty^i(s)$ goes back to π^i. Around $\hat{p}^i$, the signals are again informative, but this time favoring state A, so $\phi_\infty^i(s) \cong 1$. Finally, signals again become uninformative and $\phi_\infty^i(s)$ falls back to π^i.

Intuitively, when $\rho(s)$ is around $1-\hat{p}^i$ or $\hat{p}^i$, the individual assigns very high probability to the true state, but outside of this region, he sticks to his prior, concluding that the signals are not informative. However, he also understands that since $\delta < \left|\hat{p}^1 - \hat{p}^2\right|$, when the long-run frequency in a region where he learns that $\theta = A$, the other individual will conclude that the signals are uninformative and adhere to his prior belief; conversely, when the other individual learns, he will view the signals as uninformative. Consequently, he knows that the posterior beliefs of the other individual will always be far from his. This can be seen from the third panel of Figure 1; at each sample path in $\bar{S}$, at least one of the individuals will fail to learn,

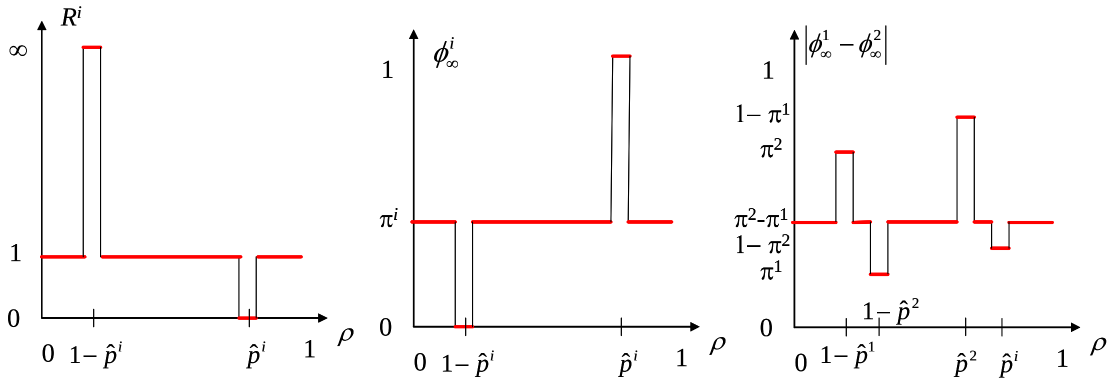

Figure 1: The three panels show, respectively, the approximate values of $R^i(\rho)$, ϕ^i_∞, and $\left|\phi^1_\infty - \phi^2_\infty\right|$ as $\epsilon \to 0$.

and the difference between their limiting posteriors will be uniformly higher than the following lower bound

$$\min\left\{\pi^1, \pi^2, 1-\pi^1, 1-\pi^2, \left|\pi^1 - \pi^2\right|\right\}.$$

When $\pi^1 = 1/3$ and $\pi^2 = 2/3$, this bound is equal to $1/3$.[9]

The next example shows an even more extreme phenomenon, whereby a high frequency of $s = a$ among the signals may reduce the individual's posterior that $\theta = A$ below his prior.

Example 2 ***(Reversal)*** Now suppose that individuals' subjective probability densities are given by

$$f^i_\theta(p) = \begin{cases} \left(1-\epsilon-\epsilon^2\right)/\delta & \text{if } \hat{p}^i - \delta/2 \le p \le \hat{p}^i + \delta/2 \\ \epsilon & \text{if } p < 1/2 \\ \epsilon^2 & \text{otherwise} \end{cases}$$

for each θ and $i = 1, 2$, where $\epsilon > 0$, $\hat{p}^i > 1/2$, and $0 < \delta < \hat{p}^1 - \hat{p}^2$. Clearly, as $\epsilon \to 0$, (4) gives:

$$R^i(\rho(s)) \cong \begin{cases} 0 & \begin{array}{l} \text{if } \rho(s) < 1 - \hat{p}^i - \delta/2, \\ \text{or } 1 - \hat{p}^i + \delta/2 < \rho(s) < 1/2, \\ \text{or } \hat{p}^i - \delta/2 \le \rho(s) \le \hat{p}^i + \delta/2 \end{array} \\ \infty & \text{otherwise.} \end{cases}$$

[9] In fact, since each agent believes that he will learn but the other agent will not, their expected difference in limit posteriors will be even higher: for each i, $\Pr^i\left(\lim_{n\to\infty}\left|\phi^1_n(s) - \phi^2_n(s)\right| \ge Z\right) \ge 1-\epsilon$, where $Z \to \min\left\{\pi^1, \pi^2, 1-\pi^1, 1-\pi^2\right\}$. This bound can be as high as $1/2$.

Hence, the asymptotic posterior probability that $\theta = A$ is

$$\phi^i_\infty(\rho(s)) \cong \begin{cases} 1 & \begin{array}{l} \text{if } \rho(s) < 1 - \hat{p}^i - \delta/2, \\ \text{or } 1 - \hat{p}^i + \delta/2 < \rho(s) < 1/2, \\ \text{or } \hat{p}^i - \delta/2 \leq \rho(s) \leq \hat{p}^i + \delta/2 \end{array} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, in this case observing a sufficiently high frequency of $s = a$ may reduce the posterior that $\theta = A$ below the prior. Moreover, the individuals assign probability $1 - \epsilon$ that there will be extreme asymptotic disagreement in the sense that $\left|\phi^1_\infty(\rho(s)) - \phi^2_\infty(\rho(s))\right| \cong 1$.

In both examples, it is crucial that the likelihood ratio R^i is not monotone. If R^i were monotone, at least one of the individuals would expect that their beliefs will asymptotically agree. To see this, take $\hat{p}^i \geq \hat{p}^j$. Now, i is almost certain that, when the state is A, $\rho(s)$ will be close to $\hat{p}^i$. He also understands that j would assign a very high probability to the event that $\theta = A$ when $\rho(s) = \hat{p}^j \geq \hat{p}^i$. If R^j were monotone, she would assign even higher probability to A at $\rho(s) = \hat{p}^i$ and thus her probability assessment on A would also converge to 1 as $\epsilon \to 0$. Therefore, in this case i will be almost certain that j will learn the true state and that their beliefs will agree asymptotically.

Theorem 1 shows that there will be asymptotic agreement under certainty. One might have thought that as $\epsilon \to 0$ and uncertainty disappears, the same conclusion would apply. In contrast, the above examples show that even as each F^i_θ converges to a Dirac distribution (that assigns a unit mass to a point), there may be significant asymptotic disagreement between the two individuals. Notably this is true not only when there is negligible uncertainty, i.e., $\epsilon \to 0$ and $\delta \to 0$, but also when the individuals' subjective distributions are nearly identical, i.e., as $\hat{p}^1 - \hat{p}^2 \to 0$. This suggests that the result of asymptotic agreement in Theorem 1 may not be a continuous limit point of a more general model of learning under uncertainty. However, it is also not the case that asymptotic agreement under approximate certainty requires the support of the distribution of each F^i_θ to converge to a set as in Theorem 2. Instead, we will see in the next subsection that whether or not there is asymptotic agreement under approximate certainty (i.e., as F^i_θ becomes more and more concentrated around a point) is determined by the tail properties of the family of distributions F^i_θ.

2.4 Agreement and Disagreement with Approximate Certainty

In this subsection, we characterize the conditions under which "approximate certainty" ensures asymptotic agreement. More specifically, we will study the behavior of asymptotic beliefs as the subjective probability distribution F_θ^i converges to a Dirac distribution and the uncertainty about the interpretation of the signals disappears. We will demonstrate that whether or not there is asymptotic agreement in the limit depends on the family of distributions converging to certainty—in particular, on their tail properties. For many natural distributions, a small amount of uncertainty about informativeness of the signals is sufficient to lead to significant differences in posteriors.

To state and prove our main result in this case, consider a *family* of subjective probability density functions $f_{\theta,m}^i$ for $i = 1, 2$, $\theta \in \{A, B\}$ and $m \in \mathbb{Z}_+$, such that as $m \to \infty$, we have that $F_{\theta,m}^i \to F_{\theta,\infty}^i$ where $F_{\theta,\infty}^i$ assigns probability 1 to $p = \hat{p}^i$ for some $\hat{p}^i \in (1/2, 1)$. In particular, we consider the following families: take a *determining* density function f, which will parameterize $\left\{ f_{\theta,m}^i \right\}$. We impose the following conditions on f:

(i) f is symmetric around zero;

(ii) there exists $\bar{x} < \infty$ such that $f(x)$ is decreasing for all $x \geq \bar{x}$;

(iii)
$$\tilde{R}(x, y) \equiv \lim_{m\to\infty} \frac{f(mx)}{f(my)} \tag{7}$$
exists in $[0, \infty]$ at all $(x, y) \in \mathbb{R}_+^2$.[10]

In order to vary the amount of uncertainty, we consider mappings of the form $x \mapsto (x - y)/m$, which scale down the real line around y by the factor $1/m$. The family of subjective densities for individuals' beliefs about p_A and p_B, $\left\{ f_{\theta,m}^i \right\}$, will be determined by f and the transformation $x \mapsto \left(x - \hat{p}^i\right)/m$. In particular, we consider the following family of densities

$$f_{\theta,m}^i(p) = c^i(m) f\left(m\left(p - \hat{p}^i\right)\right) \tag{8}$$

for each θ and i where $c^i(m) \equiv 1/\int_0^1 f\left(m\left(p - \hat{p}^i\right)\right) dp$ is a correction factor to ensure that $f_{\theta,m}^i$ is a proper probability density function on $[0, 1]$ for each m. We also define $\phi_{\infty,m}^i \equiv$

[10] Convergence will be uniform in most cases in view of the results discussed following Definition 1 below (and Egorov's Theorem, which links pointwise convergence of a family of functions to a limiting function to uniform convergence, see, for example, Billingsley, 1995, Section 13).

$\lim_{n\to\infty}\phi^i_{n,m}(s)$ as the limiting posterior distribution of individual i when he believes that the probability density of signals is $f^i_{\theta,m}$. In this family of subjective densities, the uncertainty about p_A is scaled down by $1/m$, and $f^i_{\theta,m}$ converges to unit mass at $\hat{p}^i$ as $m\to\infty$, so that individual i becomes sure about the informativeness of the signals in the limit. In other words, as $m\to\infty$, this family of subjective probability distributions leads to approximate certainty.

The next theorem characterizes the class of determining functions f for which the resulting family of the subjective densities $\left\{f^i_{\theta,m}\right\}$ leads to asymptotic learning and agreement under approximate certainty.

