Agency Problems, Screening and Increasing Credit Lines. †

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ABSTRACT. We propose a model in which an optimal dynamic financing contract for a cash-constrained entrepreneur is a credit line with a growing credit limit. This simple contract, which resembles those used in practice, presents a good benchmark to understand dynamic moral hazard and adverse selection. In our setting the moral hazard problem is that the agent, who privately observes stochastic cash flows, can manipulate them using hidden savings. The adverse selection problem is that only the agent initially knows the quality of the project. It is appealing that the credit line is an incomplete contract: it does not spell out the agent's actions, but gives him full discretion to draw and deposit funds up until the credit limit. The agent has incentives to use discretion in a way that is optimal for the principal.

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1. Introduction

This paper studies optimal dynamic financing in the presence of adverse selection and moral hazard. A risk-neutral but cash-constrained agent needs funds for start-up capital and possible operating losses of a project. The adverse selection problem is that the agent is privately informed about the distribution of potential cash flows: he knows if the project is profitable or not. The moral hazard problem is that, once the project starts running, the agent privately observes the project's actual cash flows, which he can divert for consumption or manipulate using hidden savings. There is a great complexity of history-dependent contracts in our setting. Yet, we show that the optimal contract is remarkably simple: it takes the form of a credit line with a credit limit that increases over time. In this contract the agent gets full discretion to use available credit, but reaching the credit limit results in liquidation. Such a contract gives the agent incentives to draw only to cover operating losses and pay interest on outstanding balance.

We assume that the project's stochastic cash flows arrive continuously with mean $\mu > 0$ per unit of time if the project is profitable, and $\mu' < \mu$ if the project is not profitable. The cumulative cash flows are represented by a Brownian motion with a drift. We address the following problem: how an agent with a profitable project and limited funds $R_0>0$ can borrow from the principal, if the cash flows are privately observed and he can be imitated by an agent with a bad project, possibly with unlimited funds?

How does a credit line contract create appropriate incentives in this setting, and why is an appropriate credit line optimal? Consider a credit line with interest rate r, which is also the agent's rate of return on savings and his discount rate. Then the agent's payoff depends only on the timing of default and the final outstanding debt, but not on the specifics of monetary transfers between the credit line and savings. When the credit limit is constant, it is optimal to default immediately if expected cash flows μ are smaller than the interest on the full available credit. The agent with a good project strictly prefers to use all cash flows to avoid default if the credit limit is shorter than μ/r, and has just enough incentives to pay down balance using cash flows if the credit limit is μ/r. This property of a credit line with limit μ/r makes it an optimal contract under pure moral hazard, since providing with stronger incentives than necessary is costly and inefficient.

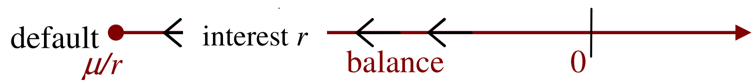

Figure 1: Credit line with limit μ/r, an optimal contract under pure moral hazard.

If the agent with a bad project got access to a credit line with limit μ/r, he would draw the entire credit and default immediately. To screen out bad projects, the principal must limit the funds available to the agent to draw, e.g. by lowering the credit limit. The *maximal amount* that the agent may be allowed to draw at any time t would make the agent with a *bad* project *just indifferent* between not financing the project and financing it to default at that moment of time t. This condition determines a credit limit that depends only on time and not the agent's actions because, as we argued earlier, the payoff of the agent (this time, with a bad project) depends only on outstanding debt at the time of default and not the movement of funds between the credit line and savings. When we superimpose this restriction on funds onto the optimal pure moral hazard contract, we get a credit line a credit that grows with time until it reaches μ/r, as shown in Figure 2. Over time the principal allows the agent to borrow more because the agent effectively 'signals' his good type by the willingness to pay interest on outstanding debt.

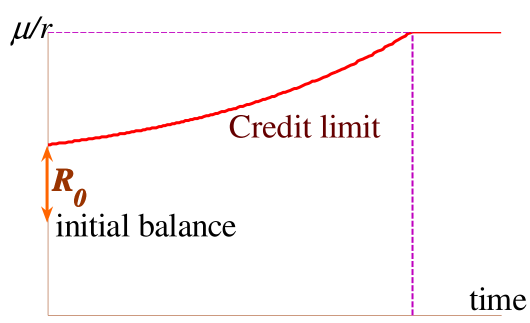

Figure 2.

It turns out that this contract is optimal, out of all possible history-dependent contracts. The formal justification of this result, which is far from straightforward, is based on two main reasons. First, this contract delays inefficient liquidation the most by making as much funds available to the agent as adverse selection allows. Second, by virtue of its simplicity, it minimizes the agent's incentives to manipulate cash flows through hidden savings. The contract involves very simple monetary accounting. The cash that the agent draws or deposits translates into balance at one-to-one ratio. The agent's final payoff depends only on the net present value of cash deposited, and not the temporal pattern of deposits. As a result, the agent's problem is straightforward and he is not tempted to find the best way to manipulate cash flows to benefit from contractual intricacies. Because the increasing credit line minimizes the agent's incentives to divert cash and manipulate future cash flows, and it is costly to provide incentives, this contract turns out to be optimal.

A contract of this form is consistent with a number of intuitions we have about dynamic lending under moral hazard and adverse selection. Mirrlees (1974) argues that punishments after poor outcomes are appropriate to deal with moral hazard. In a borrowing context, inefficient liquidation can be a natural form of punishment, as seen, for example, in the two-period model of Bolton and Scharfstein (1990). The seminal paper of Akerlof (1970) has first recognized that adverse selection may make borrowing more difficult, citing the extortionate rates charged by local moneylenders in India.[1] Diamond (1989) finds that adverse selection decreases over time in a model with reputation. In a related problem with bilateral learning, Holmstrom (1999) demonstrates that workers have particularly strong incentives to put effort initially in order to influence market's learning. A similar property is present in our contract as well: while the credit line is growing, the agent strictly prefers to use cash flows to pay down balance rather than to consume them.

A distinguishing characteristic of our paper is that it analyzes moral hazard and adverse selection when the lender can fully commit to a dynamic contract. Dynamic contracts can offer significant new insights, as exemplified by the wide use of these methods in macroeconomics (e.g. see the textbook of Ljungquist and Sargent (2000)). Most of dynamic contracting literature focuses specifically on pure moral hazard. Dynamic financing with moral hazard has been studied in such papers as Clementi and Hopenhayn (2002), DeMarzo and Fishman (2005), DeMarzo and Sannikov (2005) and Biais et al.

[1] Also, Stiglitz and Weiss (1981) who show that credit can be rationed if there is adverse selection or moral hazard and that the interest rate can be both a screening and an incentive device.

(2005). There is very limited literature that looks at adverse selection also, although elements of adverse selection are present in settings where the agent's private information is correlated over time, as in Battaglini (2005) and Tchistyi (2005).[2] In those settings, when the agent intentionally misreports his current information to the principal, their beliefs about future uncertainty diverge. Closer in spirit to our paper, Tchistyi (2005) finds that an agent must pay higher interest on debt when he is closer to default, a feature known as performance pricing.

While most of dynamic contracting literature finds unavoidably complex history-dependent contracts, our result is especially attractive because a credit line is not only a simple, but also an *incomplete* contract, which happens to be dynamically optimal. It gives the agent discretion to use available credit but does not spell out the agent's actions. Although the contract is simple, it implements complex dynamic behavior, as the agent has incentives to borrow only to cover losses and pay interest on outstanding balance. This form of a contract creates a connection with contracts used in practice and with the incomplete contracting literature, which studies how contractual outcomes arise when parties interact within a simple set of rules. Grossman and Hart (1986) argue that the allocation of ownership rights over productive assets between two parties helps when optimal production decisions are ex-ante indescribable. Similarly, Hart and Moore (1998) derive an implementation of a borrowing contract that involves renegotiation if the agent fails to pay off debt. The idea of giving the agent discretion has first appeared in DeMarzo and Fishman (2005) and DeMarzo and Sannikov (2005), who show that a credit line with a fixed credit limit optimally solves pure moral hazard.

In our setting the agent's ability to manipulate cash flows through savings is an important reason for the simplicity of the optimal contract. While hidden savings typically make agency problems nearly intractable, on rare occasions they are a key reason for simple and elegant results. Bulow and Rogoff (1989) show that international debt cannot be sustained by reputation alone when countries have access to savings. In a similar spirit, Cole and Kocherlakota (2001) show that in the presence of hidden savings, the best insurance contract against random income shocks relies exclusively on saving and borrowing.

