Dynamic Strategic Information Transmission

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Abstract

This paper studies strategic information transmission in a dynamic environment, where a privately informed expert and a decision maker interact for a finite number of periods. Our theoretical results argue that the dynamic cheap talk games are fundamentally different from static ones. First, information can be fully revealed. Second, there exist non monotonic equilibria. Third, there is no general way to Pareto rank the equlibria.

1. Introduction

The seminal paper on strategic information transmission by Crawford and Sobel (1982) has been a model of choice for a number of applications, ranging from economics and political science, to philosophy and biology.[1] One limitation of their model is that the environment is static as an expert and a decision maker interact only once. However, there are many environments in which information transmission is dynamic. Many sequential decisions have to take place, and the decision maker seeks the expert's advice prior to each one of them. Our paper extends the analysis of Crawford and Sobel to study strategic information transmission in a dynamic environment in which an expert and a decision maker interact for a finite number of periods. The main contribution of our paper is to argue that the dynamic nature of interaction significantly affects the results, characterization, and intuition from the static setup.

We analyze a mutli-period relationships between an expert and a decision maker that last finitely many periods. We assume that the state of the world remains constant over time. We maintain all other features of the Crawford and Sobel (1982) environment, in particular, that the interests of the expert and the decision maker differ.

We first show that under fairly general conditions in dynamic cheap talk models there are equilibria that are fully revealing. This result is in stark contrast to the central result of Crawford and Sobel (1982) where full information revelation is impossible in any equilibrium. In their static

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[1] For a survey of some applications accross disciplines see Sobel (2008).

environment, the conflict of interest between the expert and the decision-maker implies that at most finitely many different actions can be induced at equilibrium. The intuition there is that because of the way the decision-maker responds to information sent by the expert, nearby expert types have an incentive to pool together, precluding full information revelation.

In our dynamic environment, full information revelation can occur. We show, over time, it is possible to divide states into separable groups. A separable group is a set of types which are sufficiently far apart that each would rather reveal the truth, than mimic any other type in his group. Therefore, we create histories after which it is common knowledge that the decision maker puts probability one on a particular separable group, at which point the types in this group separate.

The construction of the proof of how to divide types into separable groups is delicate. The expert anticipates that if at some point he joins a separable group then he will forgo his informational advantage. For this reason he has to be given adequate incentives in order to do so. Right before the expert joins a separable group we have to make sure that he does not want to mimic close-by types supposed to join another separable group. The decision maker can provide adequate incentives only if he has enough leeway to choose separation-inducing actions: The separation-inducing actions can only be constructed when the expert conditions his own advice on the decision-maker's actions, and in turn the decision maker finds it optimal to choose (non-myopically optimal) actions which induce the expert to reveal the truth.

Without the delicate dependence of actions on past histories the construction of fully revealing equilibrium is impossible. Our next theorem highlights the importance of the dynamic conditioning of strategies. We show that full revelation information is impossible even in a dynamic setting when we restrict to Markov Perfect Bayesian equilibria and if the discount factor of the expert is positive. We then present an example in which if the expert is extremely impatient and discounts future at zero it is possible to have a fully revealing equilibrium. The construction of the fully revealing equilibrium here relies on the fact that when the discount factor is zero separation-inducing actions need not depend on previous actions chosen, as in the case of $\delta > 0$.

We then emphasize another necessary condition for information revelation: the equilibrium cannot be a monotonic partition equilibrium as in Crawford and Sobel (1982). In a monotonic partition equilibria the expert employs a uniform signalling rule where each type sends one message with probability one, and the set of types that choose the same message is connected. Our second main results is to prove that there exists no fully revealing monotonic partition equilibria in a dynamic game: If attention is restricted to monotonic partition equilibria learning stops. Finally, we present an example of a subclass of monotonic partition equilibria and show how the dynamic nature of the game introduces additional bias between the expert and the decision maker compared to the static case. Overall, we conclude that monotonic partition equilibria constitute a very special class of equilibria in the dynamic setup, whereas this is a canonical class of equilibria in the static setup

Our third set of results focuses on another important difference between the dynamic and the static cheap talk games whether equilibria can be easily Pareto ranked. Crawford and Sobel (1982) show that under some assumptions ex-ante both parties are better-off at the most informative equilibrium: Both the expert and the decision maker ex-ante (before the state is realized) prefer the equilibrium with the highest number of actions induced. We provide an example that shows that it is not necessarily the case for dynamic equilibria. More partitions of the possible states of the world do not guarantee higher expected payoff for the expert and the decision maker. We then show an example in which dynamic monotonic partition equilibria can be strictly Pareto superior than playing the most informative one-shot equilibrium in each period and babbling thereafter. As our examples show there is no-clear cut Pareto comparisons among equilibria exist.

The discussion up to now as fairly abstract. We now illustrate the analysis with a more general discussion and two more applied examples. First, what are the potential examples where the strategic dynamic information transmission is relevant. Think, for instance, about the manager of an R&D group of some large corporation (the expert). He has a project in mind whose probability of success (the state of the world) is known only to him. At the beginning of each new budgetary period he makes a report to the company's central administration (the decision maker) about the likely success of the project. In turn, the administration decides the amount of resources to be currently devoted to the project. In this situation there are multiple rounds of communication and decision making. Another example, is an investor (decision maker) where each month seeks his company's financial advisor's (expert) opinion on the performance of the stock of the company where he works, and decides the fraction of his monthly savings to invest in that stock. Again, here there are multiple rounds of advice and of decisions.

To get a better sense of the type of problem we are examining, as well as of the type of information revelation that is possible in a dynamic setting, let us describe some of our findings within the context of simple examples:

The piano teacher example: A parent hires a piano teacher for his daughter . He does not know how talented his daughter is in piano, but ideally he would like the child to receive the number of hours of instruction that correspond to her level of talent. The teacher knows the child's talent, but because he is a struggling musician, would like the parent to hire him for many more hours than it is actually suitable for the child's talent. Furthermore, suppose that the piano teacher is so severely biased, that if communication and the decision of the hours of instruction were one-shot as in Crawford and Sobel (1982), all equilibria would be equivalent to babbling: the parent would always end-up choosing the number of hours that are optimal prior to any communication. We show, that even in these extreme bias situations, if there multiple rounds of communication and decision (for instance the parent asks the teacher his opinion about the child's talent each month in order to decide next month's hours of instruction), some information is released even with two rounds: More precisely, we show that there is an equilibrium where the parent learns after two rounds whether the child has almost no talent, whether she has the highest possible talent, or whether her talent lies somewhere in between. This equilibrium has the feature that the decision maker learns the state quite precisely when the news is either horrific, or terrific, but remains quite agnostic for intermediate levels. This often happens in reality, where uniformed parties get to learn close to the truth when news is extremely good, or extremely bad, but remain quite ignorant otherwise.

The impatient financial advisor example: Consider, just as described in the beginning, a financial advisor (the expert) who advises one of the company's employee (the decision maker) about the value of the company stock. What is special about this advisor, is that he only cares about his current-year bonus, which depends on the amount of company stock employees currently purchase, and does not think further in the future. In this paper we show how the decision maker can exploit this short-sightedness of the financial advisor, by constructing an equilibrium, where for a large range of biases, the decision maker learns the exact true state, with just two rounds of communication.

Related Literature

Crawford and Sobel (1982) is the seminal contribution on strategic information transmission. Since that paper there is an enormous amount of work in this area and myriads of applications. Some of these works are nicely summarized in Farrell and Rabin (1996), Sobel (2008) and Krishna and Morgan (2008). In a nutshell, the basic sender-receiver game of Crawford and Sobel (1982) has been generalized within a static setup along the following dimensions: (i) more than one receivers, for instance, by Farrell and Gibbons (1989), Levy and Razin (2004) and more recently Goltzman and Pavlov (2008) and Koessler and Martimort (2008); (ii) more than one sender: Battaglini (2002), Krishna and Morgan (2004), Levy and Razin (2007); (iii) certifiable messages by Seidmann and Winter (1997); (iv) noisy communication by Blume, Board and Kawamura (2007).