Theorem 5 *Suppose that Assumption 1 holds. For each $i=1,2$, consider the family of subjective densities $\left\{f^i_{\theta,m}\right\}$ defined in (8) for some $\hat{p}^i>1/2$, with f satisfying conditions (i)-(iii) above. Suppose that $f(mx)/f(my)$ uniformly converges to $\tilde{R}(x,y)$ over a neighborhood of $\left(\hat{p}^1+\hat{p}^2-1,\left|\hat{p}^1-\hat{p}^2\right|\right)$. Then,*

1. $\lim_{m\to\infty}\left(\phi^i_{\infty,m}\left(\hat{p}^i\right)-\phi^j_{\infty,m}\left(\hat{p}^i\right)\right)=0$ *if and only if* $\tilde{R}\left(\hat{p}^1+\hat{p}^2-1,\left|\hat{p}^1-\hat{p}^2\right|\right)=0$.
2. *Suppose that $\tilde{R}\left(\hat{p}^1+\hat{p}^2-1,\left|\hat{p}^1-\hat{p}^2\right|\right)=0$. Then for every $\epsilon>0$ and $\delta>0$, there exists $\bar{m}\in\mathbb{Z}_+$ such that*
$$\Pr{}^i\left(\lim_{n\to\infty}\left|\phi^1_{n,m}(s)-\phi^2_{n,m}(s)\right|>\epsilon\right)<\delta \qquad (\forall m>\bar{m}, i=1,2).$$
3. *Suppose that $\tilde{R}\left(\hat{p}^1+\hat{p}^2-1,\left|\hat{p}^1-\hat{p}^2\right|\right)\neq 0$. Then there exists $\epsilon>0$ such that for each $\delta>0$, there exists $\bar{m}\in\mathbb{Z}_+$ such that:*
$$\Pr{}^i\left(\lim_{n\to\infty}\left|\phi^1_{n,m}(s)-\phi^2_{n,m}(s)\right|>\epsilon\right)>1-\delta \qquad (\forall m>\bar{m}, i=1,2).$$

Proof. (Proof of Part 1) Let $R^i_m(\pi)$ be the asymptotic likelihood ratio as defined in (4) associated with subjective density $f^i_{\theta,m}$. One can easily check that $\lim_{m\to\infty}R^i_m\left(\hat{p}^i\right)=0$. Hence, by (5), $\lim_{m\to\infty}\left(\phi^i_{\infty,m}\left(\hat{p}^i\right)-\phi^j_{\infty,m}\left(\hat{p}^i\right)\right)=0$ if and only if $\lim_{m\to\infty}R^j_m\left(\hat{p}^i\right)=0$. By definition, we have:

$$\begin{aligned}\lim_{m\to\infty}R^j_m\left(\hat{p}^i\right) &= \lim_{m\to\infty}\frac{f\left(m\left(1-\hat{p}^1-\hat{p}^2\right)\right)}{f\left(m\left(\hat{p}^1-\hat{p}^2\right)\right)}\\ &= \tilde{R}\left(1-\hat{p}^1-\hat{p}^2,\hat{p}^1-\hat{p}^2\right)\\ &= \tilde{R}\left(\hat{p}^1+\hat{p}^2-1,\left|\hat{p}^1-\hat{p}^2\right|\right),\end{aligned}$$

where the last equality follows by condition (i), the symmetry of the function f. This establishes that $\lim_{m\to\infty} R^i_m\left(\hat{p}^i\right) = 0$ (and thus $\lim_{m\to\infty}\left(\phi^i_{\infty,m}\left(\hat{p}^i\right) - \phi^j_{\infty,m}\left(\hat{p}^i\right)\right) = 0$) if and only if $\tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right) = 0$.

(Proof of Part 2) Take any $\epsilon > 0$ and $\delta > 0$, and assume that $\tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right) = 0$. By Lemma 1, there exists $\epsilon' > 0$ such that $\phi^i_{\infty,m}\left(\rho\left(s\right)\right) > 1-\epsilon$ whenever $R^i\left(\rho\left(s\right)\right) < \epsilon'$. There also exists x_0 such that

$$\Pr^i\left(\rho\left(s\right) \in \left(\hat{p}^i - x_0/m, \hat{p}^i + x_0/m\right) | \theta = A\right) = \int_{-x_0}^{x_0} f\left(x\right) dx > 1 - \delta. \tag{9}$$

Let $\kappa = \min_{x\in[-x_0,x_0]} f\left(x\right) > 0$. Since f monotonically decreases to zero in the tails (see (ii) above), there exists x_1 such that $f\left(x\right) < \epsilon'\kappa$ whenever $|x| > |x_1|$. Let $m_1 = \left(x_0 + x_1\right)/\left(2\hat{p}^i - 1\right) > 0$. Then, for any $m > m_1$ and $\rho\left(s\right) \in \left(\hat{p}^i - x_0/m, \hat{p}^i + x_0/m\right)$, we have $\left|\rho\left(s\right) - 1 + \hat{p}^i\right| > x_1/m$, and hence

$$R^i_m\left(\rho\left(s\right)\right) = \frac{f\left(m\left(\rho\left(s\right) + \hat{p}^i - 1\right)\right)}{f\left(m\left(\rho\left(s\right) - \hat{p}^i\right)\right)} < \frac{\epsilon'\kappa}{\kappa} = \epsilon'.$$

Therefore, for all $m > m_1$ and $\rho\left(s\right) \in \left(\hat{p}^i - x_0/m, \hat{p}^i + x_0/m\right)$, we have that

$$\phi^i_{\infty,m}\left(\rho\left(s\right)\right) > 1 - \epsilon. \tag{10}$$

Again, by Lemma 1, there exists $\epsilon'' > 0$ such that $\phi^j_{\infty,m}\left(\rho\left(s\right)\right) > 1-\epsilon$ whenever $R^j_m\left(\rho\left(s\right)\right) < \epsilon''$. Now, for each $\rho\left(s\right)$,

$$\lim_{m\to\infty} R^j_m\left(\rho\left(s\right)\right) = \tilde{R}\left(\rho\left(s\right) + \hat{p}^j - 1, \left|\rho\left(s\right) - \hat{p}^j\right|\right). \tag{11}$$

Moreover, by the uniform convergence assumption, there exists $\eta > 0$ such that $R^j_m\left(\rho\left(s\right)\right)$ uniformly converges to $\tilde{R}\left(\rho\left(s\right) + \hat{p}^j - 1, \left|\rho\left(s\right) - \hat{p}^j\right|\right)$ on $\left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$ and

$$\tilde{R}\left(\rho\left(s\right) + \hat{p}^j - 1, \left|\rho\left(s\right) - \hat{p}^j\right|\right) < \epsilon''/2$$

for each $\rho\left(s\right)$ in $\left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$. (By uniform convergence, at $\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right)$, $\tilde{R}$ is continuous and takes value of 0—by assumption.) Hence, there exists $m_2 < \infty$ such that for all $m > m_2$ and $\rho\left(s\right) \in \left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$,

$$R^j_m\left(\rho\left(s\right)\right) < \tilde{R}\left(\rho\left(s\right) + \hat{p}^j - 1, \left|\rho\left(s\right) - \hat{p}^j\right|\right) + \epsilon''/2 < \epsilon''.$$

Therefore, for all $m > m_2$ and $\rho\left(s\right) \in \left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$, we have

$$\phi^j_{\infty,m}\left(\rho\left(s\right)\right) > 1 - \epsilon. \tag{12}$$

Set $\bar{m} \equiv \max\{m_1, m_2, \eta/x_0\}$. Then, by (10) and (12), for any $m > \bar{m}$ and $\rho(s) \in (\hat{p}^i - x_0/m, \hat{p}^i + x_0/m)$, we have $\left|\phi^i_{\infty,m}(\rho(s)) - \phi^j_{\infty,m}(\rho(s))\right| < \epsilon$. Then, (9) implies that $\Pr^i\left(\left|\phi^i_{\infty,m}(\rho(s)) - \phi^j_{\infty,m}(\rho(s))\right| < \epsilon | \theta = A\right) > 1 - \delta$. By the symmetry of A and B, this establishes that $\Pr^i\left(|\phi^i_{\infty,m}(\rho(s)) - \phi^j_{\infty,m}(\rho(s))| < \epsilon\right) > 1 - \delta$ for $m > \bar{m}$.

(Proof of Part 3) Since $\lim_{m\to\infty} R^j_m\left(\hat{p}^i\right) = \tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right)$ is assumed to be strictly positive, $\lim_{m\to\infty} \phi^j_{\infty,m}\left(\hat{p}^i\right) < 1$. We set $\epsilon = \left(1 - \lim_{m\to\infty} \phi^j_{\infty,m}\left(\hat{p}^i\right)\right)/2$ and use similar arguments to those in the proof of Part 2 to obtain the desired conclusion. ■

Theorem 5 provides a complete characterization of the conditions under which approximate certainty will lead to asymptotic agreement. In particular, it shows that approximate certainty may not be enough to guarantee asymptotic learning and agreement. This contrasts with the result in Theorems 1 that there will always be asymptotic learning and agreement under full certainty. Theorem 5, instead, shows that even a small amount of uncertainty may be sufficient to cause absence of learning and disagreement between the individuals.

The first part of the theorem provides a simple condition on the tail of the distribution f that determines whether the asymptotic difference between the posteriors is small under approximate uncertainty. This condition can be expressed as:

$$\tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right) \equiv \lim_{m\to\infty} \frac{f\left(m\left(\hat{p}^1 + \hat{p}^2 - 1\right)\right)}{f\left(m\left(\hat{p}^1 - \hat{p}^2\right)\right)} = 0. \tag{13}$$

The theorem shows that if this condition is satisfied, then as uncertainty about the informativeness of the signals disappears the difference between the posteriors of the two individuals will become negligible. Notice that condition (13) is symmetric and does not depend on i.

Parts 2 and 3 of the theorem then exploit this result and the continuity of $\tilde{R}$ to show that the individuals will attach probability 1 to the event that the asymptotic difference between their beliefs will disappear when (13) holds, and they will attach probability 1 to asymptotic disagreement when (13) fails to hold. Thus the behavior of asymptotic beliefs under approximate certainty are completely determined by condition (13).

It is also informative to understand for which classes of determining distributions f condition (13) holds. Clearly, this will depend on the tail behavior of f, which, in turn, determines the behavior of the family of subjective densities $\left\{f^i_{\theta,m}\right\}$. Suppose $x \equiv \hat{p}^1 + \hat{p}^2 - 1 > \hat{p}^1 - \hat{p}^2 \equiv y > 0$. Then, condition (13) can be expressed as

$$\lim_{m\to\infty} \frac{f(mx)}{f(my)} = 0.$$

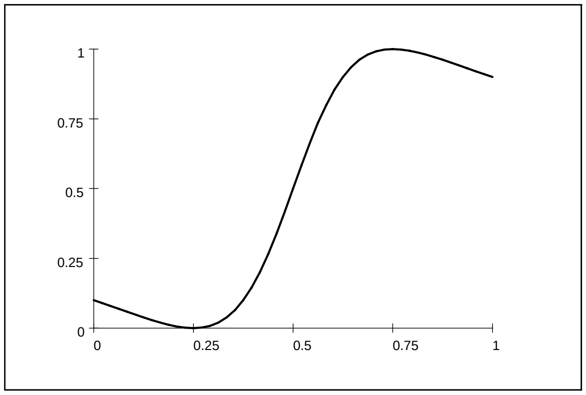

Figure 2: $\lim_{n\to\infty} \phi_n^i(s)$ for Pareto distribution as a function of $\rho(s)$ [$\alpha = 2$, $\hat{p}^i = 3/4$.]

This condition holds for distributions with exponential tails, such as the exponential or the normal distributions. On the other hand, it fails for distributions with polynomial tails. For example, consider the Pareto distribution, where $f(x)$ is proportional to $|x|^{-\alpha}$ for some $\alpha > 1$. Then, for each m,

$$\frac{f(mx)}{f(my)} = \left(\frac{x}{y}\right)^{-\alpha} > 0.$$

This implies that for the Pareto distribution, individuals' beliefs will fail to converge even when there is a negligible amount of uncertainty. In fact, for this distribution, the asymptotic beliefs will be independent of m (since R_m^i does not depend on m). If we take $\pi^1 = \pi^2 = 1/2$, then the asymptotic posterior probability of $\theta = A$ according to i is

$$\phi_{\infty,m}^i(\rho(s)) = \frac{\left(\rho(s) - \hat{p}^i\right)^{-\alpha}}{\left(\rho(s) - \hat{p}^i\right)^{-\alpha} + \left(\rho(s) + \hat{p}^i - 1\right)^{-\alpha}}$$

for any m.