[2] Also see Cvitanic and Zhang (2006) and Y. Zhang (2006).

This paper is organized as follows. Section 2 presents the model. Section 3 describes the benchmark contract, a credit line with a fixed credit limit, which is optimal when moral hazard is the only problem. Section 4 presents an optimal contract with adverse selection. Section 5 justifies that this contract is optimal. Section 6 discusses issues connected with the assumptions of the model. Section 7 concludes.

2. The Model

Consider a market with many agents who seek funding for their projects. Any project requires a verifiable start-up capital investment of K. There are good and bad projects. Good projects produce cash flows

$$dX_t = \mu dt + dZ_t,$$

and bad projects,

$$dX_t = \mu' dt + dZ_t,$$

where $\mu > \mu' > 0$ are the means flow of cash flows and $\{Z_t, t \geq 0\}$ is a standard Brownian motion. If started, a project runs from time 0 until time T, unless it is liquidated prematurely. Any project produces a liquidation value of $L \geq 0$ when it stops running. Assume that it is inefficient to liquidate a good project prematurely (i.e. $rL < \mu$, where r is the market interest rate) and unprofitable to fund a bad project altogether.

There is adverse selection and moral hazard. The adverse selection problem is that the agent privately knows if the project is good or bad. The moral hazard problem is that the agent privately observes cash flow realizations if the project is started. The agent can secretly divert cash flows to consume or manipulate them his savings. That is, if the agent has positive savings, he can fabricate cash flows that did not actually realize. Premature liquidation after unsatisfactory performance can be used as a tool to solve these informational problems.

There is an entire market of agents with different project types and different initial levels of wealth. We assume that agents discount future consumption at rate *r*, which is also their return on savings. Due to risk-neutrality, an agent with a good project who has "deep pockets" does not need outside funding. An agent with a good project and limited wealth $R_0>0$ may seek outside funding to help him cover start-up capital and possible negative cash flows. However, investors are concerned that they may fund an agent with a bad project, who may imitate an agent with a good project. Agents with a bad project who have "deep pockets" pose the greatest threat, since they have complete flexibility to fabricate cash flows using their wealth. We assume that such agents have an infinite mass in the population. Therefore, the contract designed to fund good projects must be unattractive to agents with a bad project who have "deep pockets." We interpret R_0 as collateral. Financing under adverse selection is feasible only if R_0 is sufficiently large.

While an agent with a bad project may want to conceal his initial wealth, it turns out that an agent with a good project has incentives to reveal it. This is definitely true in competitive credit markets, because the agent's initial wealth mitigates adverse selection and moral hazard and the agent captures the entire benefit of improved efficiency. In Section 6 we show this to be the case even if credit markets are not competitive.[3]

The principal can fully commit to any history-dependent contract. By the revelation principle we can focus on truth-telling contracts. Such a contract specifies a termination time $\tau(\hat{X}) \in [0,T]$ and cumulative transfers from the agent to the principal $\{D_t(\hat{X}), t \in [0, \tau(\hat{X})]\}$ as functions of the agent's reports $\{\hat{X}_s, s \geq 0\}$. Formally, $D(\hat{X})$ is random process and $\tau(\hat{X})$ is a stopping time progressively measurable with respect to the agent's reports. Since the agent's discount rate is the same as the interest rate on savings, without loss of generality we assume that the agent postpones consumption until liquidation. Then, formally, the agent's savings evolve according to

$$dS_t = rS_t dt + dX_t - dD_t.$$

[3] We derive the optimal contract under the assumption that the agent with a good project reveals his wealth, and prove that that contract provides smaller incentives to hide wealth than any other contract.

The agent's savings must stay nonnegative.

Denote by $W_0 > R_0$ the value, determined by the relative bargaining powers of the agent and the investor, which an agent with a good project must get from his contract. The problem of an investor is to find a contract (D, τ) that maximizes profit

$$E\left[\int_0^{\tau} e^{-rt} dD_t + e^{-r\tau} L\right]$$

subject to three constraints: (1) giving the agent with a good project a value of

$$E\left[e^{-r\tau} S_\tau\right] = W_0 \text{ given reports } X,$$

(2) providing incentives to the agent with a good project to tell the truth, i.e.

$$\forall \hat{X}, E\left[e^{-r\tau} S_\tau\right] \le W_0 \text{ given reports } \hat{X}$$

and (3) not letting agents with bad projects get value more than R_0 even if they have deep pockets. We say that a contract is *truth-telling* if it satisfies constraint (2) and *screening* if it satisfies constraint (3).

To solve the principal's problem, it is enough to consider only contracts where the agent chooses to keep zero savings, as shown in the following lemma (proved in the Appendix).[4]

Lemma 1. *Given any truth-telling contract, there is another truth-telling contract with the same value to the principal and the agent with a good project, in which the agent does not save until liquidation, when he may receive a payment from the principal. If the former contract screens out bad project, the latter contract can be constructed to do so as well.*

Using the terminology of Hart and Moore (1994), we focus on contracts with the *fastest* repayment path by making the agent to transfer all cash flows to the principal until liquidation. Thus, we consider only contracts with $D_t = \hat{X}_t$ for $t < \tau$.[5]

[4] A simplification of this type is standard in problems with pure moral hazard and hidden savings. For example, see Werning (2002), Kocherlakota (2004) and Williams (2005).

3. The optimal contract under pure moral hazard.

Before we move on to a full-fledged analysis of adverse selection and moral hazard, we discuss a benchmark optimal contract under pure moral hazard in this section. Here the methods similar to those of DeMarzo and Sannikov (2005) apply. We do not provide a full derivation, but present the optimal contract and sketch a proof that it is optimal.

Consider a contract in which the agent gets a credit line with limit μ/r, interest rate r and a starting balance of $M_0=\mu/r - W_0$. Then the balance evolves according to

$$dM_t = rM_t dt - d\hat{X}_t,$$

where $\hat{X} = D$ are transfers to the principal that are chosen by the agent himself. The balance M_t can be positive or negative, but if it reaches the credit limit before time T, the project is immediately liquidated without a payment to the agent. If the project survives until time T, the agent can collect $W_T = \mu / r - M_T$ from the principal.[6] Before termination the agent has full discretion to transfer funds to and from the credit line.

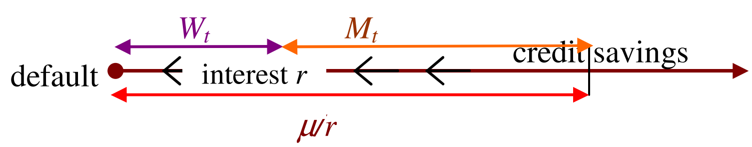

Figure 3: Credit line.

We make two claims about this contract:

1. The agent is indifferent between all strategies. In particular, it is optimal for the agent to pay all cash flows to the principal until termination.

2. This contract is optimal for the principal if the agent follows the latter strategy.

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[5] As a mathematical curiosity, the optimal credit line, being an incomplete contract, implements any optimal repayment path.

[6] This payment can be interpreted as the value of the collateral, which comes from funds R_0 that the agent contributes and an initial draw on the credit line.

The see that the first claim is true, note the agent can always "cash out" by drawing the remaining credit and triggering termination. Moreover, the agent is always indifferent between cashing out and continuing the project, independently of his current balance or actions, because

- the interest rate on the credit line is the same as on the agent's savings account and
- the interest on the maximal amount of credit equals the expected rate at which cash flows arrive

In particular, the agent maximizes utility by setting $\hat{X} = X$ until liquidation. Note that the remaining credit $W_t = \mu / r - M_t$ is the agent's future expected payoff at time *t*.

For the second claim, by Lemma 1 we only need to consider alternative contracts in which the agent is required to transfer all reported cash flows to the principal until liquidation, when the principal may make a transfer back to the agent.[7] The key property behind the optimality of the credit line is that the agent has *just enough* incentives to tell the truth. Other truth-telling contracts amplify the agent's incentives, by offering him more than a dollar of future payoff for each dollar reported to the principal. Such amplified incentives precipitate default in the event of losses and delay default in the event of gains relative to the credit line contract. Overall, this causes inefficiency because the likelihood of default is more sensitive to cash flows in the event of losses than in the event of gains.