There are also several papers that look at dynamic relations, however these papers differ from ours because the focus is on the sender's reputation (Morris (2000), Ottaviani and Sorensen (2006a), (2006b)). Other papers have a dynamic aspect because they allow for multi-round communication protocols but with a *single* round of action(s): Aumann and Hart (2003) characterize correlated equilibria when a long conversation is possible, Krishna and Morgan (2004) add a long protocol on Crawford and Sobel (1982)'s game, Goltsman, Hoerner, Pavlov and Squintani (2008) characterize optimal such protocols. Eso and Fong (2008) allow for the possibility of delay when the decision maker consults two experts. In that paper the possibility of constructing fully revealing equilibria relies on the fact that the decision maker can "punish" experts that give conflicting advice. Ivanov (2007) allows for a dynamic communication protocol in a setup where the decision maker controls the quality of information available to the expert. Finally, Forges and Koessler (2008a, 2008b) allow for a long protocol in a setup where messages can be certifiable. The main difference from these papers and ours, is that in ours there are multiple rounds of *actions* as well: at the first stage of each round, communication takes place and at the second stage, the decision maker chooses the round's action. Within each round the communication is one shot, exactly as in Crawford and Sobel (1982). The dynamic nature of action choices changes quite dramatically the nature of equilibria we study compared to standard case.

Non-monotonic equilibria in the literature

Our model bears some similarities to the static extensions with multiple receivers. In those models the expert cares about a sequence of actions, but in contrast to our model those actions are chosen by different individuals. An important difference is that in our model the receiver cares for the entire vector of actions chosen, compared to those models where each receiver cares only about his own action. Still, some of the properties of the equilibria that we obtain appear also in the models with multiple receivers. For example, our non-monotonic piano teacher example resembles example 2 of Goltzman and Pavlov (2008). Equilibria can be non-motonic also in environments where the decision maker consults two experts as mentioned in a footnote in Krishna and Morgan (2001). The same is true when a long communication protocol as in Krishna and Morgan (2004) who construct non-monotonic equilibria that strictly Pareto dominate the best equilibrium with one shot communication, when the bias is large. Their non-monotonic equilibria are similar to the one in our first example where the very low and the very high states are pooled together. The difference lies in the forces that sustain such counterintuitive pooling: In Krishna and Morgan high states are willing to pull with low ones because by doing they induce with some probability a very high action and with the residual probability they get the low pooling action. In our setup, the pooling is with probability one, meaning that very high states induce the same action as the very low states for sure. This is an equilibrium behavior because this allows these states to separate and induce a much higher action in the future.

More partitions do not necessarily imply higher welfare in the literature

In contrast to the finding of Crawford and Sobel (1982) that show that under Assumption M, more partition imply higher ex-ante welfare for both, we have an example where this is not true. A similar phenomenon occurs when the communication is noisy as is shown in an example of the working paper version of Blume, Board and Kawamura (2007) (they show that a two step partition that Pareto dominates a three-step partition). In the dynamic setup the intertemporal nature of the indifference conditions that pin down the cut-offs that characterize partition equilibria allow for a much richer set of possibilities which overturn the Pareto comparison results.

Fully revealing equilibria in the literature

Full information revelation is possible when the decision maker consults two experts as in Battaglini (2002) and in Eso and Fong (2008). When information is certifiable as in Mathis (2008), when there are lying costs and the state is unbounded as in Kartik, Ottaviani and Squintani (2007). In t he case of multiple experts, playing one against the other is the main force that supports truthful revelation. In the case of an unbounded state lying costs become large and support the truth and in the case of certifiability, one can exploit the fact that messages are state-contigent to induce truth telling. All these forces are very different from the forces behind our fully revealing construction.

2. The Environment

We extend the classical model of Crawford and Sobel (1982) to a dynamic setting. There are two players, an expert (sender) and a decision maker (receiver) who interact for finitely many periods $t = 1, ..., T$. The expert knows the state of the world θ. The state $\theta \in [0, 1]$, is constant over time, and is distributed according to the c.d.f. F. At the beginning of each period t, the expert gives his advice (sends a report) to the decision maker m_t who then updates his beliefs about the state of the world and chooses an action $y_t \in \mathbb{R}$ that affects both players' payoffs. There is a conflict of interest because for all states of the world the expert's ideal choice of action differs from the decision maker's. This is captured by the bias parameter of the expert which we denote by b. The expert and the decision maker discount the future with a discount factor $\delta_i \in [0, 1]$ for $i \in \{E, DM\}$. When the state of the world is θ and the decision maker chooses actions $y^T = (y_1, .., y_T)$, then the expert's payoff is given by

$$U^E(y^T, \theta, b) = \Sigma_{t=1}^T \delta_E^{t-1} u^E(y_t, \theta, b),$$

and the decision maker's payoff is given by

$$U^{DM}(y^T, \theta) = \Sigma_{t=1}^T \delta_{DM}^{t-1} u^{DM}(y_t, \theta).$$

We assume that $u^E(y_t, \theta)$ and $u^{DM}(y_t, \theta, b)$ satisfy the main conditions imposed by Crawford and Sobel (1982): u^i is twice differentiable and $\frac{\partial u^i(y,\theta)}{\partial y} = 0$ for some y for $i \in \{E, DM\}$ and for all θ, as well as $\frac{\partial^2 u_t^i(.,.)}{\partial y^2} < 0$. These last two conditions guarantee a unique maximum in y. We also assume a single crossing condition,$\frac{\partial^2 u_t^i(.,.)}{\partial \theta \partial y} > 0$.

Let $y^{DM}(\theta)$ and $y^E(\theta)$ denote the decision maker's and, respectively, the expert's most preferred action when the true state is known by everyone and is θ. We assume throughout that regardless of the state of the world a conflict of interest exists: $y^{DM}(\theta) \neq y^E(\theta)$ for all θ. The single crossing condition ensures that $y^{DM}(\theta)$ and $y^E(\theta)$ are increasing in θ.

The decision maker observes his payoffs only at the end of the game. He does *not* observe his payoff at each period t. Otherwise, the problem would have been trivial because the decision maker can learn the expert's information by simply inverting his payoff.

Assessments

An *assessment* consists of a strategy profile and a belief system. A *strategy profile* $\sigma = (\sigma_i)_{i=S,B}$, specifies a strategy for each player. Let h_t denote a history that contains all the reports submitted by the expert $m^{t-1} = (m_1, ..., m_{t-1})$, and all actions chosen by the decision maker $y^{t-1} = (y_1, ..., y_{t-1})$ up to but not including stage t. The set of all feasible histories at t is denoted by H_t. A *behavioral* strategy of the expert, σ_E, consists of a sequence of signaling rules that map $[0, 1] \times H_t$ to a probability distribution over reports $\mathcal{M}$. Let $q(m \,|\theta, h_t)$ denote the probability that the expert reports message m at history h_t when his type is θ. A strategy for the decision maker, σ_{DM}, is a sequence of maps from H_t to actions. We use $y_t(m \,|h_t) \in \mathbb{R}$ to denote the action that the decision maker chooses at h_t given a report m. A *belief system*, μ, maps H_t to the set of probability distributions over $[0, 1]$. Let $F(\theta|h_t)$ denote the decision maker's beliefs about the experts's type after a history of moves h_t, $t = 1, ..., T$. A strategy profile σ and a belief system μ is an assessment.