As illustrated in Figure 2, in this case $\phi_{\infty,m}^i$ is not monotone. To see the magnitude of asymptotic disagreement, consider $\rho(s) \cong \hat{p}^i$. In that case, $\phi_{\infty,m}^i(\rho(s))$ is approximately 1, and $\phi_{\infty,m}^j(\rho(s))$ is approximately $y^{-\alpha}/\left(x^{-\alpha} + y^{-\alpha}\right)$. Hence, both individuals believe that the difference between their asymptotic posteriors will be

$$\left|\phi_{\infty,m}^1 - \phi_{\infty,m}^2\right| \cong \frac{x^{-\alpha}}{x^{-\alpha} + y^{-\alpha}}.$$

This asymptotic difference is increasing with the difference $y \equiv \hat{p}^1 - \hat{p}^2$, which corresponds to the difference in the individuals' views on which frequencies of signals are most likely. It is also clear from this expression that this asymptotic difference will converge to zero as $y \to 0$ (i.e., as $\hat{p}^1 \to \hat{p}^2$). This last statement is indeed generally true when $\tilde{R}$ is continuous:

Proposition 1 *In Theorem 5, in addition, assume that $\tilde{R}$ is continuous on $D = \{(x, y) \mid -1 \le x \le 1, |y| \le \bar{y}\}$ for some $\bar{y} > 0$. Then for every $\epsilon > 0$ and $\delta > 0$, there exist $\lambda > 0$ and $\bar{m} \in (0, \infty)$ such that whenever $|\hat{p}^1 - \hat{p}^2| < \lambda$,*

$$\Pr^i \left(\lim_{n\to\infty} \left| \phi^1_{n,m} - \phi^2_{n,m} \right| > \epsilon \right) < \delta \qquad (\forall m > \bar{m}, i = 1, 2).$$

Proof. To prove this proposition, we modify the proof of Part 2 of Theorem 5 and use the notation in that proof. Since $\tilde{R}$ is continuous on the compact set D and $\tilde{R}(x, 0) = 0$ for each x, there exists $\lambda > 0$ such that $\tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right) < \epsilon''/4$ whenever $\left|\hat{p}^1 - \hat{p}^2\right| < \lambda$. Fix any such $\hat{p}^1$ and $\hat{p}^2$. Then, by the uniform convergence assumption, there exists $\eta > 0$ such that $R^j_m(\rho(s))$ uniformly converges to $\tilde{R}\left(\rho(s) + \hat{p}^j - 1, \left|\rho(s) - \hat{p}^j\right|\right)$ on $\left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$ and

$$\tilde{R}\left(\rho(s) + \hat{p}^j - 1, \left|\rho(s) - \hat{p}^j\right|\right) < \epsilon''/2$$

for each $\rho(s)$ in $\left(\hat{p}^i - \eta, \hat{p}^i + \eta\right)$. The rest of the proof is identical to the proof of Part 2 in Theorem 5. ■

This proposition implies that if the individuals are almost certain about the informativeness of signals, then any significant difference in their asymptotic beliefs must be due to a significant difference in their subjective densities regarding the signal distribution (i.e., it must be the case that $\left|\hat{p}^1 - \hat{p}^2\right|$ is not small). In particular, the continuity of $\tilde{R}$ in Proposition 1 implies that when $\hat{p}^1 = \hat{p}^2$, we must have $\tilde{R}\left(\hat{p}^1 + \hat{p}^2 - 1, \left|\hat{p}^1 - \hat{p}^2\right|\right) = 0$, and thus, from Theorem 5, there will be no significant differences in asymptotic beliefs. Notably, however, the requirement that $\hat{p}^1 = \hat{p}^2$ is rather strong. For example, Theorem 1 established that under certainty there will be asymptotic learning and agreement for all $\hat{p}^1, \hat{p}^2 > 1/2$.

It is also worth noting that the assumption that $\tilde{R}$ or $\lim_{m\to 0} R^i_m(\rho)$ is continuous in the relevant range is important for the results in Proposition 1. In particular, recall that Example 1 illustrated a situation in which this assumption failed and the asymptotic differences remained bounded away from zero, irrespective of the gap between $\hat{p}^1$ and $\hat{p}^2$.

We next focus on the case where $\hat{p}^1 \ne \hat{p}^2$ and provide a further characterization of which classes of determining functions lead to asymptotic agreement under approximate certainty. We first define:

Definition 1 *A density function f has* regularly-varying tails *if it has unbounded support and satisfies*

$$\lim_{m\to\infty} \frac{f(mx)}{f(m)} = H(x) \in \mathbb{R}$$

for any $x > 0$*.*

The condition in Definition 1 that $H(x) \in \mathbb{R}$ is relatively weak, but nevertheless has important implications. In particular, it implies that $H(x) \equiv x^{-\alpha}$ for $\alpha \in (0, \infty)$. This follows from the fact that in the limit, the function $H(\cdot)$ must be a solution to the functional equation $H(x)H(y) = H(xy)$, which is only possible if $H(x) \equiv x^{-\alpha}$ for $\alpha \in (0, \infty)$.[11] Moreover, Seneta (1976) shows that the convergence in Definition 1 holds locally uniformly, i.e., uniformly for x in any compact subset of $(0, \infty)$. This implies that if a density f has regularly-varying tails, then the assumptions imposed in Theorem 5 (in particular, the uniform convergence assumption) are satisfied. In fact, we have that, in this case, $\tilde{R}$ defined in (7) is given by the same expression as for the Pareto distribution,

$$\tilde{R}(x, y) = \left(\frac{x}{y}\right)^{-\alpha},$$

and is everywhere continuous. As this expression suggests, densities with regularly-varying tails behave approximately like power functions in the tails; indeed a density $f(x)$ with regularly-varying tails can be written as $f(x) = \mathcal{L}(x)x^{-\alpha}$ for some *slowly-varying* function $\mathcal{L}$ (with $\lim_{m\to\infty} \mathcal{L}(mx)/\mathcal{L}(m) = 1$). Many common distributions, including the Pareto, log-normal, and t-distributions, have regularly-varying densities. We also define:

Definition 2 *A density function* f *has* rapidly-varying tails *if it satisfies*

$$\lim_{m\to\infty} \frac{f(mx)}{f(m)} = x^{-\infty} \equiv \begin{cases} 0 & if \quad x > 1 \\ 1 & if \quad x = 1 \\ \infty & if \quad x < 1 \end{cases}$$

for any $x > 0$*.*

As in Definition 1, the above convergence holds locally uniformly (uniformly in x over any compact subset that excludes 1). Examples of densities with rapidly-varying tails include the exponential and the normal densities.

From these definitions, the following corollary to Theorem 5 is immediate and links asymptotic agreement under approximate certainty to the tail behavior of the determining density function.

[11] To see this, note that since $\lim_{m\to\infty} (f(mx)/f(m)) = H(x) \in \mathbb{R}$, we have

$$H(xy) = \lim_{m\to\infty} \left(\frac{f(mxy)}{f(m)}\right) = \lim_{m\to\infty} \left(\frac{f(mxy)}{f(my)} \frac{f(my)}{f(m)}\right) = H(x) H(y).$$

See de Haan (1970) or Feller (1971).

Corollary 1 *Suppose that Assumption 1 holds and $\hat{p}^1 \neq \hat{p}^2$.*

1. *Suppose that in Theorem 5 f has regularly-varying tails. Then there exists $\epsilon > 0$ such that for each $\delta > 0$, there exists $\bar{m} \in \mathbb{Z}_+$ such that*

$$\Pr^i \left(\lim_{n\to\infty} \left| \phi^1_{n,m}(s) - \phi^2_{n,m}(s) \right| > \epsilon \right) > 1 - \delta \qquad (\forall m > \bar{m}, i = 1, 2).$$

2. *Suppose that in Theorem 5 f has rapidly-varying tails. Then for every $\epsilon > 0$ and $\delta > 0$, there exists $\bar{m} \in \mathbb{Z}_+$ such that*

$$\Pr^i \left(\lim_{n\to\infty} \left| \phi^1_{n,m}(s) - \phi^2_{n,m}(s) \right| > \epsilon \right) < \delta \qquad (\forall m > \bar{m}, i = 1, 2).$$

This corollary therefore implies that whether there will be asymptotic learning and agreement depends on whether the family of subjective densities converging to "certainty" has regularly or rapidly-varying tails (provided that $\hat{p}^1 \neq \hat{p}^2$).

3 Generalizations

The previous section provided our main results in an environment with two states and two signals. In this section, we show that our main results generalize to an environment with $K \geq 2$ states and $L \geq K$ signals. The main results parallel those of Section 2, and all the proofs for this section are contained in the Appendix.

To generalize our results to this environment, let $\theta \in \Theta$, where $\Theta \equiv \left\{A^1, ..., A^K\right\}$ is a set containing $K \geq 2$ distinct elements. We refer to a generic element of the set by A^k. Similarly, let $s_t \in \left\{a^1, ..., a^L\right\}$, with $L \geq K$ signal values. As before, define $s \equiv \{s_t\}_{t=1}^{\infty}$, and for each $l = 1, ..., L$, let

$$r^l_n(s) \equiv \#\left\{t \leq n | s_t = a^l\right\}$$

be the number of times the signal $s_t = a^l$ out of first n signals. Once again, the strong law of large numbers implies that, according to both individuals, for each $l = 1, ..., L$, $r^l_n(s)/n$ almost surely converges to some $\rho^l(s) \in [0, 1]$ with $\sum_{l=1}^{L} \rho^l(s) = 1$. Define $\rho(s) \in \Delta(L)$ as the vector $\rho(s) \equiv \left(\rho^1(s), ..., \rho^L(s)\right)$, where $\Delta(L) \equiv \left\{p = \left(p^1, \ldots, p^L\right) \in [0, 1]^L \,:\, \sum_{l=1}^{L} p^l = 1\right\}$, and let the set $\bar{S}$ be

$$\bar{S} \equiv \left\{s \in S : \lim_{n\to\infty} r^l_n(s)/n \text{ exists for each } l = 1, ..., L\right\}. \qquad (14)$$

With analogy to the two-state-two-signal model in Section 2, let $\pi_k^i > 0$ be the prior probability individual i assigns to $\theta = A^k$, $\pi^i \equiv \left(\pi_1^i, ..., \pi_K^i\right)$, and p_θ^l be the frequency of observing signal $s = a^l$ when the true state is θ. When players are certain about p_θ^l's as in usual models, immediate generalizations of Theorems 1 and 2 apply. With analogy to before, we define F_θ^i as the *joint subjective probability distribution* of conditional frequencies $p \equiv \left(p_\theta^1, ..., p^L\right)$ according to individual i. Since our focus is learning under uncertainty, we impose an assumption similar to Assumption 1.

Assumption 2 *For each i and θ, the distribution F_θ^i over $\Delta(L)$ has a continuous, non-zero and finite density f_θ^i over $\Delta(L)$.*

We also define $\phi_{k,n}^i(s) \equiv \Pr^i\left(\theta = A^k \mid \{s_t\}_{t=0}^n\right)$ for each $k = 1, ..., K$ as the posterior probability that $\theta = A^k$ after observing the sequence of signals $\{s_t\}_{t=0}^n$, and

$$\phi_{k,\infty}^i(\rho(s)) \equiv \lim_{n\to\infty} \phi_{k,n}^i(s).$$

Given this structure, it is straightforward to generalize the results in Section 2. Let us now define the transformation $T_k : \mathbb{R}_+^K \to \mathbb{R}_+^{K-1}$, such that

$$T_k(x) = \left(\frac{x_{k'}}{x_k}; k' \in \{1, ..., K\} \setminus k\right).$$

Here $T_k(x)$ is taken as a column vector. This transformation will play a useful role in the theorems and the proofs. In particular, this transformation will be applied to the vector π^i of priors to determine the ratio of priors assigned the different states by individual i. Let us also define the norm $\|x\| = \max_l |x|^l$ for $x = \left(x^1, \ldots, x^L\right) \in \mathbb{R}^L$.

The next lemma generalizes Lemma 1:

Lemma 3 *Suppose Assumption 2 holds. Then for all $s \in \bar{S}$,*

$$\phi_{k,\infty}^i(\rho(s)) = \frac{1}{1 + \frac{\sum_{k' \neq k} \pi_{k'}^i f_{A^{k'}}^i(\rho(s))}{\pi_k^i f_{A^k}^i(\rho(s))}}. \tag{15}$$

Our first theorem in this section parallels Theorem 3 and shows that under Assumption 2 there will be lack of asymptotic learning, and under a relatively weak additional condition, there will also asymptotic disagreement.

Theorem 6 *Suppose Assumption 2 holds for $i = 1{,}2$, then for each $k = 1, ..., K$, and for each $i = 1{,}2$,*

1. $\Pr^i\left(\phi^i_{k,\infty}(\rho(s)) \neq 1 | \theta = A^k\right) = 1$, *and*

2. $\Pr^i\left(\left|\phi^1_{k,\infty}(\rho(s)) - \phi^2_{k,\infty}(\rho(s))\right| \neq 0\right) = 1$ *whenever* $\Pr^i((T_k(\pi^1) - T_k(\pi^2))' T_k(f^i(\rho(s)) = 0) = 0$ *and* $F^1_\theta = F^2_\theta$ *for each* $\theta \in \Theta$.