The formal argument behind optimality relies on dynamic programming. It turns out that the principal's value function

$$b(W, T-t) = E_t\left[\int_t^\tau e^{-r(s-t)}\mu ds + e^{-r(\tau-t)}L - e^{-r(\tau-t)}W_\tau \mid W_t = W\right],$$

his future expected profit when the remaining lifetime of the project is *T - t*, is concave in the agent's the agent's "continuation value" $W_t = \mu/r - M_t$ (see Lemma 2 in the Appendix). This property formalizes the fact that default is more sensitive to cash flows in the event of

[7] Lemma 1 in the Appendix shows that for any truth-telling contract, there is a payoff-equivalent contract, in which the agent transfers all cash flows to the principal until termination.

losses than in the event of gains. From the law of motion of M_t, the agent's continuation value W_t in the credit line contract follows

$$dW_t = rW_t dt + (d\hat{X}_t - \mu dt),$$

so the agent gets exactly a dollar of future payoff for each dollar of reported cash flows. There are other contracts that give the agent incentives to tell the truth, in which the agent's continuation value evolves according to

$$dW_t = rW_t dt + \beta_t (d\hat{X}_t - \mu dt),$$

where $\beta_t \geq 1$ is the marginal value the agent gets from reporting cash. The condition $\beta_t \geq 1$ is necessary for incentive compatibility, but not sufficient.[8] However, since the function $b(W,T - t)$ is concave in W, it follows that a contract with $\beta_t > 1$ is suboptimal, since extra variance in W would hurt the principal. Figure 4 shows a typical form of $b(\cdot,T)$.

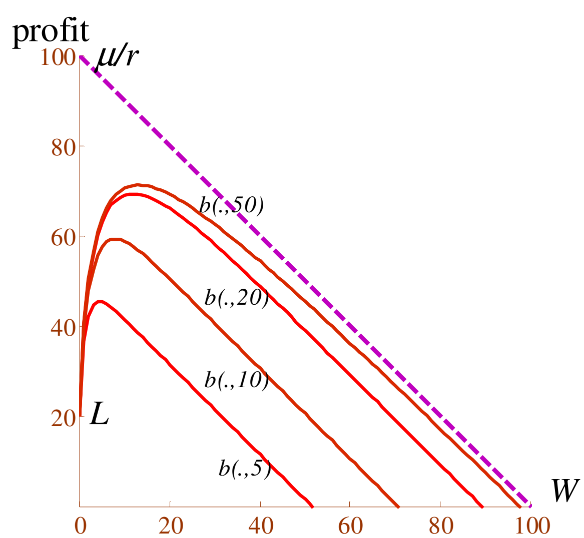

Figure 4: The principal's profit as a function of the agent's value/credit line balance for several values of project lifetime T (μ = 10, r = 0.1).[9]

[8] It would be sufficient for incentive compatibility if the agent did not have hidden savings.

[9] The computation of this figure relies on the following explicit formula: the probability that the project survives until time t is $\Phi(2re^{rt}W_0/(e^{2rt}-1)) - \Phi(-2re^{rt}W_0/(e^{2rt}-1))$, where Φ is the CDF of standard normal distribution.

To get a better sense of the significance of the preceding arguments, let us discuss the agent's incentives and the principal's profit in several alternative contracts:

1. A credit line with a fixed credit limit greater than μ/r and interest rate r

2. A credit line with a credit limit μ/r and interest $\gamma > r$ on positive balances

3. A credit line with a fixed credit limit smaller than μ/r and interest rate r

Contracts 1 and 2 do not provide the agent with proper incentives. When the credit limit is greater than μ/r, the agent prefers to draw the credit line and default immediately, since the expected cash flows fall behind the interest on the full amount of credit. The same is true when the credit limit is μ/r but the interest rate is greater than r. In contrast, in contract 3 the agent gets amplified incentives to pay down credit line balance, since drawing credit line to default gives him less value than avoiding default by paying interest. Therefore, contract 3 is truth-telling, but suboptimal.

The next section discusses the incentives of the agent with a bad project and conjectures an optimal contract under both moral hazard and adverse selection.

4. The Optimal Contract under Adverse Selection and Moral Hazard

In this section we present an optimal contract under adverse selection and moral hazard. We also prove that this contract is a truth-telling screening contract, but postpone the justification of its optimality until the next section.

Does the optimal contract under pure moral hazard, a credit line with a fixed credit limit μ/r, succeed at screening out bad projects? No. Since $W_0 > R_0$, an agent with a bad project would happily accept financing with such a credit line, and draw the entire balance to default immediately. To screen out bad projects, a contract needs to limit the amount of available funds so that an agent with a bad project finds such a contract unacceptable, even if he has infinite wealth. Specifically, there should be no history of cash flow reports

$\{\hat{X}_s, 0 \le s \le t\}$, which allows the project to survive until time *t*, and has value less than the expected value of bad project's cash flows until time *t* minus R_0. In other words, the project has to be liquidated not later than the first time when

$$\underbrace{\int_0^t e^{-rs}\mu' ds}_{\text{expected value of cash flows}} = \underbrace{R_0 + \int_0^t e^{-rs} d\hat{X}_s}_{\text{funds transferred to investor}} .$$

The following lemma, proved in the Appendix, summarizes this finding.

Lemma 3. *Let*

$$R_t = e^{rt}\left(R_0 + \int_0^t e^{-rs} d\hat{X}_s - \int_0^t e^{-rs}\mu' ds\right).$$

A contract fails to screen out bad projects if for some history of reports, for which $R_t < 0$, the project still survives, or the agent gets a payment greater than R_t during termination. A contract succeeds to screen out bad projects if liquidation happens before R_t becomes negative and the agent receives a payment of at most R_t during liquidation.

We claim that by *superimposing* the adverse selection restriction of Lemma 3 onto the optimal pure moral hazard contract, we get an optimal contract under both informational problems. Let us show that R_t reaches 0 when M_t reaches a critical level $\bar{M}_t$, so that the contract we obtain is a credit line with credit limit $\min(\mu / r, \bar{M}_t)$. Since the expression for R_t includes the value of cash flows that the agent reports, the credit line balance can be expressed in terms of R_t

$$dM_t = rM_t dt - d\hat{X}_t \Rightarrow M_t = e^{rt}\left(M_0 - \int_0^t e^{-rs} d\hat{X}_s\right) = e^{rt}\left(M_0 + R_0 - \int_0^t e^{-rs}\mu' ds\right) - R_t.$$

Therefore, R_t reaches 0 when the balance M_t reaches

$$\bar{M}_t = e^{rt}\left(M_0 + R_0 - \int_0^t e^{-rs}\mu' ds\right).$$

When $M_0+R_0 > \mu'/r$, this credit limit increases in time and reaches μ/r in a finite moment of time T^*. Figure 5 shows a typical path that the credit limit takes.

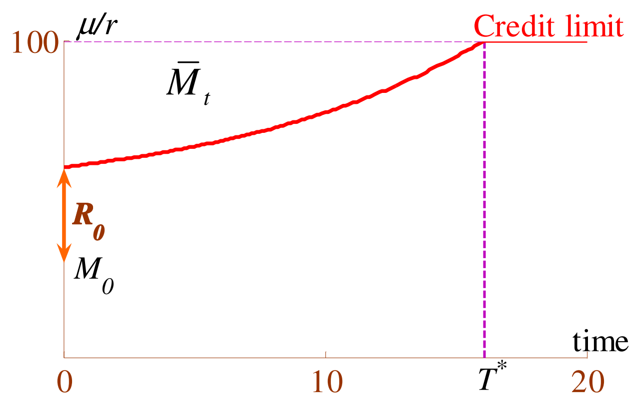

Figure 5. The evolution of the credit line when M_0 =30, R_0 =30, μ =10, μ'=5 and r =.1.

One prominent property of this contract is that the credit limit evolves deterministically in time, independently of the agent's actions. The reason for this simple property is that before time T^*, the contract makes funds available to the agent up until the limit where the agent with a bad project is just indifferent between taking financing or not. The point of indifference depends only on the lifetime of the bad project and outstanding debt, and not the movement of funds between the credit line and savings, which have the same interest rate. Therefore, the credit limit before time T^* does not depend on the agent's actions. The agent with a bad project is indifferent between not funding the project and funding it to follow any course of actions that leads to default not later than time T^*.

We are ready to formulate our main result, which is justified in the next section.

Main Theorem. *The optimal under adverse selection and moral hazard takes the form of a credit line with interest rate r and credit limit* $\bar{M}_t$ *that grows to* μ/r *until time* T^**, and stays at value* μ/r *thereafter. If the balance* M_t *reaches the credit limit before time T, liquidation occurs immediately and the agent gets no payment from the principal. If the project survives until time T, the agent gets a payment of* $\mu/r - M_T$.