Solution Concept

A *Perfect Bayesian Equilibrium, (PBE),* is a strategy profile and a belief system that satisfy:

1. For all $\theta \in [0,1]$ and h_t, $\int_{M_t} q_t(m\,|\theta, h_t)dm = 1$, where M_t is a Borel set that contains all feasible signals at stage t. If a message $\hat{m}_t$ is in the support of $q_t(m\,|\theta, h_t)$ it must be the case that
$$\hat{m} \in \arg\max_{m \in M} \Sigma_{t=1}^{T} \delta_E^{t-1} u^E(y_t, \theta, b).$$

2. Given $F_t(\theta|h_t)$ and the expert's strategy, the decision maker chooses at each h_t an optimal action: $y_t(m\,|h_t)$ solves
$$\max_{y \in \mathbb{R}} \Sigma_{\hat{t}=t}^{T} \int_0^1 \Sigma_{t=1}^{T} \delta_{DM}^{t-1} u^{DM}(y_t, \theta) F(\theta|h_t) d\theta.$$

3. $F_t(\theta|h_t)$ is derived from F_{t-1} given h_t using Bayes' rule whenever possible.

3. Uniform Signaling: Monotonic Partition Equilibria

We start the analysis by first focusing on the equilibria that are the dynamic version of the canonical class of equilibria in the static Crawford and Sobel (1982) game, the class of monotonic partition equilibria. At this class the state is divided into a finite number of subintervals, and for all states in each subinterval the expert sends the same message and induces the same action. Formally we define them as follows:

Definition 1 *We call monotonic partition equilibria where at each stage t and history h_t the expert employs a uniform signalling rule where each type sends one message with probability one, and the set of types that choose the same message is connected.*

Note that the essential feature of monotonic partition equilibria is that the set of types that choose the same message is connected and each type sends only one message. We call this simple communication uniform signaling. This kind of signaling induces actions that are increasing step-functions of the state and it is the canonical communication of Crawford and Sobel (1982).

We start by reviewing the Crawford and Sobel (1982) findings. Then we examine properties of this class of equilibria in the dynamic setup and show several results. First, if we were to restrict attention to monotonic partition equilibria, then the decision-maker would never learn the truth. The second result establishes that when the bias is so large that only one action is induced at an equilibrium of the static game, the same is true also in dynamic monotonic partition equilibria. Finally, we show in an example that dynamic partitions can differ and can be more informative compared to static ones. In the subsequent sections we show that other types of equilibria arise in dynamic games. Those equilibria not only have different properties from the monotonic partitional ones, but also allow for full information revelation.

3.1 Uniform Signaling: The Canonical Static Communication

For the one-shot communication and decision game Crawford and Sobel (1982) showed that all equilibria are equivalent to partition equilibria where the expert employs simple uniform signalling rules. Hence at the canonical communication protocol, intervals of types pool together, in the sense that they send the same message, ultimately inducing the same action. This implies that communication is coarse: even though the state takes a continuum of values, only finitely many different actions are induced.

The forces behind this result can be summarized as follows. Fix an equilibrium of the one-shot game and let $y(\theta)$ denote an action induced when the state is θ. The conflict of interest between the expert and the decision maker ($y^{DM}(\theta) \neq y^{E}(\theta)$, for all θ) implies that there will be at most finitely many actions induced. The fact that u^E is strictly concave in y, together with single-crossing and the finiteness of actions implies that there will be at most finitely many states θ where the expert is indifferent between two induced actions. This, in turn, implies that $y(\theta)$ can be without any loss taken to be single-valued. Now, from the single crossing property we also have that $y(\theta)$ is increasing in θ. This, together with the fact that $y(\theta)$ takes at most finitely many actions, implies that $y(\theta)$ is a increasing step function: the state space is divided into a finite number of subintervals and y takes a different value along each subinterval. An even deeper observation of Crawford and Sobel (1982) is to show that without loss of generality equilibrium induced actions, $y(\theta)$ can be taken to arise from uniform signaling rules: This result follows from the observation that all messages inducing the same action y can be replaced by a single message. Therefore complex signaling rule plays no role in the static setup.

In order to illustrate how these equilibria work, we consider a simple case where the state θ is uniformly distributed on $[0, 1]$, and the preferences over θ and y are represented by a quadratic loss function

$$u^E(y,\theta) = -(y-\theta-b)^2 \text{ and } u^{DM}(y,\theta,b) = -(y-\theta)^2\,, \tag{1}$$

where $b > 0$. **In what follows we will use this environment extensively to illustrate complex types of communication and their welfare properties in the dynamic setup.**

At a partition equilibrium the state space is divided in a finite number of subintervals described by a sequence of cutoffs $(\theta_1, ..., \theta_p)$. The expert uses a uniform signaling rule where for all states in $[\theta_{i-1}, \theta_i)$ he sends message m_i with probability one. Then, the optimal action by the decision maker who observes message m_i is then given by $y_i = \dfrac{\theta_{i-1}+\theta_i}{2}$. These equilibria are monotonic because a higher action is induced, the higher the state is.

At an equilibrium the cutoffs have to be such that types in $[\theta_{i-1}, \theta_i]$ prefer y_i to the actions that are induced by sending another message. Given single crossing, this is ensured if for all $i = 1, ..., p$, θ_i satisfies the following indifference condition

$$-\left(\frac{\theta_{i-1}+\theta_i}{2}-\theta_i-b\right)^2 = -\left(\frac{\theta_i+\theta_{i+1}}{2}-\theta_i-b\right)^2, \tag{2}$$

which holds if the difference equation

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} + 4b \tag{3}$$

is satisfied. The solution of this difference equation is given by $\theta_i = \frac{i}{p} + 2bi^2 - 2bip$, where p is the number of subintervals that the type space is divided into at the given equilibrium. Each cut-off θ_i must be in $[0, 1]$. Using this observation, we can obtain the largest number of subintervals that the type space can be divided into, which we denote by $p^{\max}$. This is the largest integer that satisfies

$$-2bp^2 + 2bp + 1 > 0, \tag{4}$$

whose solution is

$$\left\langle -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} \right\rangle, \tag{5}$$

and where $\langle x \rangle$ denotes the smallest integer greater than or equal to x. For each $p = 1, ..., p^{\max}$ that satisfies (4) we get a different equilibrium. Therefore, in the static strategic information game of Crawford and Sobel (1982) multiple equilibria exist, but they are simple in the sense that they are all equivalent to partition equilibria. More importantly, in this example, and more generally, under certain conditions (assumption M in Crawford and Sobel (1982)) equilibria can be easily Pareto ranked: Equilibria with more partitions are ex-ante preferred both by the expert and the decision maker to the equilibria with fewer partitions.[2]

3.2 Uniform Signaling: A Special kind of Dynamic Communication

Here we study properties of the monotonic partition equilibria. In constructing these equilibria, the main difference from the static case is that the expert's indifference conditions (the dynamic analog of (2)) are intetemporal which makes the explicit solution of the resulting difference equation difficult even in quite simple cases, such as the uniform quadratic loss case. We obtain properties of these equilibria, by noticing that at some point in the game the expert's indifference conditions reduce to the static ones and by then applying the results of Crawford and Sobel (1982).

We first show that among the class of monotonic partition equilibria, there exists no equilibrium where the decision maker eventually learns the state of the world for all $\theta \in [0, 1]$, that is there exist no fully revealing equilibria. We start by describing what is meant by full revelation:

Definition 2 *An equilibrium is fully revealing if there exists a $\hat{T} \leq T$ such that after all histories along the equilibrium path leading to $\hat{T}$, the expert sends for all $\theta \in [0, 1]$ a different message with probability one, inducing $y_t(\theta) = \theta$ from $t = \hat{T}$ on.*

By definition, it follows that in such an equilibrium, there must exist a point in the game at which a continuum of possible actions are induced.

[2] A recent, important paper by Chen, Kartik and Sobel (2008) shows that under mild conditions, the most informative equilibrium is the one that satisfies the "no incentive to separate" (NITS) condition.