The additional condition in part 2 of Theorem 6, that $\Pr^i((T_k(\pi^1) - T_k(\pi^2))' T_k(f^i(\rho(s)) = 0) = 0$, plays the role of differences in priors in Theorem 3 (here " $'$ " denotes the transpose of the vector in question). In particular, if this condition did not hold, then at some $\rho(s)$, the relative asymptotic likelihood of some states could be the same according to two individuals with different priors and they would interpret at least some sequences of signals in a similar manner and achieve asymptotic agreement. It is important to note that the condition that $\Pr^i((T_k(\pi^1) - T_k(\pi^2))' T_k(f^i(\rho(s)) = 0) = 0$ is relatively weak and holds generically—i.e., if it did not hold, a small perturbation of π^1 or π^2 would restore it.[12] The Part 2 of Theorem 6 therefore implies that asymptotic disagreement occurs *generically.*

The next theorem shows that small differences in priors can again widen after observing the same sequence of signals.

Theorem 7 *Under Assumption 2, assume* $\mathbf{1}'\left(T_k\left(\left(f^1_\theta(\rho)\right)_{\theta\in\Theta}\right) - T_k\left(\left(f^2_\theta(\rho)\right)_{\theta\in\Theta}\right)\right) \neq 0$ *for each* $\rho \in [0,1]$, *each* $k = 1, ..., K$, *where* $\mathbf{1} \equiv (1, ..., 1)'$. *Then, there exists an open set of prior vectors* π^1 *and* π^2, *such that*

$$\left|\phi^1_{k,\infty}(\rho(s)) - \phi^2_{k,\infty}(\rho(s))\right| > \left|\pi^1_k - \pi^2_k\right| \text{ for each } k = 1, ..., K \text{ and } s \in \bar{S}$$

and

$$\Pr^i\left(\left|\phi^1_{k,\infty}(\rho(s)) - \phi^2_{k,\infty}(\rho(s))\right| > \left|\pi^1_k - \pi^2_k\right|\right) = 1 \text{ for each } k = 1, ..., K.$$

The condition $\mathbf{1}'\left(T_k\left(\left(f^1_\theta(\rho)\right)_{\theta\in\Theta}\right) - T_k\left(\left(f^2_\theta(\rho)\right)_{\theta\in\Theta}\right)\right) \neq 0$ is similar to the additional condition in part 2 of Theorem 6, and as with that condition, it is relatively weak and holds generically. Finally, the following theorem generalizes Theorem 5. The appropriate construction of the families of probability densities is also provided in the theorem.

[12]More formally, the set of solutions $\mathcal{S} \equiv \{(\pi^1, \pi^2, \rho) \in \Delta(L)^2 : (T_k(\pi^1) - T_k(\pi^2))' T_k(f^i(\rho)) = 0\}$ has Lebesgue measure 0. This is a consequence of the Preimage Theorem and Sard's Theorem in differential topology (see, for example, Guillemin and Pollack, 1974, pp. 21 and 39). The Preimage Theorem implies that if y is a regular value of a map $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of X with dimension equal to $\dim X - \dim Y$. In our context, this implies that if 0 is a regular value of the map $(T_k(\pi^1) - T_k(\pi^2))' T_k(f^i(\rho))$, then the set $\mathcal{S}$ is a two dimensional submanifold of $\Delta(L)^3$ and thus has Lebesgue measure 0. Sard's theorem implies that 0 is generically a regular value.

Theorem 8 *Suppose that Assumption 2 holds. For each* $\theta \in \Theta$ *and* $m \in \mathbb{Z}_+$*, define the subjective density* $f^i_{\theta,m}$ *by*

$$f^i_{\theta,m}(p) = c(i,\theta,m) f(m(p - \hat{p}(i,\theta))) \tag{16}$$

where $c(i,\theta,m) \equiv 1/\int_{p\in\Delta(L)} f(m(p-\hat{p}(i,\theta)))\,dp$*,* $\hat{p}(i,\theta) \in \Delta(L)$ *with* $\hat{p}(i,\theta) \neq \hat{p}(i,\theta')$ *whenever* $\theta \neq \theta'$*, and* $f : \mathbb{R}^L \to \mathbb{R}$ *is a positive, continuous probability density function that satisfies the following conditions:*

(i) $\lim_{h\to\infty} \max_{\{x:\|x\|\geq h\}} f(x) = 0$,

(ii)

$$\tilde{R}(x,y) \equiv \lim_{m\to\infty} \frac{f(mx)}{f(my)} \tag{17}$$

exists at all x, y*, and*

(iii) convergence in (17) holds uniformly over a neighborhood of each $(\hat{p}(i,\theta) - \hat{p}(j,\theta'), \hat{p}(i,\theta) - \hat{p}(j,\theta))$.

Also let $\phi^i_{k,\infty,m}(\rho(s)) \equiv \lim_{n\to\infty} \phi^i_{k,n,m}(s)$ *be the asymptotic posterior of individual* i *with subjective density* $f^i_{\theta,m}$*. Then,*

1. $\lim_{m\to\infty}\left(\phi^i_{k,\infty,m}(\hat{p}(i,A^k)) - \phi^j_{k,\infty,m}(\hat{p}(i,A^k))\right) = 0$ *if and only if* $\tilde{R}\left(\hat{p}(i,A^k) - \hat{p}\left(j,A^{k'}\right), \hat{p}(i,A^k) - \hat{p}(j,A^k)\right) = 0$ *for each* $k' \neq k$.

2. *Suppose that* $\tilde{R}(\hat{p}(i,\theta) - \hat{p}(j,\theta'), \hat{p}(i,\theta) - \hat{p}(j,\theta)) = 0$ *for each distinct* θ *and* θ'*. Then for every* $\epsilon > 0$ *and* $\delta > 0$*, there exists* $\bar{m} \in \mathbb{Z}_+$ *such that*

$$\Pr{}^i\left(\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| > \epsilon\right) < \delta \qquad (\forall m > \bar{m}, i = 1,2).$$

3. *Suppose that* $\tilde{R}(\hat{p}(i,\theta) - \hat{p}(j,\theta'), \hat{p}(i,\theta) - \hat{p}(j,\theta)) \neq 0$ *for each distinct* θ *and* θ'*. Then there exists* $\epsilon > 0$ *such that for each* $\delta > 0$*, there exists* $\bar{m} \in \mathbb{Z}_+$ *such that*

$$\Pr{}^i\left(\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| > \epsilon\right) > 1 - \delta \qquad (\forall m > \bar{m}, i = 1,2).$$

These theorems therefore show that the results about lack of asymptotic learning and asymptotic agreement derived in the previous section do not depend on the assumption that there are only two states and binary signals. It is also straightforward to generalize Proposition 1 and Corollary 1 to the case with multiple states and signals; we omit this to avoid repetition.

The results in this section are stated for the case in which both the number of signal values and states are finite. They can also be generalized to the case of a continuum of signal values and states, but this introduces a range of technical issues that are not central to our focus here.

4 Applications

In this section we discuss a number of applications of the results derived so far. The applications are chosen to show various different economic consequences from learning and disagreement under uncertainty. Throughout, we strive to choose the simplest examples. The first example illustrates how learning under uncertainty can overturn some simple insights from basic game theory. The second example shows how such learning can act as an equilibrium selection device as in Carlsson and van Damme (1993). The third example is the most substantial application and shows how learning under uncertainty affects speculative asset trading. The fourth example illustrates how learning under uncertainty can affect the timing of agreement in bargaining. Finally, the last example shows how a special case of our model of learning under uncertainty can arise when there is information transmission by a potentially biased media outlet.[13]

4.1 Value of Information in Common-Interest Games

Consider a common-interest game in which the players have identical payoff functions. Typically in common interest games information is valuable in the sense that with more information about underlying parameters, the value of the game in the best equilibrium will be higher. Consequently, we would expect players to collect or at least wait for the arrival of additional

[13]In this section, except for the example on equilibrium selection and the last example of the game of belief manipulation, we will study complete-information games with possibly non-common priors. Formaly, information and belief structure in these games can be described as follows. Fix the state space $\Omega = \Theta \times \bar{S}$, and for each $n < \infty$ consider the information partition $I^n = \left\{ I^n(s) = \{(\theta, s') \,|\, s'_t = s_t \forall t \leq n\} \,|\, s \in \bar{S} \right\}$ that is common for both players. For $n = \infty$, we introduce the common information partition $I^\infty = \left\{ I^\infty(s) = \Theta \times \{s\} \,|\, s \in \bar{S} \right\}$. At each $I^n(s)$, player $i = 1, 2$ assigns probability $\phi^i_n(s)$ to the state $\theta = A$ and probability $1 - \phi^i_n(s)$ to the sate $\theta = B$. Since the players have a common partition at each s and n, their beliefs are common knowledge. Notice that, under certainty, $\phi^1_\infty(s) = \phi^2_\infty(s) \in \{0, 1\}$, so that after observing s, both players assign probability 1 to the same θ. In that case, there will be *common certainty* of θ, or loosely speaking, θ becomes "common knowledge." This is not necessarily the case under uncertainty.

information before playing such games. In contrast, we now show that when there is learning under uncertainty, additional information can be harmful in common-interest games, and thus the agents may prefer to play the game *before* additional information arrives.

To illustrate these issues, consider the payoff matrix

	α	β
α	θ,θ	$1/2,1/2$
β	$1/2,1/2$	$1,1$

where $\theta \in \{0,2\}$, and the agents have a common prior on θ according to which probability of $\theta = 2$ is $\pi \in (1/2, 1)$. When there is no information, there are two equilibria in pure strategies: (α,α)—the good equilibrium—and (β,β)—the bad equilibrium. The good equilibrium here is both Pareto- and risk-dominant, and hence, it is plausible to presume that the players will indeed choose to play this good equilibrium. In this equilibrium, each player would receive θ, with expected payoff of $2\pi > 1$.

First, consider the implications of learning under certainty. Suppose that the agents are allowed to observe an infinite sequence of signals $s = \{s_t\}_{t=1}^{\infty}$, where each agent thinks that $\Pr^i(s_t = \theta|\theta) = p^i > 1/2$. Theorem 1 then implies that after observing the signal, the agents will learn θ. If the frequency $\rho(s)$ of signal with $s_t = 2$ is greater than $1/2$, they will learn that $\theta = 2$; otherwise they will learn that $\theta = 0$. If $\rho(s) \leq 1/2$, β strictly dominates α, and hence (β,β) is the only equilibrium. If $\rho(s) > 1/2$, as before, we have a good equilibrium (α,α), which is Pareto- and risk-dominant, and a bad equilibrium (β,β). Assuming that they will also play the good equilibrium in this game, we can conclude that information benefits both agents; they will choose the best strategy profile at each state and each will receive a payoff of $\max\{\theta,1\}$ or an expected payoff of $2\pi + (1-\pi)$. Consequently, in this case we would expect the players to wait for the arrival of public information before playing the game.

Let us next turn to learning under uncertainty. In particular, suppose that the agents do not know the signal distribution and their subjective densities are similar to those in Example 2:

$$f_\theta^i(p) = \begin{cases} \left(1-\epsilon-\epsilon^2\right)/\delta & \text{if } \hat{p}^i - \delta/2 \leq p \leq \hat{p}^i + \delta/2 \\ \epsilon & \text{if } p < 1/2 \\ \epsilon^2 & \text{otherwise} \end{cases} \tag{18}$$

for each θ, where $0 < \delta < \hat{p}^1 - \hat{p}^2$ and ϵ is taken to be arbitrarily small. Given these subjective densities, we will see that according to both agents, with probability greater than $1-\epsilon$, β will be the *unique rationalizable* action, yielding the low payoff of 1. Hence, as $\epsilon \to 0$, the arrival of public information will decrease each agent's payoff to 1. Consequently, both agents would prefer to play the game before the information arrives.[14]

To show this, recall from Example 2 that when $\epsilon \cong 0$ (i.e., when $\epsilon \to 0$), the asymptotic posterior probability of $\theta = 2$ is

$$\phi^i_\infty(\rho(s)) \cong \begin{cases} 1 & \begin{array}{l} \text{if } \rho(s) < 1 - \hat{p}^i - \delta/2, \\ \text{or } 1 - \hat{p}^i + \delta/2 < \rho(s) < 1/2, \\ \text{or } \hat{p}^i - \delta/2 \le \rho(s) \le \hat{p}^i + \delta/2, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for any $\rho(s) > 1/2$, at least one of the agents will assign posterior probability $\phi^i_\infty(\rho(s)) \cong 0$ to the event that $\theta = 2$, and therefore, for this agent, β will strictly dominate α. This implies that (β, β) must be the unique rationalizable action profile. When $\rho(s) \in \left(1 - \hat{p}^i - \delta/2, 1 - \hat{p}^i + \delta/2\right)$, agent i assigns probability $\phi^i_\infty(\rho(s)) \cong 0$ to $\theta = 2$, and again (β, β) is the unique rationalizable action profile for any such $\rho(s)$. The probability of the remaining set of frequencies is less than $1 - \epsilon$ according to both agents. This implies that each agent (correctly) expects that if they wait for the arrival of public information, their payoff will be approximately 1. He would therefore prefer to play the game before the arrival of the public information.