We will refer to this contract as the *increasing credit line* contract for values *(W_0, R_0)*. Unlike in a credit line with a fixed limit μ/r, the precise relationship between the starting balance M_0 and the value of the agent with a good project W_0 is somewhat complicated. Since an initial restriction on funds available to the agent hurts his value, $W_0 < \mu/r - M_0$. We denote by $U(R_0, T^*)$ the function that gives the agent's value W_0 as a function of the amount of funds R_0 initially available to the agent and the amount of time T^* it takes the credit line to increase fully. Function U plays an important role in our justification of the optimal contract: it tells us the continuation value $W_t = U(R_t, T^*-t)$ of the agent with a good project at any time $t \leq T^*$ and his marginal value of cash flows $U_1(R_t, T^*-t) > 1$. That is, the agent gets *more* than a dollar of payoff for each dollar of cash flows he receives before time T^*, because a reported dollar not only reduces balance, but also allows the credit line to grow longer. The reason for this property is adverse selection: extra funds help the agent signal good project quality by paying interest on the outstanding balance. Figure 6 shows a typical form of function U, which is increasing in T^* and concave in R_0. The range of W_0 for which the principal's problem has a solution is obtained by varying T^* from 0 to T.

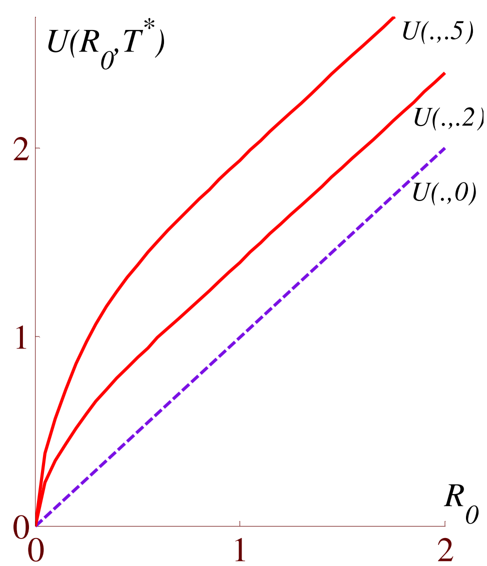

Figure 6: The agent's value as a function of initially available funds R_0, for T^*=0, 2 and 5 ($\mu = 10$, $\mu' = 8$, $r = 0.1$).[10]

In the remainder of this section, we only demonstrate that agents with both types of projects have proper incentives in the increasing credit line contract. Regarding bad

[10] This figure is computed using Monte Carlo simulations.

projects, Lemma 3 already showed that the increasing credit line contract successfully screens them out. Lemma 4, which is proved in the appendix, shows that this contract also gives the agent with a good project incentives to tell the truth.

Lemma 4. *In the increasing credit line contract for values (W_0, R_0) the agent with a good project gets value W_0 from any strategy, under which he never reaches default before time T^* with positive savings. If the agent sometimes ends up in default with positive savings before time T^*, he gets value less than W_0 in expectation.*

In particular, Lemma 4 implies that if the agent makes truthful reports, he maximizes his expected payoff under the increasing credit line contract and obtains a value of W_0. The proof of Lemma 4 relies on three observations. The first two imply that any strategy in which the agent exhausts his savings before time T^* achieves the same payoff for the agent.

1. Conditional upon avoiding default until time T^*, the payoff that the agent gets for his money is independent of his strategy, as if the credit limit was μ/r throughout.

2. Before time T^* default is determined only by the total value of reported cash flows and not the timing of reports.

Therefore, the time of default before time T^* (as a function of true cash flows) and the agent's payoff if the project survives until time T^* are the same for any strategy that exhausts the agent's savings to report cash flows in the event of a default before time T^*.

Lastly, it is strictly optimal for the agent to use savings fully in order to avoid default before time T^*:

3. When the balance on the credit line approaches $\bar{M}_t$ and $t < T^*$, the agent strictly prefers to avoid default (e.g. by paying interest) whenever he has funds. The reason is that expected cash flows exceed interest payments, and in addition the agent gets an extension to the credit line.[11]

[11] Here the logic is the same as in the previous section, in which we argued that the agent gets amplified incentives in a credit line with a limit shorter than μ/r.

5. Justification.

This section proves that the increasing credit line contract described in Section 4 is optimal when there is moral hazard and adverse selection.

Consider an arbitrary truth-telling screening contract. Two dynamically evolving variables play an important role in such a contract:[12]

- the *limit on funds* R_t that adverse selection imposes on the contract and
- the *continuation payoff* of the agent with a good project, defined by

$$W_t = E_t[e^{-r(\tau-t)}W_\tau],$$

where W_τ is the payment that the agent receives during liquidation. Initially $W_0 > R_0$, but at some time before liquidation W_t and R_t must become equal. Indeed, W_t and R_t follow continuous paths, and the agent cannot get a payment of $W_t > R_t$ during liquidation by Lemma 3. When W_t becomes equal to R_t for the first time, the adverse selection problem goes away, since the continuation value of the good agent no longer exceeds the upper bound on the value of the bad agent. At that time, it is optimal for the "continuation contract" to be a pure moral hazard contract with value $W_t = R_t$. Such a continuation contract would give the bad agent value of at most R_t, and would also minimize incentives of the good type to build up savings to manipulate cash flows after time t. (The good agent gets exactly a dollar of value for each dollar of saved cash flows after time t). From now on, we can assume that the contract becomes a pure moral hazard contract from the time when W_t and R_t become equal, which always happens before liquidation.

Variables R_t and W_t evolve dynamically, and the difference $W_t - R_t$ reflects the degree of adverse selection that remains to be dealt with. As in the case of pure moral hazard, in general W_t evolves as

$$dW_t = rW_t dt + \beta_t (d\hat{X}_t - \mu dt)$$

[12] Following Fernandez and Phelan (2000), these two variables do not present a full recursive structure of our problem, as they do not summarize the agent's incentives to save. However, by focusing on W and R, we can prove that the increasing credit line contract is optimal.

for some process β_t. With adverse selection, we should not expect the marginal value to the agent of project's cash flows β_t to be equal to 1. Indeed, at least in the increasing credit line contract the agent gets more than a dollar of value for each dollar of cash flows before time T^*: cash flows serve not only to reduce balance, but also to allow the credit line to grow longer. How about in an alternative contract? In fact, it turns out that *for any pair of* (W_t, R_t), β_t *is weakly lower in an increasing credit line than in any other contract*. Proposition 1 formalizes these statements.

Why is this the case? The answer lies behind the simple accounting of the increasing credit line contract. In that contract, the agent's payoff, conditional on being able to avoid default until time T^*, depends only on the monetary value of cash flows and not the manner in which they are reported. In an alternative contract the agent's value could depend on reported cash in a complex manner, enticing the agent to take advantage of contractual intricacies. The agent is tempted to divert cash in order to fabricate histories that give him the greatest value for reported cash. As a result, the agent can benefit at least as much from diverted cash flows in an alternative contract as in an increasing credit line. In order for the agent to have incentives to tell the truth, the marginal payoff from reported cash flows β_t (as a function of W_t and R_t) must be weakly greater in any alternative contract than in an increasing credit line.

Proposition 1, whose proof relies heavily on the agent's ability to save secretly and manipulate cash flows using savings, formalizes this reasoning. For any pair (W, R) with $W = U(R, s)$ for some $s \in [0, \infty)$ let

$$\sigma(W, R) = U_1(R, s).$$

Proposition 1. *Consider an arbitrary contract. Under truthful reports, the agent's continuation value evolves according to*

$$dW_t = rW_t dt + \beta_t (dX_t - \mu dt),$$

where the process $\{\beta_t, t \le \tau\}$ *in* L^* *is the sensitivity of continuation value to cash flows.*[13] *In an increasing credit line contract,* $\beta_t = \sigma(W_t, R_t)$. *In any alternative truth-telling screening contract* $\beta_t \ge \sigma(W_t, R_t)$.

Proof. Note that $e^{-rt}W_t$ is a martingale. By the Martingale Representation Theorem, there exists a process $\{\beta_t, t \le \tau\}$ in L^* such that

$$d(e^{-rt}W_t) = e^{-rt}\beta_t dZ_t = e^{-rt}\beta_t(dX_t - \mu dt).$$

This expression is equivalent to the desired representation above. Moreover, since $W_t = U(R_t, T^*-t)$ in the increasing credit line contract, by Ito's lemma the volatility of W_t is $\sigma(W_t, R_t) = U_1(R_t, T^* - t)$. The remainder of the proof is in the Appendix. QED

The necessary incentive-compatibility condition $\beta_t \ge \sigma(W_t, R_t)$, which is satisfied with equality in the increasing credit line contract, is similar to the condition $\beta_t \ge 1$ that was used to justify the optimality of a credit line under pure moral hazard. Under pure moral hazard, we argued that it is inefficient to amplify the agent's incentives by setting $\beta_t > 1$, because the principal's profit $b(W,s)$ is concave in the agent's continuation value W. In other words, giving the agent access to *more funds* in the event when he reports good cash flows, and *taking away funds* in the event that he reports losses causes inefficiency overall.