Proposition 1 *For all $T < \infty$ there exist no fully revealing monotonic partition equilibria.*

This result follows almost immdediately from the ones of Crawford and Sobel (1982). A short sketch of the argument is as follows: Suppose that there exists a fully revealing dynamic monotonic partition equilibrium. Then, there exists some period $\hat{T} \leq T$ where the last subdivision occurs and where $y_t(\theta) = \theta$ from $t = \hat{T}$. This implies that a subinterval of states at $\hat{T}$ is partitioned into an uncountably many subintervals contradicting Crawford and Sobel's result.

Another straightforward implication of the Crawford and Sobel (1982) results is the following Proposition:

Proposition 2 *If all static equilibria are equivalent to the babbling equilibrium, than all dynamic monotonic partition equilibria are equivalent to babbling.*

A short sketch of the argument is as follows: Suppose otherwise. Then, as we have argued before there will be a point where the last subdivision occurs. From that point on we can use the static indifference condition. However, from Corollary 1 of Crawford and Sobel (1982) we can easily see that if the static equilibrium is babbling, the same must be true for all monotonic partition equilibria in the dynamic setup.

Finally, it is easy to see that all static equilibria are also equilibria in the dynamic games. Consider any static equilibrium. We can find an "equivalent" dynamic equilibrium, in which in the first period all agents follow the same strategies as in the static equilibrium, and for all $t > 1$ experts of all types "babble" and the decision maker repeats the same action she took in period 1. It is trivial to verify that these strategies constitute an equilibrium. In the next example we show that the reverse is not true by constructing a monotonic partition equilibria in the dynamic game that cannot arise in the static one.

Example 1 *Monotonic partition equilibria with more partitions in a dynamic game*

Suppose that $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0, 1]$ and preferences satisfy (1). Suppose that bias $b = \frac{1}{12}$. From (5) static game has only two equilibria, a babbling equilibrium and an equilibrium with two partitions, where the space is divided in two sub-intervals: $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$ inducing two actions $\frac{1}{6}$ and $\frac{4}{6}$. Now we show that when $T = 2$, there exists a monotonic partition equilibrium where in the end the state space is divided in three sub-intervals.

We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0, \theta_1] \text{ send message sequence } A = (m_{1(1)}, m_{2(1)}) \\
&\text{types in } [\theta_1, \theta_2] \text{ send message sequence } B = (m_{1(2)}, m_{2(2)}) \\
&\text{types in } [\theta_2, 1] \text{ send message sequence } C = (m_{1(2)}, m_{2(3)}).
\end{aligned}$$

With this signaling rule in the first period the interval $[0, 1]$ is partitioned into $[0, \theta_1]$ and $[\theta_1, 1]$. At stage 2, $[\theta_1, 1]$ is partitioned further into $[\theta_1, \theta_2]$ and $[\theta_2, 1]$. We can express θ_2 in terms of θ_1 as follows:[3]

$$\theta_2 = \left(\frac{1-\theta_1}{2} - \frac{4}{12}\right) + \frac{2}{12} + \theta_1 = \frac{1}{2}\theta_1 + \frac{1}{3}. \tag{6}$$

Then second period actions induced then are

$$y_{2(1)} = \frac{\theta_1}{2}, y_{2(2)} = \frac{3}{4}\theta_1 + \frac{1}{6} \text{ and } y_{2(3)} = \frac{1}{4}\theta_1 + \frac{2}{3},$$

whereas the first period actions induced are

$$y_{1(1)} = \frac{\theta_1}{2} \text{ and } y_{1(2)} = \frac{1+\theta_1}{2}.$$

After any out of equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of equilibrium beliefs it is immediate to see that no type has an incentive to send an out-of equilibrium message.

At an equilibrium θ_1 must satisfy the following indifference condition:

$$-(\frac{1+\theta_1}{2} - \theta_1 - \frac{1}{12})^2 - (\frac{\frac{1}{2}\theta_1 + \frac{1}{3} + \theta_1}{2} - \theta_1 - \frac{1}{12})^2 = -2(\frac{\theta_1}{2} - \theta_1 - \frac{1}{12})^2$$

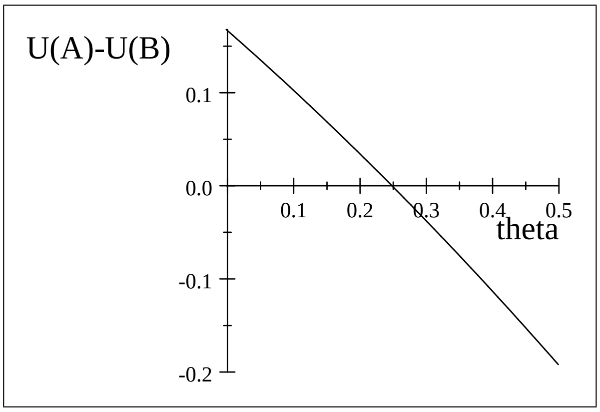
which with the help of (6) gives us 3 partitions with cut offs $\theta_1 = 0.25$ and in turn $\theta_2 = 0.458\,33$.

Using this value of θ_1 we get that:

$$y_{1(1)} = y_{2(1)} = 0.125, \quad y_{1(2)} = 0.625 \quad y_{2(2)} = 0.354\,17, \quad y_{2(3)} = 0.729\,17.$$

In constructing this strategy profile we have imposed that type θ_2 is indifferent between inducing action $y_{2(2)}$ and $y_{2(3)}$ at $t = 2$ and that type θ_1 is indifferent between inducing action sequence $y_{1(1)}$ and $y_{2(1)}$ and $y_{1(2)}$ and $y_{2(2)}$. Now we want to verify that these conditions are sufficient for global incentive compatibility. At $t = 2$ the game is isomorphic to the static one, where the fact that θ_2 is indifferent between $y_{2(2)}$ and $y_{2(3)}$ implies that all types above θ_2 prefer $y_{2(3)}$ and all types below θ_2 prefer $y_{2(2)}$. To verify that types below θ_1 prefer message sequence A inducing $y_{1(1)}$ and $y_{2(1)}$, and types aboev θ_1 prefer message sequence B inducing $y_{1(2)}$ and $y_{2(2)}$, we plot the difference $U(A, \theta) - U(B, \theta)$ and show that it is positive for all $\theta < \theta_1$ and negative for $\theta > \theta_1$:

[3] This can be done using (11) derived later. The construction relies on solving for states on the boundaries that are indifferent between different sequences of messages. Given that the quadratic preferences satisfy the single-crossing condition, it is immediate that all the incentive constraints of the expert are satisfied.



It is useful to compare the players' ex-ante expected payoffs in these two cases: The expert's, and respectively the decision maker's ex-ante expected welfare by playing the most informative static equilibrium in the first period and babbling in the second, is -0.069 and -0.055. Now at the dynamic equilibrium we constructed here the expert's welfare is -0.065 and the decision maker's welfare is given by -0.0517. Both the expert's and the decision maker's welfare is strictly higher at our dynamic equilibrium compared to the equilibrium that induces the same partitions as the most informative static equilibrium.

In Appendix C we provide an explicit characterization of a subclass of monotonic partition equilibria in the uniform quadratic loss case.

In contrast to the static case, there are many equilibria that do not fit this class. In fact, in a dynamic setup monotonic partition equilibria are not only a subclass of all equilibria, but also a subclass of all monotonic equilibria. To understand this, note that in a dynamic setup in all monotonic equilibria the same action is induced by a connected set of states (an interval - possibly degenerate). However, this observation does not imply that we can without any loss assume that all monotonic equilibria can be replicated by monotonic equilibria where the expert employs a simple uniform signaling rule. The reason is that different signaling rules generating the same action at some period t, may be generating dramatically different posteriors, leading to different continuation equilibria. This is not an issue in a one-shot setup because there only the current action matters. This point is nicely illustrated in example 3, which has a non-partitional, albeit monotonic equilibrium.