4.2 Selection in Coordination Games

The initial difference in players' beliefs about the signal distribution need not be due to lack of common prior; it may be due to private information. Building on an example by Carlsson and van Damme (1993), we now illustrate that when the players are uncertain about the signal distribution, small differences in beliefs, combined with learning, may have a significant effect on the outcome of the game and may select one of the multiple equilibria of the game.

Consider a game with the payoff matrix

	I	N
I	θ, θ	$\theta - 1, 0$
N	$0, \theta - 1$	$0, 0$

where $\theta \sim \mathcal{N}(0, 1)$. The players observe an infinite sequence of public signals $s \equiv \{s_t\}_{t=0}^\infty$, where $s_t \in \{0, 1\}$ and

$$\Pr(s_t = 1|\theta) = 1/\left(1 + \exp\left(-(\theta + \eta)\right)\right), \tag{19}$$

[14]Throughout the section we use "approximately" interchangeably with "as $\epsilon \to 0$" or "as $\epsilon \cong 0$".

with $\eta \sim \mathcal{N}(0,1)$. In addition, each player observes a *private* signal

$$x_i = \eta + u_i$$

where u_i is uniformly distributed on $[-\epsilon/2, \epsilon/2]$ for some small $\epsilon > 0$.

Let us define $\kappa \equiv \log(\rho(s)) - \log(1-\rho(s))$. Equation (19) implies that after observing s, the players infer that $\theta + \eta = \kappa$. For small ϵ, conditional on x_i, η is distributed approximately uniformly on $[x_i - \epsilon/2, x_i + \epsilon/2]$ (see Carlsson and van Damme, 1993). This implies that conditional on x_i and s, θ is approximately uniformly distributed on $[\kappa - x_i - \epsilon/2, \kappa - x_i + \epsilon/2]$. Now note that with the reverse order on x_i, the game is supermodular. Therefore, there exist extremal rationalizable strategy profiles, which also constitute monotone, symmetric Bayesian Nash Equilibria. In each equilibrium, there is a cutoff value, x^*, such that the equilibrium action is I if $x_i < x^*$ and N if $x_i > x^*$. This cutoff, x^*, is defined such that player i is indifferent between the two actions, i.e.,

$$\kappa - x^* = \Pr(x_j > x^* | x_i = x^*) = 1/2 + O(\epsilon),$$

where $O(\epsilon)$ is such that $\lim_{\epsilon \to 0} O(\epsilon) = 0$. This establishes that

$$x^* = \kappa - 1/2 - O(\epsilon).$$

Therefore, when ϵ is small, the game is dominance solvable, and each player i plays I if $x_i < \kappa - 1/2$ and N if $x_i > \kappa + 1/2$.

In this game, learning under certainty has very different implications from those above. Suppose instead that the players knew the conditional signal distribution (i.e., they knew η), so that we are in a world of learning under certainty. Then after s is observed, θ would become common knowledge, and there would be multiple equilibria whenever $\theta \in (0,1)$. This example therefore illustrates how learning under uncertainty can lead to the selection of one of the equilibria in a coordination game.

4.3 A Simple Model of Asset Trade

One of the most interesting applications of the ideas developed here is to models of asset trading. Models of assets trading with different priors have been studied by, among others, Harrison and Kreps (1978) and Morris (1996). These works assume different priors about the dividend process and allow for learning under certainty. They establish the possibility of "speculative asset trading". We now investigate the implications of learning under uncertainty for the pattern of speculative asset trading.

Consider an asset that pays 1 if the state is A and 0 if the state is B. Assume that Agent 2 owns the asset, but Agent 1 may wish to buy it. We have two dates, $\tau = 0$ and $\tau = 1$, and the agents observe a sequence of signals between these dates. For simplicity, we again take this to be an infinite sequence $s \equiv \{s_t\}_{t=1}^{\infty}$. We also simplify this example by assuming that Agent 1 has all the bargaining power: at either date, if he wants to buy the asset, Agent 1 makes a take-it-or-leave-it price offer P_τ, and trade occurs at price P_τ if Agent 2 accepts the offer. Assume also that $\pi^1 > \pi^2$, so that Agent 1 is more optimistic. This assumption ensures that Agent 1 would like to purchase the asset. We are interested in subgame-perfect equilibrium of this game.

Let us start with the case in which there is learning under certainty. Suppose that each agent is certain that $p_A = p_B = p^i$ for some number $p^i > 1/2$. In that case, from Theorem 1, both agents recognize at $\tau = 0$ that at $\tau = 1$, for each $\rho(s)$, the value of the asset will the same for both of them: it will be worth 1 if $\rho(s) > 1/2$ and 0 if $\rho(s) < 1/2$. Hence, at $\tau = 1$ the agents will be indifferent between trading the asset (at price $P_1 = \phi^1_\infty(\rho(s)) = \phi^2_\infty(\rho(s))$) at each history $\rho(s)$. Therefore, if trade does not occur at $\tau = 0$, the continuation value of Agent 1 is 0, and the continuation value of Agent 2 is π^2. If they trade at price P_0, then the continuation value of agents 1 and 2 will be $\pi^1 - P_0$ and P_0, respectively. This implies that at date 0, Agent 2 accepts an offer if and only if $P_0 \geq \pi^2$. Since $\pi^1 > \pi^2$, Agent 1 is happy to offer the price $P_0 = \pi^2$ at date $\tau = 0$ and trade takes place. Therefore, with learning under certainty, there will be immediate trade at $\tau = 0$.

We next turn to the case of learning under uncertainty and suppose that the agents do not know p_A and p_B. Unlike with learning under certainty, the agents have a strong incentive to delay trading. To illustrate this, we first consider a simple example where subjective densities are as in Example 1, with $\epsilon \to 0$. Now, at date 1, if $\hat{p}^1 - \delta/2 < \rho(s) < \hat{p}^1 + \delta/2$, then the value of the asset for Agent 2 is $\phi^2_\infty(\rho(s)) = \pi^2$, and the value of the asset for Agent 1 is approximately 1. Hence, at such $\rho(s)$, Agent 1 buys the asset from Agent 2 at price $P_1(\rho(s)) = \pi^2$, enjoying gains from trade equal to $1 - \pi^2$. Since the equilibrium payoff of Agent 1 must be non-negative in all other contingencies, this shows that when they do not trade at date 0, his continuation value is at least

$$\pi^1 \left(1 - \pi^2\right)$$

(when $\epsilon \to 0$). The continuation value of Agent 2 must be at least π^2, as he has the option of never selling his asset. Therefore, they can trade at date 0 only if the total payoff from trading, which is π^1, exceeds the sum of these continuation values, $\pi^1 \left(1 - \pi^2\right) + \pi^2$. Since this is impossible, there will be no trade at $\tau = 0$. Instead, Agent 1 will wait for the information to buy the asset at date 1 (provided that $\rho(s)$ turns out to be in a range where he concludes that the asset pays 1).

This example exploits the general intuition discussed after Theorem 4: if the agents are uncertain about the informativeness of the signals, each agent may expect to *learn more* from the signals than the other agent. In fact, this example has the extreme feature whereby each agent believes that he will definitely learn the true state, but the other agent will fail to do so. This induces the agents to wait for the arrival of additional information before trading. This contrasts with the intuition that observation of common information should take agents towards common beliefs and make trades less likely. This intuition is correct in models of learning under certainty and is the reason why previous models have generated speculative trade at the beginning (Harrison and Kreps, 1978, and Morris, 1996). Instead, here there is delayed speculative trading.

The next result characterizes the conditions for delayed asset trading more generally:

Proposition 2 *In any subgame-perfect equilibrium, trade is delayed to $\tau = 1$ if and only if*

$$\mathbb{E}^2 \left[\phi^2_\infty\right] = \pi^2 > \mathbb{E}^1 \left[\min \left\{\phi^1_\infty, \phi^2_\infty\right\}\right].$$

That is, when $\pi^2 > \mathbb{E}^1 \left[\min \left\{\phi^1_\infty, \phi^2_\infty\right\}\right]$, Agent 1 does not buy at $\tau = 0$ and buys at $\tau = 1$ if $\phi^1_\infty(\rho(s)) > \phi^2_\infty(\rho(s))$; when $\pi^2 < \mathbb{E}^1 \left[\min \left\{\phi^1_\infty, \phi^2_\infty\right\}\right]$, Agent 1 buys at $\tau = 0$.

Proof. In any subgame-perfect equilibrium, Agent 2 is indifferent between trading and not, and hence his valuation of the asset is $\Pr^2(\theta = A|\text{Information})$. Therefore, trade at $\tau = 0$ can take place at the price $P_0 = \pi^2$, while trade at $\tau = 1$ will be at the price $P_1(\rho(s)) = \phi^2_\infty(\rho(s))$. At date 1, Agent 1 buys the asset if and only if $\phi^1_\infty(\rho(s)) \geq \phi^2_\infty(\rho(s))$, yielding the payoff of $\max\left\{\phi^1_\infty(\rho(s)) - \phi^2_\infty(\rho(s)), 0\right\}$. This implies that Agent 1 is willing to buy at $\tau = 0$ if and only if

$$\begin{aligned}\pi^1 - \pi^2 &\geq \mathbb{E}^1\left[\max\left\{\phi^1_\infty(\rho(s)) - \phi^2_\infty(\rho(s)), 0\right\}\right] \\ &= \mathbb{E}^1\left[\phi^1_\infty(\rho(s)) - \min\left\{\phi^1_\infty(\rho(s)), \phi^2_\infty(\rho(s))\right\}\right] \\ &= \pi^1 - \mathbb{E}^1\left[\min\left\{\phi^1_\infty(\rho(s)), \phi^2_\infty(\rho(s))\right\}\right],\end{aligned}$$

as claimed. ■

Since $\pi^1 = \mathbb{E}^1\left[\phi^1_\infty\right] \geq \mathbb{E}^1\left[\min\left\{\phi^1_\infty, \phi^2_\infty\right\}\right]$, this result provides a cutoff value for the initial difference in beliefs, $\pi^1 - \pi^2$, in terms of the differences in the agents' interpretation of the signals. The cutoff value is $\mathbb{E}^1\left[\max\left\{\phi^1_\infty(\rho(s)) - \phi^2_\infty(\rho(s)), 0\right\}\right]$. If the initial difference is lower than this value, then agents will wait until $\tau = 1$ to trade; otherwise, they will trade immediately. Consistent with the above example, delay in trading becomes more likely when the agents interpret the signals more differently, which is evident from the expression for the cutoff value. This reasoning also suggests that if $F^1_\theta = F^2_\theta$ for each θ (so that the agents interpret the signals in a similar fashion),[15] then trade should occur immediately. The next lemma shows that each agent believes that additional information will bring the other agent's expectations closer to his own and will be used to prove that $F^1_\theta = F^2_\theta$ indeed implies immediate trading.