The case of adverse selection is based on similar logic. We must show that $f(W,R,s)$, the profit from the increasing credit line contract for values (W,R) when the lifetime of the project is s, is the maximal profit that the principal can achieve. That is, we need to show that a contract with $\beta_t > \sigma(W_t, R_t)$ achieves lower profit than the increasing credit line. Because variable R_t evolves dynamically together with W_t, this conclusion follows formally not from the concavity of $f(W,R,s)$ in W, but from the inequality

$$\sigma(W,R) f_{11}(W,R,s) + f_{12}(W,R,s) \le 0,$$

---

[13] L^* is the space of processes β for which $E[\int_0^t \beta_s^2 ds] < \infty$ for all $t < \infty$.

as can be seen from the proof of Proposition 2. Thus, the level of algebra required to prove that the increasing credit line contract is optimal is significantly more complex. Nevertheless, it is clear intuitively why a contract with $\beta_t > \sigma(W_t,R_t)$ is suboptimal. Such a contract would take the agent further from default in the already good event of high cash flows at the expense of precipitating default in the already bad event of low cash flows. Overall, this causes inefficiency.

Proposition 2. *Consider an alternative truth-telling screening contract with value W_0 to an agent with a good project and R_0 to an agent with a bad project. The principal's profit from this contract is at most $f(W_0,R_0,T)$. Therefore, the increasing credit line contract is optimal.*

Proof.[14] Let $\hat{\tau}$ be the earliest time when $W_t = R_t$. As we argued earlier, $\hat{\tau} \le \tau$. In an arbitrary truth-telling screening contract, the processes W_t and R_t follow

$$dW_t = rW_t dt + (\sigma(W_t,R_t) + \varsigma_t)dZ_t \quad \text{and} \quad dR_t = (rR_t + \mu - \mu')dt + dZ_t$$

with $\varsigma_t \ge 0$. In an increasing credit line contract we have $\varsigma_t = 0$.

Let the process $\{G_t,\ t \le \hat{\tau}\}$ be defined by $G_t = \int_0^t e^{-rs}\mu ds + e^{-rt} f(W_t,R_t,T-t)$.

In an increasing credit line, G_t is time-t expectation of the principal's profit, and thus a martingale, i.e. the drift of G_t is 0. Then Ito's lemma implies that

$$\mu - rf + rWf_1 + (rR + \mu - \mu')f_2 - f_3 + \tfrac{1}{2}\sigma(W,R)^2 f_{11} + \sigma(W,R)f_{12} + \tfrac{1}{2} f_{22} = 0.$$

In an alternative contract, G_t is a *submartingale* because

$$\mu - rf + rWf_1 + (rR + \mu - \mu')f_2 - f_3 + \tfrac{1}{2}(\sigma(W,R) + \varsigma_t)^2 f_{11} + (\sigma(W,R) + \varsigma_t)f_{12} + \tfrac{1}{2} f_{22} \le 0.$$

[14] The proof is technically quite involved, and may be skipped. The first part of the proof follows a standard dynamic-programming approach to show that the value function *f(W, R, T)* is unimprovable, while the second half (Lemma 6) demonstrates a key property of *f* using non-standard techniques. The argument of the proof is very much connected with the verbal discussion at the beginning of the section, hence the choice to leave the proof in text.

(We used the facts that $f_{11}, \sigma(W,R) f_{11} + f_{12} \le 0$, see Lemma 5 in the Appendix).
Therefore, the principal's profit from an alternative contract is less than or equal to

$$E\left[\int_0^{\hat{\tau}} e^{-rs}\mu ds + e^{-r\hat{\tau}} \underbrace{f(W_{\hat{\tau}}, W_{\hat{\tau}}, T-\hat{\tau})}_{=b(W_{\hat{\tau}}, T-\hat{\tau})}\right] = E[G_{\hat{\tau}}] \le G_0 = f(W_0, R_0, T).$$

This completes the proof. QED

6. Market with Agents and Investors.

In this section we investigate the issues connected investor's problem. We formulated the problem assuming that principal's full knowledge of the wealth level of the agent with a good project, and an infinite mass of agents with bad projects who have deep pockets. We also ignored other important issues that can arise in a market with borrowers and lenders, such as the agent's ability to write side-contracts with other investors, renegotiation, etc. Here we show that some of these assumptions, which appear important, are actually satisfied naturally given the problem's solution. Other assumptions matter somewhat, but the solution of our problem provides useful information and intuition even when those assumptions are violated.

The agent with a good project will always reveal his initial wealth *R_0*. Generally the principal faces a population of agents with good projects who have different wealth levels. The agents' wealth levels are their private information. In competitive credit markets agents with good projects would be willing to reveal their wealth truthfully, because larger collateral improves efficiency. In general, when the principal is able to get some rents, he would offer a menu of contracts that screen out bad projects and offer the agents with good projects a schedule of values $W(R_0)$. Would the principal ever offer the agent with wealth R_0 a contract other than an increasing credit line? There is only one possible reason to distort the contract offered to an agent with wealth R_0: it is to discourage agents with higher wealth levels from imitating him. However, as shown in Section 5, the increasing credit line contract gives the agent the least benefit from hidden wealth of any alternative contract for a given pair ($W(R_0)$, R_0). Therefore, our formulation of the principal's problem, which ignores the agent's incentives to reveal his wealth at time 0, in fact produces a valid solution.

An infinite mass of agents with deep pockets and bad projects simplifies the problem. With a general distribution over wealth levels and project types and when the investor has some bargaining power, one could imagine a situation in which some bad projects become screened out, some get funded, and some are bribed *not* to obtain funding, depending on the wealth of the agent with a bad project. Nevertheless, while the increasing credit line contract may be, strictly speaking, suboptimal, it comes close to achieving the optimal profit and possesses attractive robustness properties.

Also, our increasing credit line contract is optimal under a broader set of conditions than one may imagine at first. For example, in a setting with competitive credit markets, using the logic of Rothschild and Stiglitz (1976), the only equilibrium, if it exists, is fully separating. In that equilibrium unprofitable projects are screened out, and contracts that fund good projects are increasing credit lines.

The increasing credit line is not renegotiation-proof, as typical in settings with adverse selection. Since the principal knows that only good types of the agent accept financing, there is a temptation to renegotiate to a pure moral hazard contract immediately after the agent accepts financing through an increasing credit line. Even when adverse selection goes away at time T^*, the pure moral hazard contract that follows is not renegotiation-proof if the principal's profit $b(W, t)$ increases in the agent's value near default (as it happens in Figure 4). Therefore, the principal's ability to fully commit to a contract at time 0 plays an important role in our setting. How about the agent? Is it a concern that the agent may be able to sign a secret side-contract with a third party once his type is known to be good?

Side-contracts cannot harm the original investor. There are a number of good reasons why side contracts would be unfeasible or unprofitable for the agent. First, the agent can never benefit from a side-contract if he runs a pure moral hazard contract. Without side-contracts we know from Section 3 that agent's future payoff in a pure moral hazard contract is the remaining credit W_t, independently of the strategy the agent chooses to follow. If the agent cooperates with a third party with wealth $\tilde{w}$, then, by a similar logic, the joint value of the agent and the third party is $W_t + \tilde{w}$, independently of the agent's strategy or the characteristics of the side-contract. Since the third party must get at least $\tilde{w}$ from this arrangement, the agent cannot benefit from a side-contract. Second, one could imagine a third party signing a side contract with an agent, who has revealed his good type, to help him avoid default before time T^*. Such a contract could involve an up-front payment from the agent in exchange for financial help in the event the agent gets close to default before time T^*. The agent can never benefit from such a contract (because he would be better-off just using savings) unless sometimes he gets more funds back than his initial payment. These funds are beneficial only if the agent receives them after bad histories of cash flows. However, since it is easy to report a bad history of cash flows, the agent would be able to cheat the third party, yielding any beneficial side-contract infeasible in the first place. Such arrangements are infeasible because the agent has little to offer, since the project's assets are already collateralized with the original investor.