In the following sections we will show that dynamic equilibrium properties can be very different from the static ones. First, we show that there exist non-monotonic equilibria, where experts of types far apart from each other bunch together at some periods, while the types in the middle separate. These equilibria rely on complex (non-uniform) signaling rules have very different qualitative properties from the monotonic particition equilibria. We show that when preferences are quadratic there always exists equilibria in which the expert after a finite number of periods fully learns the state, provided that the bias of the decision maker is not too large. These equilibria are non-monotonic. Finally we show that Pareto comparisons of equilibria is more complicated in dynamic than in static games. Non-monotonic equilibria can be Pareto superior to monotonic equilibria, and equilibria with more partitions can be Pareto inferior to the equilibria with fewer partitions.

4. Complex Signaling and Dynamic Information Revelation

We saw that all static equilibria are equivalent to ones where the state space is partitioned in a finite number of intervals, and types in each subinterval induce the same action. This implies that experts of similar types tell the same advice, and the actions induced in equilibrium vary monotonically with the state. We now show that very different types of equilibria arise in dynamic settings, where the induced action varies non-monotonically with the state of the world. We will show that such equilibria allow for a greater amount of information revelation and they are the key for our main result on the existence of the fully revealing equilibria. We also will show that non-monotonic equilibria can be Pareto superior to the best monotonic equilibria.

For the rest of the paper we will make the following assumption about preferences and distribution of types.

Assumption 1. Types θ are uniformly distrubuted on $[0, 1]$ and preferences satisfy (1)

This assumption is not necessary to show that dynamic cheap talk model has equilibria with the very different properties from the static cheap talk models, e.g. that full information revelation is possible in a finite number of periods. For distribution and preferences satisfying Assumption 1 we can obtain much tighter characterization about such equilibria. In Section 5. we show that if Assumption 1 holds then there always exists a fully revealing equilibrium provided that the game lasts at least 4 periods and the bias is not too large. Furthermore we show that all such equilibria are non-Markovian.

Example 2 *Dynamic equilibria can be non-monotonic*

Consider the following two-period example. Suppose assumption Assumption 1 holds. Let the expert use the following signaling rule. At stage 1, the expert for states in $[0, \underline{\theta}) \cup (\bar{\theta}, 1]$ sends $m_{1(1)}$ with probability 1. For states in $[\underline{\theta}, \bar{\theta}]$ sends a message $m_{1(2)}$ with probability 1. At stage 2, the expert adopts the following signaling rule. For states in $[0, \underline{\theta})$ sends a message $m_{2(1)}$ with probability 1, for states in $[\underline{\theta}, \bar{\theta}]$ sends a message $m_{2(2)}$ with probability 1, and finally for states in $(\bar{\theta}, 1]$ sends a message $m_{2(3)}$ with probability 1. To sum up, we look for an equilibrium with the following signaling rule:

types in $[0, \underline{\theta})$ send message sequence $A = (m_{1(1)}, m_{2(1)})$
types in $[\underline{\theta}, \bar{\theta}]$ send message sequence $B = (m_{1(2)}, m_{2(2)})$
types in $(\bar{\theta}, 1]$ send message sequence $C = (m_{1(1)}, m_{2(3)})$.

With this signaling rule, the optimal actions of the decision maker are respectively given by

$$\text{first period: } y_{1(1)} = \frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)},\ y_{1(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}$$
$$\text{second period: } y_{2(1)} = \frac{\underline{\theta}}{2},\ y_{2(2)} = \frac{\underline{\theta} + \bar{\theta}}{2},\ y_{2(3)} = \frac{1 + \bar{\theta}}{2}.$$

After any out of equilibrium message the decision maker assigns probability one to the state belonging to $[\underline{\theta}, \bar{\theta}]$, inducing an action equal to $y^{out} = \frac{\underline{\theta}+\bar{\theta}}{2}$. With these out-of equilibrium beliefs no type of the expert has an incentive to send an out-of equilibrium message.

In order for this to be an equilibrium, it must be the case that both types $\underline{\theta}$ and $\bar{\theta}$ are indifferent between messages $m_{1(1)}$ and $m_{1(2)}$ at $t = 1$, in other words, the following indifference equations must hold. For type $\underline{\theta}$:

$$-(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \underline{\theta} - b)^2 - (\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 = -2(\frac{\underline{\theta} + \bar{\theta}}{2} - \underline{\theta} - b)^2 \tag{7}$$

and for type $\bar{\theta}$:

$$-(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \bar{\theta} - b)^2 - (\frac{1 + \bar{\theta}}{2} - \bar{\theta} - b)^2 = -2(\frac{\underline{\theta} + \bar{\theta}}{2} - \bar{\theta} - b)^2. \tag{8}$$

At $t = 2$ it must also be the case that type $\underline{\theta}$ prefers $m_{2(1)}$ to $m_{2(3)}$ and the reverse for type $\bar{\theta}$, in other words:

$$-(\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 \geq -(\frac{1 + \bar{\theta}}{2} - \underline{\theta} - b)^2$$
$$-(\frac{1 + \bar{\theta}}{2} - \underline{\theta} - b)^2 \geq -(\frac{\underline{\theta}}{2} - \bar{\theta} - b)^2.$$

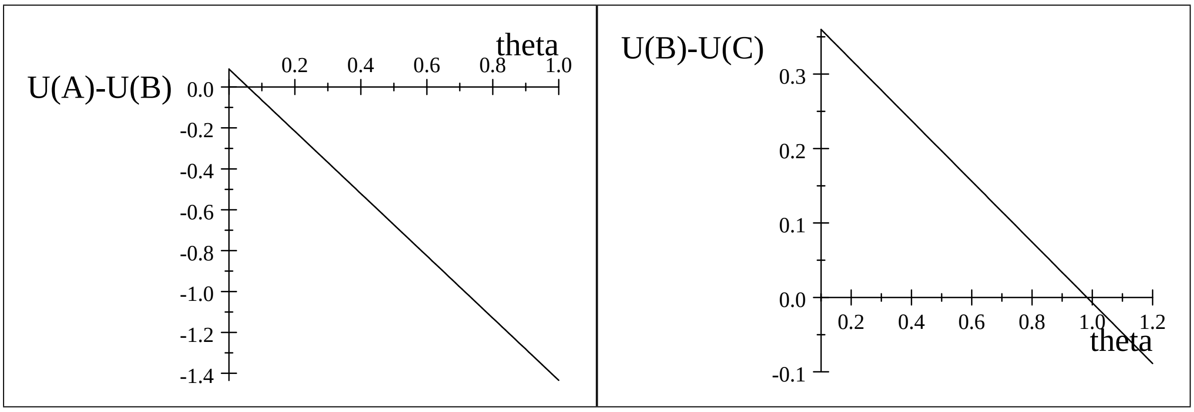
The highest bias that makes our non-monotonic equilibrium survive is 0.256. For this bias we have that $\underline{\theta} = 0.0581$, $\bar{\theta} = 0.9823$, from which we get the corresponding optimal actions

$$\text{first period: } y_{1(1)} = 0.253,\ y_{1(2)} = 0.52,$$
$$\text{second period: } y_{2(1)} = 0.029,\ y_{2(2)} = 0.52,\ y_{2(3)} = 0.991.$$

This equilibrium is non-monotonic as at stage 1, action $y_{1(1)} = 0.253$ is induced for states in $[0.982, 1]$, whereas action $y_{1(2)} = 0.52$ is induced for states in $[0.058, 0.982]$.