Lemma 4 *If $\pi^1 > \pi^2$ and $F^1_\theta = F^2_\theta$ for each θ, then*

$$\mathbb{E}^1\left[\phi^2_\infty\right] \geq \pi^2.$$

Proof. Recall that ex ante expectation of individual i regarding ϕ^j_∞ can be written as

$$\begin{aligned}\mathbb{E}^i\left[\phi^j_\infty\right] &= \int_0^1 \left[\pi^i f^i_A(\rho)\,\phi^j_\infty(\rho) + \left(1-\pi^i\right) f^i_B(1-\rho)\,\phi^j_\infty(\rho)\right] d\rho \qquad (20) \\ &= \int_0^1 \frac{\pi^i f_A(\rho) + \left(1-\pi^i\right) f_B(1-\rho)}{\pi^j f_A(\rho) + \left(1-\pi^j\right) f_B(1-\rho)} f_A(\rho)\, d\rho,\end{aligned}$$

where the first line uses the definition of ex ante expectation under the probability measure $\Pr^i$, while the second line exploits equations (3) and (4) and the fact that since $F^1_\theta = F^2_\theta$, $f^1_\theta(\rho) = f^2_\theta(\rho) = f_\theta(\rho)$ for all ρ. Now define

$$I(\pi) \equiv \int_0^1 \frac{\pi f_A(\rho) + (1-\pi) f_B(1-\rho)}{\pi^2 f_A(\rho) + \left(1-\pi^2\right) f_B(1-\rho)} f_A(\rho)\, d\rho.$$

[15] Recall from Theorem 3 that even when $F^1_\theta = F^2_\theta$, agents interpret signals differently because $\pi^1 \neq \pi^2$.

From (20), $\mathbb{E}^1\left[\phi^2_\infty\right] = I\left(\pi^1\right)$ and $\pi^2 = \mathbb{E}^2\left[\phi^2_\infty\right] = I\left(\pi^2\right)$. Hence, it suffices to show that I is increasing in π. Now,

$$I'(\pi) = \int_0^1 \frac{f_A(\rho)}{\pi^2 f_A(\rho) + (1-\pi^2) f_B(1-\rho)} \left(f_A(\rho) - f_B(1-\rho)\right) d\rho.$$

Moreover, $f_A(\rho) / \left[\pi^2 f_A(\rho) + \left(1-\pi^2\right) f_B(1-\rho)\right] \geq 1$ if and only if $f_A(\rho) \geq f_B(1-\rho)$. Hence,

$$\begin{aligned} I'(\pi) &= \int_{f_A \geq f_B} \frac{f_A(\rho)}{\pi^2 f_A(\rho) + (1-\pi^2) f_B(1-\rho)} \left(f_A(\rho) - f_B(1-\rho)\right) d\rho \\ &\quad - \int_{f_A < f_B} \frac{f_A(\rho)}{\pi^2 f_A(\rho) + (1-\pi^2) f_B(1-\rho)} \left(f_B(1-\rho) - f_A(\rho)\right) d\rho \\ &\geq \int_{f_A \geq f_B} \left(f_A(\rho) - f_B(1-\rho)\right) d\rho - \int_{f_A < f_B} \left(f_B(1-\rho) - f_A(\rho)\right) d\rho \\ &= \int_0^1 \left(f_A(\rho) - f_B(1-\rho)\right) d\rho = 0. \end{aligned}$$

■

Together with the previous proposition, this lemma yields the following result establishing that delay in asset trading can only occur when subjective probability distributions differ across individuals.

Proposition 3 *If $F^1_\theta = F^2_\theta$ for each θ, then in any subgame-perfect equilibrium, trade occurs at $\tau = 0$.*

Proof. Since $\pi^1 > \pi^2$ and $R^1 = R^2$, Lemma 1 implies that $\phi^1_\infty(\rho(s)) \geq \phi^2_\infty(\rho(s))$ for each $\rho(s)$. Then, $\mathbb{E}^1\left[\min\left\{\phi^1_\infty, \phi^2_\infty\right\}\right] = \mathbb{E}^1\left[\phi^2_\infty\right] \geq \pi^2$, where the last inequality is by Lemma 4. Therefore, by Proposition 2, Agent 1 buys at $\tau = 0$. ■

This proposition establishes that when the two agents have the same subjective probability distributions, there will be no delay in trading. However, as the example above illustrates, when $F^1_\theta \neq F^2_\theta$, delayed speculative trading is possible. The intuition is given by Lemma 4: when agents have the same subjective probability distribution but different priors, each will believe that additional information will bring the other agent's beliefs closer to his own. This leads to early trading. However, when the agents differ in terms of their subjective probability distributions, they expect to learn more from new information (because, as discussed after Theorem 4 above, they believe that they have the "correct model of the world"). Consequently, they delay trading.

Learning under uncertainty does not necessarily lead to additional delay in economic transactions, however. Whether it does so or not depends on the effect of the extent of disagreement on the timing of economic transactions. We will next see that, in the context of bargaining, the presence of learning under uncertainty may be a force towards immediate agreement rather than delay.

4.4 Bargaining With Outside Options

Consider two agents bargaining over the division of a dollar. There are two dates, $\tau \in \{0, 1\}$, and Agent 2 has an outside option $\theta \in \{\theta_L, \theta_H\}$ that expires at the end of date 1, where $\theta_L < \theta_H < 1$ and the value of θ is initially unknown. Between the two dates, the agents observe an infinite sequence of public signals $s \equiv \{s_t\}_{t=1}^{\infty}$ with $s_t \in \{a_L, a_H\}$, where the signal a_L can be thought to be more likely under θ_L.

Bargaining follows a simple protocol: at each date τ, Agent 1 offers a share w_τ to Agent 2. If Agent 2 accepts the offer, the game ends, Agent 2 receives the proposal, w_τ, and Agent 1 receives the remaining $1 - w_\tau$. If Agent 2 rejects the offer, she decides whether to take her outside option, terminating the game, or wait for the next stage of the game. We assume that delay is costly, so that if negotiations continue until date $\tau = 1$, Agent 1 incurs a cost $c > 0$.

Finally, as in Yildiz (2003), the agents are assumed to be "optimistic," in the sense that

$$y \equiv \mathbb{E}^2[\theta] - \mathbb{E}^1[\theta] > 0.$$

In other words, they differ in their expectations of θ on the outside option of Agent 2—with Agent 2 believing that her outside option is higher than Agent 1's assessment of this outside option—and y parameterizes the extent of optimism in this game.

We assume that the game form and beliefs are common knowledge and look for the subgame-perfect equilibrium of this simple bargaining game.

By backward induction, at date $\tau = 1$, for any $\rho(s)$, the value of outside option for Agent 1 is $\mathbb{E}^2[\theta|\rho(s)] < 1$, and hence she accepts an offer w_1 if and only if $w_1 \geq \mathbb{E}^2[\theta|\rho(s)]$. Agent 2 therefore offers $w_1 = \mathbb{E}^2[\theta|\rho(s)]$. If there is no agreement at date 0, the continuation values of the two agents are:

$$V^1 = 1 - c - \mathbb{E}^1\left[\mathbb{E}^2[\theta|\rho(s)]\right] \quad \text{and} \quad V^2 = \mathbb{E}^2\left[\mathbb{E}^2[\theta|\rho(s)]\right] = \mathbb{E}^2[\theta],$$

which uses the fact that there is no cost of delay for Agent 2. Since they have 1 dollar in total, the agents will delay the agreement to date $\tau = 1$ if and only if

$$\mathbb{E}^2[\theta] - \mathbb{E}^1\left[\mathbb{E}^2[\theta|\rho(s)]\right] > c.$$

Here, $\mathbb{E}^1\left[\mathbb{E}^2[\theta|\rho(s)]\right]$ is Agent 1's expectation about how Agent 2 will update her beliefs after observing the signals s. If Agent 1 expects that the information will reduce Agent 2's expectation of her outside option more than the cost of waiting, then Agent 1 is willing to wait. This description makes it clear that whether there will be agreement at date $\tau = 0$ depends on Agent 1's assessment of how Agent 2 will interpret the (public) signals.

When each agent is certain about the informativeness of the signals, they agree ex ante that they will interpret the information correctly. Consequently, as in Lemma 4, Agent 1's Bayesian updating will indicate that the public information will reveal him to be right. Yildiz (2004) has shown that this reasoning gives Agent 1 an incentive to "wait to persuade" Agent 2 that her outside option is relatively low. More specifically, assume that each agent i is certain that $\Pr^i(s_t = \theta|\theta) = \hat{p}^i > 1/2$ for some $\hat{p}^1$ and $\hat{p}^2$, where $\hat{p}^1$ and $\hat{p}^2$ may differ. Then, from Theorem 1, the agents agree that Agent 2 will learn her outside option, i.e., $\Pr^i\left(\mathbb{E}^2[\theta|\rho(s)] = \theta\right) = 1$ for each i. Hence, $\mathbb{E}^1\left[\mathbb{E}^2[\theta|\rho(s)]\right] = \mathbb{E}^1[\theta]$. Therefore, Agent 1 delays the agreement to date $\tau = 1$ if and only if

$$y > c,$$

i.e., if and only if the level of optimism is higher than the cost of waiting. This discussion therefore indicates that the arrival of public information can create a reason for delay in bargaining games.

We now show that when agents are uncertain about the informativeness of the signals, this motive for delay is reduced and there can be immediate agreement. Intuitively, each agent understands that the same signals will be interpreted differently by the other agent and thus expects that they are less likely to persuade the other agent. This decreases the incentives to delay agreement.

This result is illustrated starkly here, with an example where a small amount of uncertainty about the informativeness of signals removes all incentives to delay agreement. Suppose that the agents' beliefs are again as in Example 1 with ϵ small. Now Agent 1 assigns probability more than $1 - \epsilon$ to the event that that $\rho(s)$ will be either in $\left[\hat{p} - \delta/2, \hat{p}^1 + \delta/2\right]$ or in $\left[1 - \hat{p} - \delta/2, 1 - \hat{p}^1 + \delta/2\right]$, inducing Agent 2 to stick to her prior. Hence, Agent 1 expects that Agent 2 will not update her prior by much. In particular, we have

$$\mathbb{E}^1\left[\mathbb{E}^2\left[\theta|\rho\left(s\right)\right]\right]=\mathbb{E}^2\left[\theta\right]+O\left(\epsilon\right).$$

Thus

$$\mathbb{E}^2\left[\theta\right]-\mathbb{E}^1\left[\mathbb{E}^2\left[\theta|\rho\left(s\right)\right]\right]=-O\left(\epsilon\right)<c.$$

This implies that agents will agree at the beginning of the game. Therefore, the same forces that led to delayed asset trading in the previous subsection can also induce immediate agreement in bargaining when agents are "optimistic".

4.5 Manipulation and Uncertainty

Our final example is intended to show how the pattern of uncertainty used in the body of the paper can result from game theoretic interactions between an agent and an informed party, for example as in cheap talk games (Crawford and Sobel, 1982). Since our purpose is to illustrate this possibility, we choose the simplest environment to communicate these ideas and limit the discussion to the single agent setting—the generalization to the case with two or more agents is straightforward.

The environment is as follows. The state of the world is $\theta \in \{0,1\}$, and the agent starts with a prior belief $\pi \in (0,1)$ that $\theta = 1$ at $t = 0$. At time $t = 1$, this agent has to make a decision $x \in [0,1]$, and his payoff is $-(x-\theta)^2$. Thus the agent would like to form as accurate an expectation about θ as possible.

The other player is a media outlet, M, which observes a large (infinite) number of signals $s' \equiv \{s'_t\}_{t=1}^{\infty}$ with $s'_t \in \{0,1\}$, and makes a sequence of reports to the agent $s \equiv \{s_t\}_{t=1}^{\infty}$ with $s_t \in \{0,1\}$. The reports s can be thought of as contents of newspaper articles, while s' correspond to the information that the newspaper collects before writing the articles. Since s' is an exchangeable sequence, we can represent it, as before, with the fraction of signals that are 1's, denoted by $\rho' \in [0,1]$, and similarly s is represented by $\rho \in [0,1]$. This is convenient as it allows us to model the mixed strategy of the media as a mapping

$$\sigma_M : [0,1] \to \Delta\left([0,1]\right),$$

where $\Delta\left([0,1]\right)$ is the set of probability distributions on $[0,1]$. Let $\mathbf{i}$ be the strategy that puts probability 1 on the identity mapping, thus corresponding to M reporting truthfully.

Otherwise, i.e., if $\sigma_M \neq \mathbf{i}$, there is manipulation (or misreporting) on the part of the media outlet M.