Finally, ignoring issues related to the feasibility of a side-contract, consider what happens if the agent gets full access to extra wealth $\tilde{w}$ from a third party at time 0. Assuming that the agent can somehow commit to repay debt $\tilde{w}$ at time T, he would benefit from this extra wealth. However, the original investor would benefit as well, since it is optimal for the agent to report $\tilde{w}$ as a cash flow at time 0, which improves the principal's profit.

A continuum of types μ. One very natural way to generalize our model is to consider the case with a continuum of agents' types μ, as opposed to just two types. Even though this case poses a very complex problem, we can gain important insights about it through the prism of our solution. Here, let us assume competitive credit markets, so that unprofitable projects have to be screened out and not financed.

Denote by μ^* the type with a barely profitable, who would be just indifferent between financing the project himself or not if he had deep pockets. We have

$$\int_0^T e^{-rt}\mu^* dt + e^{-rT} L = K.$$

Then a cash-constrained type μ^* should not get funded, because informational problems make his project unprofitable. Types below μ^* should not get funded either, but types above μ^* may get outside funding with a large enough collateral. How much is enough? To answer this question, let us plot the principal's profit as a function of the agent's value *b(W,T)* for several values of $\mu \geq \mu^*$.

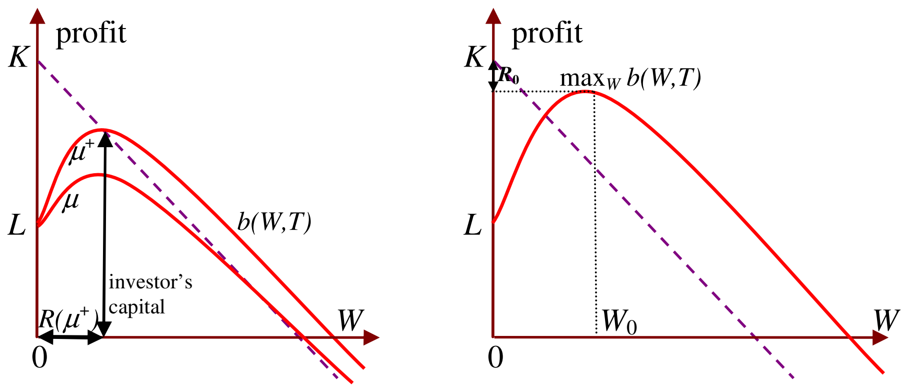

Figure 7.

On the left panel of Figure 7, let us look at points where the principal's profit intersects a dashed 45-degree line *K-W*. If the principal's profit at the intersection point is non-positive, that point corresponds to the minimal amount of collateral $R : (\mu^*, \mu^+] \rightarrow [R^+, \infty)$ for which funding is feasible (in Figue 7 the profit for type μ^+ intersects the 45-degree line at slope 0). The contract for type $\mu \in (\mu^*, \mu^+]$ with wealth $R(\mu)$ takes the form of a credit line with a fixed credit limit μ/r and a starting balance of $\mu/r - R(\mu)$. No type $\mu' < \mu$ would want to imitate type μ with such a contract. With this contract type μ' would get exactly the size of the collateral back if he defaults immediately, and defaulting immediately is optimal since the interest μ on full credit exceeds the expected cash flows.

Without adverse selection, the minimal size of collateral with which a type $\mu > \mu^+$ can get funded is $K - \max_W b(W,T)$ (no collateral is needed if $K < \max_W b(W,T)$). This funding is provided via a credit line with a fixed limit μ/r and starting balance $\mu/r - W_0$, where W_0 maximizes $b(W_0,T)$. With adverse selection such a contract would fail, because at least type μ^* with deep pockets would take this contract and draw the entire credit immediately. To screen out just that type the investor would need to limit funds initially available to the agent, and allow the credit line to grow according to $\bar{M}_t = e^{rt}(M_0 + R_0 - \int_0^t e^{-rs}\mu^* ds)$. However, this credit limit grows too fast to screen out types $\mu' \in (\mu^*, \mu^+]$, who have wealth $R(\mu')$ and, thus, zero outside option: they would gain back more than their collateral by financing the project and trying to avoid default while the credit line is growing. Thus, the principal must limit the credit line growth rate further. We feel intuitively that the growth rate should be faster for histories of reports that are more difficult to create for a cash-constrained type μ'. In other words, a credit line should grow faster when the agent makes larger payments to the principal. This intuition matches a casual observation that credit lines grow faster for customers who never miss monthly payments.

We conclude that in general, the growth of a credit limit may depend not only on time, but also the agent's actions. However, our main result has a great value as it isolates assumptions that drive a very simple contract, and helps us understand situations with a greater complexity by comparison.

7. Conclusions.

This paper considers a simple dynamic principal-agent setting with moral hazard, adverse selection and hidden savings. We derive a remarkably simple optimal contract: a credit line with a credit limit that increases deterministically in time. This form of a contract resembles many arrangements used in practice, and can serve as a natural benchmark to understand them. Also, like actual contracts, a credit line is an *incomplete* contract. It specifies interest and credit limit, and gives the agent full discretion to use available credit. Although the specification of the contract is very simple, it implements a complex outcome, as liquidation and payments to the agent become functions of the entire past history of cash flows.

Appendix.

Proof of Lemma 1. Consider an arbitrary contract, in which the agent is required to make transfers $\{D_t(\hat{X})\}$ such that

$$\int_0^t e^{-rs} dD_s(\hat{X}) \leq \int_0^t e^{-rs} d\hat{X}_s$$

as long as the project survives. When the agent has incentives to report truthfully, he consumes

$$S_{\tau(X)} = e^{r\tau(X)} \left(\int_0^{\tau(X)} e^{-rs} dX_s - \int_0^{\tau(X)} e^{-rs} dD_s(X) \right)$$

Consider an alternative contract, which requires the agent to transfer all reported cash flows to the principal until termination, when the principal gives the agent a payment of

$$e^{r\tau(\hat{X})} \left(\int_0^{\tau(\hat{X})} e^{-rs} d\hat{X}_s - \int_0^{\tau(\hat{X})} e^{-rs} dD_s(\hat{X}) \right).$$

Effectively, the principal modifies the contract to do savings for the agent.

This change extracts the same or greater amount of funds from the agent after all histories, which implies that the agent (with a good or a bad project) cannot become better off. Yet, if the agent with a good project tells the truth, he ends up with zero savings before liquidation, but the same amount $S_{\tau(X)}$ to consume at the liquidation time as before. Therefore, the modified contract gives the agent the same payoff as the old contract and incentives to report truthfully. Moreover, because the liquidation occurs at the same time $\tau(X)$ under both contracts, the principal's profit is the same. QED

Lemma 2. The function $b(W,T-t)$ is concave in W.

Proof. First, note that $b(W,T-t)$ is weakly decreasing in t since

$$b(W,T-t)+W = E_t\left[\int_t^{\tau} e^{-r(s-t)}\mu ds + e^{-r(\tau-t)}L \mid W_t = W\right]$$

and the project has less time to run in expectation for larger t. Moreover, since $b(0, T\text{-}t) = L$, it follows that $b_1(0, T\text{-}t)$ is weakly decreasing in t. Note that since $\int_0^t e^{-rs}\mu ds + e^{-rt} b(W_t, T-t)$ is a martingale and $dW_t = rW_t dt + dZ_t$, Ito's lemma implies that $b(W,T\text{-}t)$ satisfies the partial differential equation

$$rb(W,s) = \mu + rWb_1(W,s) - b_2(W,s) + \tfrac{1}{2} b_{11}(W,s).$$

Differentiating with respect to W, we obtain

$$0 = rWb_{11}(W,s) - b_{12}(W,s) + \tfrac{1}{2} b_{111}(W,s),$$

so $b_1(W_t, T-t)$ is a martingale and $b_1(W,T-t) = E_t[b_1(W_\tau, T-\tau) \mid W_t = W]$. Since $b_1(0,T\text{-}t)$ is weakly decreasing in t and $b_1(W, 0) = -1$ for all W, it follows that $b_1(W_\tau, T\text{-}\tau)$ is weakly decreasing in the initial condition $W_t = W$ for all paths of X. Therefore, $b_1(W,T\text{-}t)$ is decreasing in W and $b(W,T\text{-}t)$ is concave in W. QED

Proof of Lemma 3. Suppose that for some history of reports, the project survives until time t when $R_t(\hat{X}) < 0$. Then, if the agent with a bad project produces this history of reports, he collects an expected value of $\int_0^t e^{-rs}\mu' ds$ from the project's cash flows while transferring a value of $R_0 + \int_0^t e^{-rs} d\hat{X}_s$ to the principal. As a result, the agent's payoff is $-e^{-rt}R_t(\hat{X}) > 0$ and the contract fails to screen out bad projects. Similarly, if he gets a payment of $W_\tau > R_\tau$ after a history of reports $\hat{X}$, then agent with a bad project would get an expected payoff of $e^{-r\tau}W_\tau - R_0 - \int_0^\tau e^{-rs} d\hat{X}_s > 0$ by fabricating that history.