In costructing this equilibrium we used the indifference conditions (7) and (8). Given the dynamic nature of the game and the non-monotonicity of actions we need to check whether these two conditions suffice for global incentive compatibility. In other words we need to check that all types in $[0, 0.058)$ prefer message sequence A, types in $[0.058, 0.982]$ prefer message sequence B and types in $(0.982, 1]$ prefer message sequence C. The following graphs verify that this is indeed the case, by showing that $U(A, \theta) - U(B, \theta) > 0$ for all $\theta \in [0, 0.058)$, whereas $U(A, \theta) - U(B, \theta) < 0$ for

$\theta > 0.0580$, and that $U(B,\theta) - U(C,\theta) > 0$ for all $\theta \in [0.0580, 0.982]$, whereas $U(B,\theta) - U(C,\theta) < 0$ for all $\theta > 0.982$:



An interesting feature of the constructed non-monotonic equilibrium is that it exists even though the only monotonic partition equilibrium in this game is a trivial babbling equilibrium. To see that this is the case notice that for the bias $b = 0.256$ equation (5) implies that there can be only babbling equilibrium in the static game, and Proposition 2 then implies that there can be only a babbling monotonic partition equilibrium in the dynamic game.

4.1 When is the Truth Learnt?

In order to understand what are the key elements of the construction of a fully revealing equilibrium, one has to understand what prevents the truth to be learnt in the Crawford and Sobel setup. The main difficulty arises from the fact that the conflict of interest makes the expert induce the same action (which is essentially equivalent to giving the same advice) for states that are close by. This suggests that if the set of states that are possible from the perspective of decision maker are sufficiently far apart, then the expert can find it in his best interest to tell the truth (separate). In other words, if the decision maker's posterior puts strictly positive weight only on states that are sufficiently far apart, it is incentive compatible to separate. In an dynamic setup the decision maker's posteriors depend on the previous rounds of communication and are therefore endogenous. Hence the question that arises is whether it is possible to generate posteriors that put positive weight on states that are far apart so that they are willing to separate. We call such a groups of states *separable*. We show that generating a posterior that puts strictly positive weight on a separable group of states is possible.

In order to have a fully revealing equilibrium we need to divide *all* states into separable groups, which implies that we must form a continuum of such groups. At the same time we have to incentivize the expert appropriately in order to be in his best interest to join a separable group. This is difficult because with a finite number of periods it means at there must be at least one stage in the game where we are generating a continuum of such groups and have to make it incentive compatible for an expert type not to be willing to join an adjacent separable group. This can be done by designing appropriate rewards for the expert. In our game where the are no transfers the rewards can take the form only of actions chosen by the decision maker.

We start with providing an example of a fully revealing equilibrium when has $\delta_E = 0$. This example is instructive because it shows that to divide experts in the separable groups so that eventually they reveal all the information. When $\delta_E > 0$ this construction is significantly more complex and will be discussed in Sections 5. and 6..

Simple Case: Learning the Truth when the Expert is Myopic We first show how one can go about constructing a fully-revealing equilibrium in the simple case where the expert cares only about his current-period payoff in the sense that $\delta = 0$. In this case one can induce the expert to join a separable group easily. [elaborate on that]. The construction of the example that follows relies on the fact that there were disconnected sets of types sending the same message in period one. This results in period-two posteriors where the decision maker put positive probability on states that are so far apart, that they have no incentive to pool with each other.

There are two essential ingredients of this example. First, the first period action is flat – irrespective of the state, all experts induce the same action. Second, that the set of types that choose the same message at $t = 1$ are sufficiently far apart so they can be separated at $t = 2$. These two requirements say that a way to have a separating equilibrium when $\delta = 0$, is by grouping types together in such a way such that all groups have the same expectation, which implies that they lead to the same action in an equilibrium where the decision maker choose the action that is optimal given his posterior beliefs) and all groups consist of types that are sufficiently far apart.

Example 3 *Fully revealing equilibrium with impatient experts ($\delta_E = 0$).*

Consider the uniform quadratic loss case as in (1)) when the bias is $b < \frac{1}{16}$. At stage 1, the expert for states $\{\frac{1}{8}-\varepsilon; \frac{3}{8}+\varepsilon, \frac{4}{8}+\varepsilon, 1-\varepsilon\}$ sends message m_ε, for $\varepsilon \in [0, \frac{1}{8}]$, and for states $\{\frac{1}{8}+\tilde{\varepsilon}, \frac{7}{8}-\tilde{\varepsilon}\}$ sends a message $m_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon} \in (0, \frac{1}{4})$.

Given this signalling rule, the first period action that is a best response of the decision maker is given by

$$y(m_\varepsilon) = \frac{\frac{1}{8} - \varepsilon + \frac{3}{8} + \varepsilon + \frac{4}{8} + \varepsilon + 1 - \varepsilon}{4} = 0.5 \text{ for all } \varepsilon \in [0, \frac{1}{8}]$$
$$y(m_{\tilde{\varepsilon}}) = \frac{\frac{1}{8} + \tilde{\varepsilon} + \frac{7}{8} - \tilde{\varepsilon}}{2} = 0.5 \text{ for all } \tilde{\varepsilon} \in (0, \frac{1}{4}).$$

Notice that all messages induce the same action at $t = 1$. At least from the $t = 1$ perspective all types are indifferent among them. This is the first essential ingredient of the example.

At stage 2, the expert after each history sends with probability 1 a different message for each state. To see that this is an equilibrium, consider, for example, the history after m_ε was chosen at stage 1. At this history the posterior of the decision maker at $t = 2$ puts weight $\frac{1}{4}$ on the following types: $\frac{1}{8} - \varepsilon; \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon$ and $1 - \varepsilon$. If at stage 2 each of these types sends a different message, namely $m_{2(k)}$, where $k \in \{\frac{1}{8}-\varepsilon; \frac{3}{8}+\varepsilon, \frac{4}{8}+\varepsilon, 1-\varepsilon\}$, then after each of these messages, the decision maker chooses

$$y(m_{2(k)}) = k.$$

After any out-of-equilibrium message, at $t = 1$ the decision maker assigns equal probability to all states, leading to action $y^{out} = 0.5$, whereas at $t = 2$ after an out-of equilibrium message, the decision maker believes that the state is equal to zero, leading to an action equal to $y^{out} = 0$. With these out-off equilibrium beliefs, no type has an incentive to deviate at $t = 1$, given that they have nothing to gain. The same is true at $t - 2$, since $y_2(\theta) = \theta$ and is always preferred to $y^{out} = 0$ because it satisfies $y_t(\theta) < 2(\theta + b)$.

If the expert is impatient ($\delta = 0$), then for any $b < \frac{1}{16}$ this is an equilibrium. To see that at stage 2 the expert has no incentive to deviate, we consider, for instance, type θ and show that he does not have incentive to mimic type $\theta + \frac{1}{8}$:

$$-(\theta - \theta - b)^2 > -(\theta + \frac{1}{8} - \theta - b)^2,$$
$$(\frac{1}{8} - b - b)\frac{1}{8} > 0,$$

which is satisfied if $b < \frac{1}{16}$. The situation after a message $m_{\tilde{\varepsilon}}$ is analogous.

5. Main Result: Full information revelation in dynamic cheap talk games

One of the most stark results of the static cheap talk game is that there is no equilibrium with full information revelation: although the state can take a continuum of values, the expert sends at most finitely many signals to the decision maker, so that substantial amount of information is not transmitted. In this section, we show that our dynamic cheap talk model there exist equilibria that are fully revealing.

In the previous section we argued that fully revealing equilibria must have a feature that expert types group in separable groups before eventually separating. It is relatively straightforward to do when the expert is impatient ($\delta_E = 0$), but for positive discount factors for expert it is harder to do. When the expert is forward looking, he anticipates that once he separates, the decision maker will choose the action that maximizes decision maker's utility until the end of the game. The decision maker who learns with certainty in period $\hat{T}$ that expert type is θ will choose action θ in all $t \geq \hat{T}$. But this implies that from peroid $\hat{T}$ the decision maker chooses an ideal action from the point of view of an expert of type $\theta + b$. Therefore for the first $\hat{T}$ periods the decision maker should choose actions that provide incentives for the type $\theta + b$ (and all other types in the neighborhood of θ) not to join type-θ separable group. This is possible if type $\theta + b$ gets a higher payoff in the first $\hat{T} - 1$ periods from joining type $\theta + b$ group rather than type θ's group.