We also assume for simplicity that ρ' has a continuous distribution with density g_1 when $\theta = 1$ and g_0 when $\theta = 0$, such that $g_1(\rho) = 0$ for all $\rho \leq \bar{\rho}$ and $g_1(\rho) > 0$ for all $\rho > \bar{\rho}$, while $g_0(\rho) > 0$ for all $\rho \leq \bar{\rho}$ and $g_0(\rho) = 0$ for all $\rho > \bar{\rho}$. This assumption implies that if M reports truthfully, i.e., $\sigma_M = \mathbf{i}$, then Theorem 2 applies and there will be asymptotic learning (and also asymptotic agreement when there are more than one agent).

Now suppose instead that there are three different types of player M (unobservable to the agent). With probability $\lambda_H \in (0,1)$, the media is honest and can only play $\sigma_M^H = \mathbf{i}$ (where the superscript is for type H—honest). With probability $\lambda_\alpha \in (0, 1-\lambda_H)$, the media outlet is of type α and is biased towards 1. Type α media outlet receives utility equal to x irrespective of ρ', and hence would like to manipulate the agent to choose high values of x. With the complementary probability $\lambda_\beta = 1 - \lambda_\alpha - \lambda_H$, the media outlet is of type β and is biased towards 0, and receives utility equal to $1 - x$.

Let us now look for the perfect Bayesian equilibrium of the game between the media outlet and the agent. The perfect Bayesian equilibrium can be represented by two reporting functions $\sigma_M^\alpha : [0,1] \to \Delta([0,1])$ and $\sigma_M^\beta : [0,1] \to \Delta([0,1])$ for the two biased types of M, and updating function $\phi : [0,1] \to [0,1]$, which determines the belief of the agent that $\theta = 1$ when the sequence of reports is ρ, and an action function $x : [0,1] \to [0,1]$, which determines the choice of the agent as a function of ρ (there is no loss of generality here in restricting to pure strategies).

In equilibrium, x must be optimal for the agent given ϕ; ϕ must be derived from Bayes rule given σ_M^α, σ_M^β and the prior π; and σ_M^α and σ_M^β must be optimal for the two biased media outlets given x.

Note first that since the payoff to the biased media outlet does not depend on the true ρ', without loss of generality, we can restrict σ_M^α and σ_M^β not to depend on ρ'. Then, with a slight abuse of notation, let $\sigma_M^\alpha(\rho)$ and $\sigma_M^\beta(\rho)$ be the respective densities with which these two types report ρ.

Second, the optimal choice of the agent after observing a sequence of signals with fraction ρ being equal to 1 is

$$x(\rho) = \phi(\rho),$$

for all $\rho \in [0,1]$, i.e., the agent will choose an action equal to his belief $\phi(\rho)$.

Third, an application of Bayes' rule implies the following belief for the agent:

$$\phi(\rho) = \begin{cases} \frac{\left(\lambda_\alpha \sigma_M^\alpha(\rho) + \lambda_\beta \sigma_M^\beta(\rho)\right)\pi}{(1-\pi)\lambda_H g_0(\rho) + \lambda_\alpha \sigma_M^\alpha(\rho) + \lambda_\beta \sigma_M^\beta(\rho)} & \text{if } \rho \leq \bar{\rho} \\ \frac{\left(\lambda_H g_1(\rho) + \lambda_\alpha \sigma_M^\alpha(\rho) + \lambda_\beta \sigma_M^\beta(\rho)\right)\pi}{\pi\lambda_H g_1(\rho) + \lambda_\alpha \sigma_M^\alpha(\rho) + \lambda_\beta \sigma_M^\beta(\rho)} & \text{if } \rho > \bar{\rho}. \end{cases} \tag{21}$$

The following lemma shows that any (perfect Bayesian) equilibrium has a very simple form:

Lemma 5 *In any equilibrium, there exist $\phi_A > \pi$ and $\phi_B < \pi$ such that $\phi(\rho) = \phi_B$ for all $\rho < \bar{\rho}$ and $\phi(\rho) = \phi_A$ for all $\rho > \bar{\rho}$.*

Proof. From (21), $\phi(\rho) < \pi$ when $\rho < \bar{\rho}$, and $\phi(\rho) > \pi$ when $\rho > \bar{\rho}$. Since the media type α maximizes $x(\rho) = \phi(\rho)$, we have $\sigma_M^\alpha(\rho) = 0$ for $\rho < \bar{\rho}$. Now suppose that the lemma is false and there exists $\rho_1, \rho_2 \leq \bar{\rho}$ such that $\phi(\rho_1) > \phi(\rho_2)$. Then we also have $\sigma_M^\beta(\rho_1) = 0$—since media type β minimizes $x(\rho) = \phi(\rho)$. But in that case, equation (21) implies that $\phi(\rho_1) = 0$, contradicting the hypothesis. Therefore, $\phi(\rho)$ is constant over $\rho \in [0, \bar{\rho})$. The proof for $\phi(\rho)$ being constant over $\rho \in (\bar{\rho}, 1]$ is analogous. ■

It follows immediately from this lemma that equilibrium beliefs will take the form given in the next proposition:

Proposition 4 *Suppose that $\rho \neq \bar{\rho}$, then the unique equilibrium actions and beliefs are:*

$$\sigma_M^\alpha(\rho) = g_1(\rho) \tag{22}$$

$$\sigma_M^\beta(\rho) = g_0(\rho) \tag{23}$$

$$x(\rho) = \phi(\rho) = \begin{cases} \frac{\lambda_\beta \pi}{(1-\pi)\lambda_H + \lambda_\beta} & \text{if } \rho < \bar{\rho} \\ \frac{\pi(\lambda_H + \lambda_\alpha)}{\pi\lambda_H + \lambda_\alpha} & \text{if } \rho > \bar{\rho}. \end{cases} \tag{24}$$

Proof. Consider the case $\rho < \bar{\rho}$. As in the proof of Lemma 5, $\sigma_M^\alpha(\rho) = 0$. Since $\phi(\rho)$ is constant over $\rho \in [0, \bar{\rho})$ (by Lemma 5), equation (21) implies that σ_M^β is proportional to g_0 on this range. Since this range is the common support of the densities σ_M^β and g_0, it must be that $\sigma_M^\beta = g_0$. Similarly, $\sigma_M^\alpha = g_1$. Substituting these equalities in (21), we obtain (24). ■

The interesting implication of this proposition is that the unique equilibrium of the game between the media outlet and the agent leads to a special case of our model of learning under uncertainty. In particular, the beliefs in (24) can be obtained by the appropriate choice of the functions $f_A(\cdot)$ and $f_B(\cdot)$ from equation (3) in Section 2. This illustrates that the type of learning under uncertainty analyzed in this paper is likely to emerge in game-theoretic situations where one of the players is trying to manipulate the beliefs of others.

5 Concluding Remarks

A key assumption of most theoretical analyses is that individuals have a "common prior," meaning that they have beliefs consistent with each other regarding the game forms, institutions, and possible distributions of payoff-relevant parameters. This presumption is often justified by the argument that sufficient common experiences and observations, either through individual observations or transmission of information from others, will eliminate disagreements, taking agents towards common priors. This presumption receives support from a number of well-known theorems in statistics and economics, for example, Savage (1954) and Blackwell and Dubins (1962).

However, existing theorems apply to environments in which learning occurs under *certainty*, that is, individuals are certain about the meaning of different signals. In many situations, individuals are not only learning about a payoff-relevant parameter but also about the interpretation of different signals. This takes us to the realm of environments where learning takes place under *uncertainty.* For example, many signals favoring a particular interpretation might make individuals suspicious that the signals come from a biased source. We show that learning in environments with uncertainty may lead to a situation in which there is lack of *full identification* (in the standard sense of the term in econometrics and statistics). In such situations, information will be useful to individuals but may not lead to full learning.

This paper investigates the conditions under which learning under uncertainty will take individuals towards common priors (or asymptotic agreement). We consider an environment in which two individuals with different priors observe the same infinite sequence of signals informative about some underlying parameter. Our environment is one of learning under uncertainty, since individuals have non-degenerate subjective probability distribution over the likelihood of different signals given the values of the parameter. We show that when subjective probability distributions of both individuals have full support, they will never agree, even after observing the same infinite sequence of signals. Perhaps even more importantly, we show that this corresponds to a result of "agreement to eventually disagree"; individuals will agree, before observing the sequence of signals, that their posteriors about the underlying parameter will not converge. This common understanding that more information may not lead to similar beliefs for agents has important implications for a variety of games and economic models. Instead, when there is no full support in subjective probably distributions, asymptotic learning and agreement may obtain.

An important implication of this analysis is that after observing the same sequence of signals, two Bayesian individuals may end up disagreeing more than they originally did. This result contrasts with the common presumption that shared information and experiences will take individuals' assessments closer to each other.

We also systematically investigate whether asymptotic agreement obtain as the amount of uncertainty in the environment diminishes (i.e., as we look at families of subjective probability distributions converging to degenerate limit distributions with all their mass at one point). We provide a complete characterization of the conditions under which this will be the case. Asymptotic disagreement may prevail even under "approximate certainty," as long as the family of subjective probability distributions converging to a degenerate distribution (and thus to an environment with certainty) has regularly-varying tails (such as for the Pareto, the log-normal or the t-distributions). In contrast, with rapidly-varying tails (such as the normal and the exponential distributions), convergence to certainty leads to asymptotic agreement.

Lack of common beliefs and common priors has important implications for economic behavior in a range of circumstances. We illustrate how the type of learning outlined in this paper interacts with economic behavior in various different situations, including games of coordination, games of common interest, bargaining, asset trading and games of communication. For example, we show that contrary to standard results, individuals may wish to play common-interest games *before* rather than after receiving more information about payoffs. Similarly, we show how the possibility of observing the same sequence of signals may lead to "speculative delay" in asset trading among individuals that start with similar beliefs. We also provide a simple example illustrating why individuals may be uncertain about informativeness of signals—the strategic behavior of other agents trying to manipulate their beliefs.

The issues raised here have important implications for statistics and econometrics as well as learning in game-theoretic situations. As noted above, the environment considered here corresponds to one in which there is lack of full identification. Nevertheless, Bayesian posteriors are well-behaved and converge to a limiting distribution. Studying the limiting properties of these posteriors more generally and how they may be used for inference in under-identified econometric models is an interesting area for research.

6 Appendix: Omitted Proofs

Proof of Theorem 1. Under the hypothesis of the theorem and with the notation in (2), we have

$$\frac{\Pr^i(r_n|\theta=B)}{\Pr^i(r_n|\theta=A)} = \frac{\left(\hat{p}^i\right)^{n-r_n}(1-\hat{p}^i)^{r_n}}{\left(\hat{p}^i\right)^{r_n}(1-\hat{p}^i)^{n-r_n}} = \left[\left(\frac{\hat{p}^i}{1-\hat{p}^i}\right)^{1-2r_n/n}\right]^n,$$

which converges to 0 or ∞ depending on $\lim_{n\to\infty} r_n/n$ is greater than 1/2 or less than 1/2. If $\lim_{n\to\infty} r_n(s)/n > 1/2$, then by (2), $\lim_{n\to\infty}\phi_n^1(s) = \lim_{n\to\infty}\phi_n^2(s) = 1$, and if $\lim_{n\to\infty} r_n(s)/n < 1/2$, then $\lim_{n\to\infty}\phi_n^1(s) = \lim_{n\to\infty}\phi_n^2(s) = 0$. Since $\lim_{n\to\infty} r_n(s)/n = 1/2$ occurs with probability zero, this shows the second part. The first part follows from the fact that, according to each i, conditional on $\theta = A$, $\lim_{n\to\infty} r_n(s)/n = \hat{p}^i > 1/2$. ■

Proof of Lemma 3. The proof is identical to that of Lemma 1. ■

Proof of Theorem 6.

(Part1) This part immediately follows from Lemma 3, as each $\pi_{k'}^i f_{A^{k'}}(\rho(s))$ is positive, and $\pi_k^i f_{A^k}(\rho(s))$ is finite.