Conversely, if liquidation always happens at or before the time when R_t reaches 0, and if the agent receives payment at most R_t at that time, then the value he gets from transfers is at most

$$e^{-rt}R_t - R_0 - \int_0^t e^{-rs} d\hat{X}_s.$$

Letting τ be the time when the project is liquidated for a given strategy of an agent with a bad project, his total expected value is at most

$$E[e^{-rt}R_t - R_0 - \int_0^t e^{-rs} d\hat{X}_s + \int_0^\tau e^{-rt} dX_t] = 0.$$

Therefore, the contract succeeds at screening out bad projects. QED

Proof of Lemma 4. Consider a strategy such that whenever liquidation happens before time T^*, the agent has zero savings. Then the value of reported cash flows must equal the value of true cash flows at the liquidation time $\tau(\hat{X})$ when $\tau(\hat{X}) < T^*$, i.e.[15]

$$\int_0^{\tau(\hat{X})} e^{-rs} d\hat{X}_s = \int_0^{\tau(\hat{X})} e^{-rs} dX_s \Rightarrow R_{\tau(\hat{X})}(\hat{X}) = R_{\tau(\hat{X})}(X) = 0.$$

Thus, $\tau(\hat{X}) = \tau(X)$ when $\tau(\hat{X}) < T^*$. Then the agent's payoff from strategy $\hat{X}$ is

$$E\left[e^{-rT^*} 1_{\tau(X)\geq T^*} \overbrace{\left(\underbrace{\frac{\mu}{r} - e^{rT^*}\left(M_0 - \int_0^{T^*} e^{-rs} d\hat{X}_s \right)}_{\text{remaining credit } M_{T^*}(\hat{X})} + \underbrace{\int_0^{T^*} e^{r(s-T^*)}(dX_s - d\hat{X}_s)}_{\text{savings } S_{T^*}(\hat{X})} \right)}^{W_{T^*}(X)} \right] = W_0.$$

Now, consider an alternative strategy, under which there is a history of reports $\{\hat{X}_s, s \in [0,t]\}$ that results in termination at time $t < T$ with positive savings S_t. Let us show that by depositing savings immediately and reporting cash flows truthfully thereafter, the agent gets value greater than S_t. If the agent does so, then $M_s - \bar{M}_s$ evolves as

$$d(\bar{M}_s - M_s) = r(\bar{M}_s - M_s)ds + (dX_s - \mu' ds),$$

[15] Recall the assumption that the agent does not consume until liquidation, which we maintain throughout.

and so

$$\bar{M}_{\hat{\tau}} - M_{\hat{\tau}} = e^{r(\hat{\tau}-t)} S_t + \int_t^{\hat{\tau}} e^{r(s-t)} (dX_s - \mu' ds) \Rightarrow E[e^{-r(\hat{\tau}-t)} (\bar{M}_{\hat{\tau}} - M_{\hat{\tau}})] > S_t$$

for any stopping time $\hat{\tau}$ after *t*, e.g. the earlier of the liquidation time or time T^*. Therefore, if with positive probability the agent ends up in default with positive savings before time T^*, he is following a suboptimal strategy. QED

Proof of Proposition 1 (continued). Let us show that an alternative screening contract with $\beta_t < \sigma(W_t, R_t)$ on a set of positive measure does not provide the agent with incentives to tell the truth. Denote by

$$V(W, R, S) = U(R + S, t^*), \text{ where } t^* \text{ is given by } W = U(R, t^*),$$

the agent's value in an increasing credit line contract when $W_t = W$, $R_t = R$ and the agent has savings $S_t = S$. Consider an arbitrary fabrication strategy $\{\gamma_t\}$, defined by

$$dS_t = (rS_t - \gamma_t)dt \quad \text{and} \quad d\hat{X}_t = \gamma_t dt + dX_t.$$

By Lemma 4, in an increasing credit line contract $e^{-rt} V(W_t, R_t, S_t)$ is a martingale for any strategy γ_t, as long as the agent exhausts savings before liquidation. Using Ito's lemma and setting the drift of $e^{-rt} V(W_t, R_t, S_t)$ to 0, we get

$$\begin{aligned} &(rW_t + \sigma(W_t, R_t)\gamma_t)V_1 + (rR_t + (\mu - \mu') + \gamma_t)V_2 + (rS_t - \gamma_t)V_3 - rV + \\ &\tfrac{1}{2}(\sigma(W_t, R_t)^2 V_{11} + 2\sigma(W_t, R_t)V_{12} + V_{22}) = 0 \Rightarrow \sigma(W_t, R_t)V_1 + V_2 - V_3 = 0. \end{aligned}$$

Consider an alternative contract, where $\beta_t < \sigma(W_t, R_t)$ on a set of positive measure. If the agent spends his savings at rate γ_t, then the drift of $e^{-rt} V(W_t, R_t, S_t)$ is

$$e^{-rt} \left(\gamma_t (\beta_t V_1 + V_2 - V_3) + rW_t V_1 + (rR_t + (\mu - \mu'))V_2 + rS_t V_3 - rV + \tfrac{1}{2}(\beta_t^2 V_{11} + 2\beta_t V_{12} + V_{22}) \right)$$

Let us show that the agent can always make $e^{-rt} V(W_t, R_t, S_t)$ a submartingale by an appropriate choice of γ. Note that $V_1(W, R, S) = U_2(R + S, t^*) / U_2(R, t^*) \big|_{W = U(R+S, t^*)} > 0.$

When $\beta_t < \sigma(W_t, R_t)$ (and so $\beta_t V_1 + V_2 - V_3 < 0$) the agent can make the drift of $e^{-rt}V(W_t, R_t, S_t)$ positive by making γ_t sufficiently negative. Similarly, unless $S_t=0$, when $\beta_t > \sigma(W_t, R_t)$ (and so $\beta_t V_1 + V_2 - V_3 < 0$) the agent can make the drift of $e^{-rt}V(W_t, R_t, S_t)$ positive by making γ_t sufficiently large and positive. Finally, (1) if $\beta_t = \sigma(W_t, R_t)$ then $e^{-rt}V(W_t, R_t, S_t)$ is driftless for all γ_t, and (2) when $S_t=0$ and $\beta_t > \sigma(W_t, R_t)$ then $e^{-rt}V(W_t, R_t, 0) = e^{-rt}W_t$ is driftless when the agent sets $\gamma_t=0$.

If the agent sets γ_t according to the guidelines above until time $\hat{\tau}$ when W_t becomes equal to R_t for the first time (this must happen before liquidation, as we argued at the beginning of Section 5), then he would earn a value of at least

$$E[e^{-r\hat{\tau}}(W_{\hat{\tau}} + S_{\hat{\tau}})] = E[e^{-r\hat{\tau}}V(W_{\hat{\tau}}, \underbrace{R_{\hat{\tau}}}_{W_{\hat{\tau}}}, S_{\hat{\tau}})] > V(W_0, R_0, \underbrace{S_0}_{0}) = W_0,$$

where $W_{\hat{\tau}} + S_{\hat{\tau}} = V(W_{\hat{\tau}}, W_{\hat{\tau}}, S_{\hat{\tau}})$ is the agent's value in a fixed credit line when the remaining credit is $W_{\hat{\tau}}$ and the agent's savings are $S_{\hat{\tau}}$. We conclude that $\beta_t \geq \sigma(W_t, R_t)$ is a necessary condition for any screening contract to be truth-telling. QED

Lemma 5. For all $W \in [R, U(R,s)]$, $\sigma(W,R) f_{11}(W,R,s) + f_{12}(W,R,s) \leq 0$ and $f_{11}(W,R,s) \leq 0$.