In Section 6. we show that this is possible only when the decision maker abstains from taking the actions in periods $t \leq \hat{T}-1$ that maximize her period t utility. Therefore, the fully revealing equilibrium necessarily requires trigger strategies. Thus the equilibrium involves a complex combination of punishments. The decision maker is prevented initially from taking a myopically optimal action with the threat that the experts won't separate if such deviation occurs. The experts are incentivized to separate since if they do not the decision maker switches to the myopically optimal actoins. The separable groups are chosen in such a way that utility losses from the myopically optimal actions are higher for the expert if he stays in the group rather than separate from the other types (which can be done if types are far enough and the bias is not too big).

From this informal description of our fully revealing equilibrium construction one can see the main ingredients we need. First the division of the state space into separable groups requires complex signalling rules, in the sense that disconnected sets of states send the same recommendation. Second, actions have to be observable in order for the expert to be able to punish the decision maker who deviates from his recommendations. This second point is more subtle, but we prove that it is essential. In Theorem 4 we show that no matter how long the game lasts, there is no way (no matter how complex) to make it incentive compatible for the all types of experts to join separable groups, establishing that when actions are not observable it is impossible to construct a fully revealing equilibrium.

Theorem 3 Suppose Assumption 1 holds and $\delta_E = \delta_{DM} = 1$.Then there exists b^* such that for any $b < b^*$ and $T \geq 4$, there is a fully revealing equilibrium.

Proof. In the appendix. ■

5.1 Proof Outline and Intuition

In the appendix, we first construct a 4-period fully revealing equilibrium which works for any bias $b < \frac{1}{19.718}$. The idea is to initially pair each expert type $\theta \in [0,1]$ with exactly one "partner", a far-away type $p(\theta)$. For a specified number of periods (the "pre-separation" phase), the pair $\{\theta, p(\theta)\}$ recommends a sequence of actions to the decision-maker, from which he can infer that the true state is either θ or $p(\theta)$. If the decision-maker follows the expert's advice in the pre-separation phase, he is rewarded with the truth: following the pre-separation phase, types $\{\theta, p(\theta)\}$ separate, sending different recommendations which reveal the true state. If, however, the decision-maker rejects the expert's advice, and chooses (in some period) a different action than that recommended by types $\{\theta, p(\theta)\}$, then the expert refuses to provide any further information after the prescribed pre-separation phase, so the decision-maker remains uncertain about whether the true state is θ or $p(\theta)$.So, we simply need to first choose action recommendation functions which incentivize truth-telling by the expert - i.e., such that each pair $\{\theta, p(\theta)\}$ prefers to send the prescribed recommendations in the pre-separation phase, rather than mimic the advice of any other pair $\{\theta', p(\theta')\}$. This advice will not be myopically optimal from the DM's perspective (that is, if the DM always chooses an action equal to the expected type among those in his information set, then full revelation is impossible - we prove this in the following section on MPBE). However, the DM knows that if he fails to choose a recommended action, then the expert will never reveal the true state. Therefore, we need to make sure that at each information set $\{\theta, p(\theta)\}$, the DM prefers to choose the recommended actions and eventually learn the truth, than to choose the expected type conditional on this information set (in which case, he never learns whether the true state is θ or $p(\theta)$). Finally, we need to make sure that types $\{\theta, p(\theta)\}$ are willing to separate following the pre-separation phase: this requires that they be sufficiently far apart that type θ prefers to induce the action θ to the action $p(\theta)$, and type $p(\theta)$ has the reverse preference.

More precisely, the strategies in our construction for bias b are as follows, for two parameters $x_1 \geq 2$ and $\gamma \geq x_1$ which will be explained following the description of the strategies:

Expert Strategy: Each type $\theta \leq x_1 b$ initially pools with the type $\theta + \gamma b$, and each type $\theta \in [x_1 b, \gamma b]$ initially pools with the type $h\left(\frac{\theta}{b}\right) b$, where $h(\frac{\theta}{b}) \equiv \gamma + x_1 - 1 + \sqrt{\frac{\theta}{b} + 1 - x_1}$. The pair $\{\theta, \theta + \gamma b\}$ recommends action $u_1\left(\frac{\theta}{b}\right) b$ in period 1, then $u_2\left(\frac{\theta}{b}\right) b$ in period 2; if the DM chooses all actions that the expert recommends, then these two types separate in period 3; while, if the DM rejects advice and chooses an action other than $u_t\left(\frac{\theta}{b}\right)$ in some period $t \in \{1, 2\}$, the two types continue to pool together, both sending the recommendation θ in periods 3,4. Each pair $\{\theta, h\left(\frac{\theta}{b}\right) b\}$ recommends the action $v_1\left(\frac{\theta}{b}\right) b$ in period 1, then $v_2\left(\frac{\theta}{b}\right) b$ in both periods 2 and 3. If the DM follows all recommended actions, the two types separate in the final period (4); if he rejects advice, they remain pooled together for the last period also, both recommending action θ.

Decision-Maker's Strategy: Choose the recommended actions in all periods, and ignore any out-of-equilibrium messages as uninformative noise. In particular: (i) if a recommendation $u_1\left(\frac{\theta}{b}\right) b$ is followed by a recommendation $y_2 \neq u_2\left(\frac{\theta}{b}\right) b$, believe that the 2nd message is a mistake, and continue to assign probability 1 to the information set $\{\theta, \theta + \gamma b\}$; (ii) similarly if a recommendation $v_1\left(\frac{\theta}{b}\right) b$ is followed by a recommendation y_2 or $y_3 \neq v_2\left(\frac{\theta}{b}\right) b$, believe that the 2nd message is a mistake, and assign probability 1 to the information set $\{\theta, h\left(\frac{\theta}{b}\right) b\}$; (iii) after an out-of-equilibrium initial message (i.e, if the 1st-period recommendation lies outside of the range $[u_1(0)b, u_1(x_1) b] \cup [v_1(x_1)b, v_1(\gamma)b]$), assign probability 1 to the information set $\{0, \gamma b\}$, and subsequently behave as if the initial message was sent by this pair; (iv) after an out-of-equilibrium message in the separation phase, assign probability 1 to the lowest type in the current information set. (In particular, if the DM himself deviated in the pre-separation phase - for example, if he received an initial recommendation $u_1\left(\frac{\theta}{b}\right) b$ but chose some other action, in which case the expert's strategy is now to send the message θ regardless of whether the true state is θ or $\theta + \gamma b$ - then the DM assigns probability 1 to state θ even if the expert deviates to some message other than θ).

With these strategies, the "partner" function $p(\theta)$ described in the first paragraph of this outline is therefore $p(\theta) = \begin{cases} \theta + \gamma b & \text{if } \theta \in [0, x_1 b] \\ \left(\gamma + x_1 - 1 + \sqrt{\frac{\theta}{b} + 1 - x_1}\right) b & \text{if } \theta \in [x_1 b, \gamma b] \end{cases}$; note that $x_1 \geq 2$ and $\gamma \geq$

x_1 are then sufficient to imply that types $\theta, p(\theta)$ are always at least 2b units apart, which implies that they strictly prefer to separate from each other. (Nevertheless, following a deviation by the DM in the pre-separation phase, it is optimal for the two types to continue to pool together, as the DM's strategy is to not believe any attempts to separate: if he has received the advice prescribed for types $\{\theta, p(\theta)\}$, but deviated from following this advice in the pre-separation phase, he expects to receive the message θ, and assigns probability 1 to true state θ even if he receives an out-of-equilibrium message $m \neq \theta$).