(Part 2) Assume $F_\theta^1 = F_\theta^2$ for each $\theta \in \Theta$. Then, by Lemma 3, $\phi_{k,\infty}^1(\rho) - \phi_{k,\infty}^2(\rho) = 0$ if and only if $\left(T_k(\pi^1) - T_k(\pi^2)\right)' T_k\left(\left(f_\theta^1(\rho)\right)_{\theta\in\Theta}\right) = 0$. The latter inequality has probability 0 under both probability measures $\Pr^1$ and $\Pr^2$ by hypothesis. ■

Proof of Theorem 7. Define $\bar{\pi} = (1/K, \ldots, 1/K)$. First, take $\pi^1 = \pi^2 = \bar{\pi}$. Then,

$$\frac{\sum_{k'\neq k}\pi_{k'}^1 f_{A^{k'}}^1(\rho(s))}{\pi_k^1 f_{A^k}^1(\rho(s))} - \frac{\sum_{k'\neq k}\pi_{k'}^2 f_{A^{k'}}^1(\rho(s))}{\pi_k^2 f_{A^k}^1(\rho(s))} = \mathbf{1}'\left(T_k\left(\left(f_\theta^1(\rho(s))\right)_{\theta\in\Theta}\right) - T_k\left(\left(f_\theta^2(\rho(s))\right)_{\theta\in\Theta}\right)\right) \neq 0,$$

where $\mathbf{1} \equiv (1,\ldots,1)'$, and the inequality follows by the hypothesis of the theorem. Hence, by Lemma 3, $\left|\phi_{k,\infty}^1(\rho(s)) - \phi_{k,\infty}^2(\rho(s))\right| > 0$ for each $\rho(s) \in [0,1]$. Since $[0,1]$ is compact and $\left|\phi_{k,\infty}^1(\rho(s)) - \phi_{k,\infty}^2(\rho(s))\right|$ is continuous in $\rho(s)$, there exists $\epsilon > 0$ such that $\left|\phi_{k,\infty}^1(\rho(s)) - \phi_{k,\infty}^2(\rho(s))\right| > \epsilon$ for each $\rho(s) \in [0,1]$. Now, since $\left|\phi_{k,\infty}^1(\rho(s)) - \phi_{k,\infty}^2(\rho(s))\right|$ is continuous in π^1 and π^2, there exists a neighborhood $N(\bar{\pi})$ of $\bar{\pi}$ such that

$$\left|\phi_{k,\infty}^1(\rho(s)) - \phi_{k,\infty}^2(\rho(s))\right| > \left|\pi_k^1 - \pi_k^2\right| \text{ for each } k = 1, \ldots, K \text{ and } s \in \bar{S}$$

for all π^1, $\pi^2 \in N(\bar{\pi})$. Since $\Pr^i(\bar{S}) = 1$, the last statement in the theorem follows. ■

Proof of Theorem 8. Our proof utilizes the following two lemmas.

Lemma A.

$$\lim_{m\to\infty}\phi_{k,\infty,m}^i(p) = \frac{1}{1+\sum_{k'\neq k}\frac{\pi_{k'}^i}{\pi_k^i}\tilde{R}\left(p-\hat{p}(i,A^{k'}), p-\hat{p}(i,A^k)\right)}.$$

Proof. By condition (i), $\lim_{m\to\infty} c\left(i, A^k, m\right) = 1$ for each i and k. Hence, for every distinct k and k',

$$\lim_{m\to\infty}\frac{f_{A^{k'}}^i(p)}{f_{A^k}^i(p)} = \lim_{m\to\infty}\frac{c\left(i,A^{k'},m\right)}{c\left(i,A^k,m\right)}\lim_{m\to\infty}\frac{f\left(m\left(p-\hat{p}\left(i,A^{k'}\right)\right)\right)}{f\left(m\left(p-\hat{p}\left(i,A^k\right)\right)\right)} = \tilde{R}\left(p-\hat{p}\left(i,A^{k'}\right), p-\hat{p}\left(i,A^k\right)\right).$$

Then, Lemma A follows from Lemma 3. ■

Lemma B. *For any $\tilde{\varepsilon} > 0$ and $h > 0$, there exists $\tilde{m}$ such that for each $m > \tilde{m}$, $k \leq K$, and each $\rho(s)$ with $\left\|\rho(s) - \hat{p}\left(i, A^k\right)\right\| < h/m$,*

$$\left|\phi^i_{k,\infty,m}\left(\rho\left(s\right)\right) - \lim_{m\to\infty} \phi^i_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right)\right| < \tilde{\varepsilon}. \tag{25}$$

Proof. Since, by hypothesis, $\tilde{R}$ is continuous at each $\left(\hat{p}\left(i,\theta\right) - \hat{p}\left(j,\theta'\right), \hat{p}\left(i,\theta\right) - \hat{p}\left(j,\theta\right)\right)$, by Lemma A, there exists $h' > 0$, such that

$$\left|\lim_{m\to\infty} \phi^i_{k,\infty,m}\left(\rho\left(s\right)\right) - \lim_{m\to\infty} \phi^i_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right)\right| < \tilde{\varepsilon}/2 \tag{26}$$

and by condition (iii), there exists $\tilde{m} > h/h'$ such that

$$\left|\phi^i_{k,\infty,m}\left(\rho\left(s\right)\right) - \lim_{m\to\infty} \phi^i_{k,\infty,m}\left(\rho\left(s\right)\right)\right| < \tilde{\varepsilon}/2. \tag{27}$$

holds uniformly in $\left\|\rho\left(s\right) - \hat{p}\left(i, A^k\right)\right\| < h'$. The inequalities in (26) and (27) then imply (25). ■

(Proof of Part 1) Since $\tilde{R}\left(\hat{p}\left(i, A^k\right) - \hat{p}\left(i, A^{k'}\right), 0\right) = 0$ for each $k' \neq k$ (by condition (i)), Lemma A implies that $\lim_{m\to\infty} \phi^i_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right) = 1$. Hence, $\lim_{m\to\infty}\left(\phi^i_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right) - \phi^j_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right)\right) = 0$ if and only if $\lim_{m\to\infty} \phi^j_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right) = 1$. Since each ratio $\pi^j_{k'}/\pi^j_k$ is positive, by Lemma A, the latter holds if only if $\tilde{R}\left(\hat{p}\left(i, A^k\right) - \hat{p}\left(j, A^{k'}\right), \hat{p}\left(i, A^k\right) - \hat{p}\left(j, A^k\right)\right) = 0$ for each $k' \neq k$, establishing Part 1.

(Proof of Part 2) Fix $\epsilon > 0$ and $\delta > 0$. Fix also any i and k. Since each $\pi^j_{k'}/\pi^j_k$ is finite, by Lemma 3, there exists $\epsilon' > 0$, such that $\phi^i_{k,\infty,m}\left(\rho\left(s\right)\right) > 1 - \epsilon$ whenever $f^i_{A^{k'}}\left(\rho\left(s\right)\right)/f^i_{A^k}\left(\rho\left(s\right)\right) < \epsilon'$ holds for every $k' \neq k$. Now, by (i), there exists $h_{0,k} > 0$, such that

$$\Pr{}^i\left(\left\|\rho\left(s\right) - \hat{p}\left(i, A^k\right)\right\| \leq h_{0,k}/m | \theta = A^k\right) = \int_{\|x\| \leq h_{0,k}} f\left(x\right) dx > \left(1 - \delta\right).$$

Let

$$Q_{k,m} = \left\{p \in \Delta\left(L\right) : \left\|p - \hat{p}\left(i, A^k\right)\right\| \leq h_{0,k}/m\right\}$$

and $\kappa \equiv \min_{\|x\| \leq h_{0,k}} f\left(x\right) > 0$. By (i), there exists $h_{1,k} > 0$ such that, whenever $\|x\| > h_{1,k}$, $f\left(x\right) < \epsilon'\kappa/2$. There exists a sufficiently large constant $m_{1,k}$ such that for any $m > m_{1,k}$, $\rho\left(s\right) \in Q_{k,m}$, and any $k' \neq k$, we have $\left\|\rho\left(s\right) - \hat{p}\left(i, A^{k'}\right)\right\| > h_{1,k}/m$, and

$$\frac{f\left(m\left(\rho\left(s\right) - \hat{p}\left(i, A^{k'}\right)\right)\right)}{f\left(m\left(\rho\left(s\right) - \hat{p}\left(i, A^k\right)\right)\right)} < \frac{\epsilon'\kappa}{2}\frac{1}{\kappa} = \frac{\epsilon'}{2}.$$

Moreover, since $\lim_{m\to\infty} c\left(i, \theta, m\right) = 1$ for each i and θ, there exists $m_{2,k} > m_{1,k}$ such that $c\left(i, A^{k'}, m\right)/c\left(i, A^k, m\right) < 2$ for every $k' \neq k$ and $m > m_{2,k}$. This implies

$$f^i_{A^{k'}}\left(\rho\left(s\right)\right)/f^i_{A^k}\left(\rho\left(s\right)\right) < \epsilon',$$

establishing that

$$\phi^i_{k,\infty,m}\left(\rho\left(s\right)\right) > 1 - \epsilon. \tag{28}$$

Now, for $j \neq i$, assume that $\tilde{R}\left(\hat{p}\left(i,\theta\right) - \hat{p}\left(j,\theta'\right), \hat{p}\left(i,\theta\right) - \hat{p}\left(j,\theta\right)\right) = 0$ for each distinct θ and θ'. Then, by Lemma A, $\lim_{m\to\infty} \phi^j_{k,\infty,m}\left(\hat{p}\left(i, A^k\right)\right) = 1$, and hence by Lemma B, there exists $m_{3,k} > m_{2,k}$ such that for each $m > m_{3,k}$, $\rho\left(s\right) \in Q_{k,m}$,

$$\phi^j_{k,\infty,m}\left(\rho\left(s\right)\right) > 1 - \epsilon. \tag{29}$$

Notice that when (28) and (29) hold, we have $\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| < \epsilon$. Then, setting $\bar{m} = \max_k m_{4,k}$, we obtain the desired inequality for each $m > \bar{m}$:

$$\begin{aligned}
\Pr^i\left(\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| < \epsilon\right) &= \sum_{k\leq K} \Pr^i\left(\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| < \epsilon | \theta = A^k\right) \Pr^i\left(\theta = A^k\right) \\
&\geq \sum_{k\leq K} \Pr^i\left(\rho(s) \in Q_{k,m} | \theta = A^k\right) \Pr^i\left(\theta = A^k\right) \\
&\geq \sum_{k\leq K} (1-\delta)\pi^i_k \\
&= 1-\delta.
\end{aligned}$$

(Proof of Part 3) Assume that $\tilde{R}\left(\hat{p}(i,\theta) - \hat{p}\left(j,\theta'\right), \hat{p}(i,\theta) - \hat{p}(j,\theta)\right) \neq 0$ for each distinct θ and θ'. Then, since each $\pi^j_{k'}/\pi^j_k$ is positive, Lemma A implies that $\lim_{m\to\infty} \phi^j_{k,\infty,m}\left(\hat{p}\left(i,A^k\right)\right) < 1$ for each k. Let

$$\epsilon = \min_k \left\{1 - \lim_{m\to\infty} \phi^j_{k,\infty,m}\left(\hat{p}\left(i,A^k\right)\right)\right\}/3 > 0.$$

Then, by part 2, for each k, there exists $m_{2,k}$ such that for every $m > m_{2,k}$ and $\rho(s) \in Q_{k,m}$, we have $\phi^i_{k,\infty}(\rho(s)) > 1-\epsilon$. By Lemma B, there also exists $m_{5,k} > m_{2,k}$ such that for every $m > m_{5,k}$ and $\rho(s) \in Q_{k,m}$,

$$\phi^j_{k,\infty,m}(\rho(s)) < \lim_{m\to\infty} \phi^j_{k,\infty,m}\left(\hat{p}\left(i,A^k\right)\right) + \epsilon \leq 1 - 2\epsilon < \phi^i_{k,\infty}(\rho(s)) - \epsilon.$$

This implies that $\left\|\phi^1_{\infty,m}(\rho(s)) - \phi^2_{\infty,m}(\rho(s))\right\| > \epsilon$. Setting $\bar{m} = \max_k m_{5,k}$ and changing $\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| < \epsilon$ at the end of the proof of Part 2 to $\left\|\phi^1_{\infty,m}(s) - \phi^2_{\infty,m}(s)\right\| > \epsilon$, we obtain the desired inequality. ■

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