Proof. Letting t^* be the time such that $W=U(R,t^*)$, we have

$$\underbrace{\sigma(W,R)}_{U_1(R,t^*)} f_{11}(W,R,s) + f_{12}(W,R,s) = \frac{df_1(U(R,t^*),R,s)}{dR}.$$

We need to show that $f_1(U(R,t^*),R,s)$ is decreasing in R. Note that the principal's profit in an increasing credit line contract for values $(U(R,t^*),R)$ is

$$f(U(R,t^*),R,s) = E\left[L + \int_0^{t^*} e^{-ru} 1_{\tau \geq u}(\mu - rL)du + e^{-rt^*} 1_{\tau \geq t^*}(b(R_{t^*}, s - t^*) - L) \mid R_0 = R \right]$$

and $U(R,t^*) = E\left[e^{-rt^*} 1_{\tau \geq t^*} R_{t^*} \right]$.

We have $f_1(U(R,t^*),R,s)=\dfrac{\partial f(U(R,t^*),R,s)/\partial t^*}{\partial U(R,t^*)/\partial t^*}$, where

$$\frac{\partial f(U(R,t^*),R,s)}{\partial t^*}=E[e^{-rt^*}1_{\tau\ge t^*}(\mu-rL-r(b(R_{t^*},s-t^*)-L)+(rR_{t^*}+\mu-\mu')b_1(R_{t^*},s-t^*)$$
$$-b_2(R_{t^*},s-t^*)+\tfrac{1}{2}b_{11}(R_{t^*},s-t^*)\mid R_0=R]=E\left[e^{-rt^*}1_{\tau\ge t^*}(\mu-\mu')b_1(R_{t^*},s-t^*)\mid R_0=R\right],$$

from $dR_t = (rR_t+\mu-\mu')\,dt+dZ_t$, Ito's lemma and equation $\mu-rb+rWb_1-b_2+\frac{1}{2}b_{11}=0$. We also have $\partial U(R,t^*)/\partial t^*=E\left[e^{-rt^*}1_{\tau\ge t^*}(\mu-\mu')\mid R_0=R\right]$. It follows that

$$f_1(U(R,t^*),R,s)=E\left[b_1(R_{t^*},s-t^*)\mid R_0=R,\tau\ge t^*\right].$$

Since b_1 is decreasing in the first argument by Lemma 2 and, as shown in Lemma 6, the conditional distributions of R_{t^*} given $R_0=R,\tau\ge t^*$ are ordered by first order stochastic dominance in R, it follows that $f_1(U(R,T^*),R,T)$ is decreasing in R, as required.

Finally, $f_{11}(W,R,s)=\dfrac{\partial f_1(U(R,t^*),R,s)/\partial t^*}{\partial U(R,t^*)/\partial t^*}$, and since $\partial U(R,t^*)/\partial t^*>0,$ we need to show that $\partial f_1(U(R,t^*),R,s)/\partial t^*<0.$ We have

$$\frac{\partial f_1(U(R,t^*),R,s)}{\partial t^*}=\frac{E\left[b_1(R_{t^*},s-t^*)1_{\tau\ge t^*}\mid R_0=R\right]}{\Pr(\tau\ge t^*)}=$$
$$E[\underbrace{((rR+(\mu-\mu'))b_{11}(R_{t^*},s-t^*)-b_{12}(R_{t^*},s-t^*)+\tfrac{1}{2}b_{111}(R_{t^*},s-t^*))}_{(\mu-\mu')b_{11}(R_{t^*},s-t^*)<0}1_{\tau\ge t^*}\mid R_0=R]/\Pr(\tau\ge t^*)$$
$$+\frac{b_1(0,s-t^*)}{\Pr(\tau\ge t^*)}\frac{d\Pr(\tau\ge t^*)}{dt^*}-\frac{E\left[b_1(R_{t^*},s-t^*)\mid R_0=R\right]}{\Pr(\tau\ge t^*)^2}\frac{d\Pr(\tau\ge t^*)}{dt^*}<0$$

since

$$b_1(0,s-t^*)>E\left[b_1(R_{t^*},s-t^*)\mid R_0=R,\tau\ge t^*\right]=\frac{E\left[b_1(R_{t^*},s-t^*)\mid R_0=R\right]}{\Pr(\tau\ge t^*)}$$

by Lemma 2 and $d\Pr(\tau\ge t^*)/dt^*<0.$ QED

Lemma 6. If $0<R'<R$, the conditional distribution of R_{T^*} given $R_0=R,\tau\ge T^*$ first order stochastically dominates the conditional distribution of R_{T^*} given $R_0=R',\tau\ge T^*$.

Proof. Let $\bar{R}$ be an arbitrary positive number. Define a process $\tilde{R}$ by

$$d\tilde{R}_t = (r\tilde{R}_t + \text{sign}(\tilde{R}_t)(\mu - \mu'))dt + dZ_t.$$

Then $P_t = E_t[1_{\tilde{R}_{T^*}>0} - 1_{\tilde{R}_{T^*}<0}]$ and $Q_t = E_t[1_{\tilde{R}_{T^*}>\bar{R}} - 1_{\tilde{R}_{T^*}<-\bar{R}}]$ are martingales. Define functions P and Q by

$$P(\tilde{R}_t, T^* - t) = E_t[P_{T^*} \mid \tilde{R}_t] \text{ and } Q(\tilde{R}_t, T^* - t) = E_t[Q_{T^*} \mid \tilde{R}_t].$$

Then

$$P(R, T^*) = E[P_{\min(\tau, T^*)}] = \Pr[\tau \geq T^* \big| R_0 = R] \text{ and}$$

$$Q(R, T^*) = E[Q_{\min(\tau, T^*)}] = \Pr[R_{T^*} \geq \bar{R}, \tau \geq T^* \big| R_0 = R],$$

since P_τ and Q_τ are 0 in the event that $\tau < T^*$. We need to show that

$$h(R, T^*) = Q(R, T^*) / P(R, T^*) = \Pr[R_{T^*} \geq \bar{R} \mid \tau \geq T^*, R_0 = R]$$

is monotonically increasing in R for $R \geq 0$. Since $P(\tilde{R}_t, T^* - t)$ and $Q(\tilde{R}_t, T^* - t)$ are martingales, P and Q satisfy the same partial differential equations

$$(rR + \text{sign}(R)(\mu - \mu'))P_1 - P_2 + \frac{1}{2}P_{11} = 0 \text{ and } (rR + \text{sign}(R)(\mu - \mu'))Q_1 - Q_2 + \frac{1}{2}Q_{11} = 0$$

but with different boundary conditions at $t = T^*$: $P(R,0) = 1_{R>0} - 1_{R<0}$ and $Q(R,0) = 1_{R>\bar{R}} - 1_{R<-\bar{R}}$. Plugging $Q(R,s) = h(R,s)P(R,s)$ into the equation for Q, we get

$$(rR + \text{sign}(R)(\mu - \mu'))(h_1 P + hP_1) - h_2 P - hP_2 + \frac{1}{2}(h_{11}P + 2h_1 P_1 + hP_{11}) = 0 \Rightarrow$$

$$\underbrace{h((rR + \text{sign}(R)(\mu - \mu'))P_1 - P_2 + \frac{1}{2}P_{11})}_{0} + \left(rR + \text{sign}(R)(\mu - \mu') + \frac{P_1}{P}\right)h_1 - h_2 + \frac{1}{2}h_{11} = 0.$$

Also, h satisfies the boundary conditions $h(R,0) = 1_{R \notin (-\bar{R}, \bar{R})}$. It follows that $h(\hat{R}_t, T^* - t)$ is a martingale when $\hat{R}$ follows

$$d\hat{R}_t = \left(r\hat{R}_t + \text{sign}(\hat{R}_t)(\mu - \mu') + P_1(\hat{R}_t, T^* - t) / P(\hat{R}_t, T^* - t) \right) dt + dZ_t \,.$$

Now, let $\hat{R}$ and $\hat{R}'$ be two such processes that start from a initial conditions $\hat{R}_0 = R$ and $\hat{R}_0' = R'$. Then $\hat{R}_t \geq \hat{R}_t'$ pathwise for all $t \in [0, T^*]$. Let $\hat{\tau}$ the earliest time when $\hat{R}_t' = -\hat{R}_t$ for the first time. Then

$$h(R,T^*) - h(R',T^*) = E[1_{\hat{\tau}<T^*} (\underbrace{h(\hat{R}_{\hat{\tau}}, T^* - \hat{\tau}) - h(\hat{R}^{'}_{\hat{\tau}}, T^* - \hat{\tau})}_{=0, \text{ by symmetry}}) + 1_{\hat{\tau}\geq T^*} (\underbrace{h(\hat{R}_{T^*}, 0) - h(\hat{R}^{'}_{T^*}, 0)}_{>0, \text{ since } \hat{R}_{T^*} > |\hat{R}^{'}_{T^*}|})] > 0.$$

QED

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