The first step of the proof is to choose the functions u_1, u_2, v_1, v_2 (the recommended actions described in the expert's strategy) such that in the pre-separation phase, no expert type wishes to mimic the advice of another type. For example: if type $\theta \in [0, x_1 b]$ initially sends the recommendation $u_1(z)\, b$ for some $z \in [0, x_1]$, his expected payoff, noting that this will lead to the action sequence $(u_1(z)b, u_2(z)b, zb, zb)$, is

$$b^2 \cdot \left[\left(u_1(z) - \frac{\theta}{b} - 1 \right)^2 + \left(u_2(z) - \frac{\theta}{b} - 1 \right)^2 + 2\left(z - \frac{\theta}{b} - 1 \right)^2 \right]$$

Equilibrium requires that, for all $\theta \in [0, x_1 b]$, this expression is maximized at $z = \frac{\theta}{b}$, resulting in the following differential equation:[4]

$$u_1'\left(\frac{\theta}{b}\right)\left(u_1(z) - \frac{\theta}{b} - 1\right) + u_2'\left(\frac{\theta}{b}\right)\left(u_2(z) - \frac{\theta}{b} - 1\right) - 2 = 0 \qquad (*)$$

We obtain analogous differential equations to make sure that no type $\theta + \gamma b \in [x_1 b, (x_1+\gamma)b]$ wishes to mimic any other type from this interval (a second differential equation involving u_1, u_2), no type $\theta \in [xb, \gamma b]$ wishes to mimic any other type wishes to mimic any other type from this interval (differential equation involving v_1, v_2), and no type $\theta \in [(x_1+\gamma)b, 1]$ wishes to mimic any other type from this interval (also involving v_1, v_2). This pins down the functional forms for u_1, u_2, v_1, v_2. To make sure that no type wishes to mimic a type drawn from another interval, we make the three types at interval endpoints - namely, types $x_1 b, \gamma b$, and $(x_1+\gamma)b$ - indifferent between the strategies prescribed at the right endpoint of the interval to their left, and the left endpoint of the interval of their right. The resulting three indifference equations pin down all four action functions up to one constant (k_1 in the appendix), and also imply a relationship which must hold between x_1 and γ. Then, setting the highest type $\left(\gamma + x_1 - 1 + \sqrt{\gamma + 1 - x_1}\right) b$ in our construction equal to 1 determines our values of x_1, γ as functions of the bias b; the (separation) constraint $x_1 \geq 2$ determines the largest bias b for which our construction works.

It remains only to show that with our chosen partner function $p(\theta)$, the DM finds it optimal to follow the advice provided by types $\{\theta, p(\theta)\}$ (for all $\theta \in [0, \gamma b]$). Since the expert only reveals information in periods 1 and in the period immediately following the pre-separation phase (period

[4] This is the F.O.C., obtained by differentiating w.r.t. z, evaluating at $z = \frac{\theta}{b}$, and setting the resulting expression equal to zero. In the appendix, we show that this F.O.C. is in fact both necessary and sufficient to imply that no type $\theta \in [0, x_1 b]$ wishes to mimic any other type θ' from this interval.

3 for pairs $\{\theta, p(\theta)\}$ with $\theta \in [0, x_1 b]$ and period 4 for pairs $\{\theta, p(\theta)\}$ with $\theta \in [x_1 b, \gamma b]$), and a deviation at any time results in types $\{\theta, p(\theta)\}$ remaining pooled together in all four periods, the strongest incentive to deviate is in period 1, and the best possible deviation is to choose the expected type (conditional on information set $\{\theta, p(\theta)\}$) in all four periods. Therefore, we need to show that at all information sets $\{\theta, p(\theta)\}$, the DM's equilibrium payoff is higher than what he would obtain by choosing the expected type in all four periods. In the appendix, we show that this yields an upper bound on the constant k_1 (the only thing which was not pinned down by the expert's incentive compatibility conditions); choosing any constant k_1 below this upper bound then generates a fully revealing equilibrium for any bias $b < \frac{1}{19.718}$.

We assume that at the information set $\{\theta, p(\theta)\}$, the DM assigns probabilities $\left(\frac{1}{1+p'(\theta)}, \frac{p'(\theta)}{1+p'(\theta)}\right)$ to types $(\theta, p(\theta))$: this is so that the relative weight placed on type θ to $p(\theta)$ is (with μ the uniform distribution) is

$$\frac{\Pr(\theta)}{\Pr(p(\theta))} = \lim_{\varepsilon \to 0} \frac{\mu\left([\theta - \varepsilon, \theta + \varepsilon]\right)}{\mu\left([p(\theta - \varepsilon), p(\theta + \varepsilon)]\right)}$$

Intuitively, this is because if $p'(\theta) < 1$, then the interval of $[\theta_1, \theta_2]$ is larger than its partner "interval" $[p(\theta_1), p(\theta_2)]$; so, in essence we are pairing each type in the smaller interval with several copies of a type in the larger interval, and therefore assign a smaller probability weight $p'(\theta)$ to the type $p(\theta)$. Note also that under this assumption, the DM's expected equilibrium payoff is

$$\int_0^{\gamma b} \left(1 + p'(\theta)\right) \cdot E\left[u^{DM}(x) \mid x \in \{\theta, p(\theta)\}\right] d\theta$$

It can be proven (though we do not show it in the paper) that among all fully revealing equilibria in which each expert type pools with exactly one other type, at least two distinct actions must be chosen by each type in the pre-separation phase, and the partner function must be non-linear over some interval. Our specific choice of the function h was chosen to work conveniently for a particular range of biases, but is not the only function that works. Moreover, we have constructed examples of equilibria which work with linear partner functions, but all such equilibria require "bunching" expert types into some initial groups containing more than two types (i.e., some expert types initially send the same advice as at least three other types, rather than having a single partner). In other words, we do not claim that full revelation is impossible for $b > \frac{1}{19.718}$, only that the equilbrium construction for larger biases is more complicated and requires a different choice for the function h (which correspondingly changes the advice functions v_1, v_2). However, there is an upper bound on the bias b, above which a fully revealing equilibra do not exist (in particular, full revelation is impossible for $b > \frac{1}{4}$).

Our equilibrium requires exactly 4 periods to implement. If $T > 4$, full revelation can be achieved via our construction either by starting out with a babbling phase (and playing our equilibrium in the final four periods), and/or for T a multiple of 4, by scaling up the number of periods in our construction proportionally. (For example, if $T = 8$, our construction would work if the action functions u_1, u_2, v_1 are each recommended for two rather than one periods, v_2 for four rather than two periods, and separation occuring in period 5 for pairs $\{\theta, p(\theta)\}$ with $\theta \leq x_1 b$, period 7 for pairs $\{\theta, p(\theta)\}$ with $\theta \in [x_1 b, \gamma b]$.)

6. Markov perfect equilibria

Our fully revealing construction in the previous section is based on trigger strategies. In the early periods of the game, the expert sends a message which reveals a finite set of possibilities for the true state. The first step of our construction proves that it is possible to choose action functions which provide the expert with a sufficient incentive to eventually reveal the truth exactly. These action functions typically require that the decision-maker initially choose actions which are far away from the expected state given the finite set of possibilities in his information set, and hence cannot be part of a static or dynamic Markov equilibrium. However, we can sustain these action functions in a dynamic non-Markovian equilibrium, as outlined above. The decision maker is willing to choose the prescribed actions because if he does not, the expert will stop revealing information. Hence, we need a large enough horizon that the reward to the decision maker from choosing the prescribed action functions (ultimately learning the exact state, hence reducing his disutility to zero for the last part of the game) exceeds the cost (initially choosing actions which differ from the myopically optimal choice, in order to provide the expert with truth-telling incentives).

Our next theorem highlights the importance of having the expert choose actions that differ from the ones that maximize his current payoff given his posterior beliefs. We prove that full revelation information is impossible even in a dynamic setting when we restrict attention to Markov Perfect Bayesian equilibria. In order to establish this result, we show that no matter how long the game lasts, and no matter how sophisticated signalling the expert employs, it is never possible to incentivize the expert for all states to join a separable group if the decision maker chooses actions that are equal to the myopic best responses.

Definition 3 *A* Markov Perfect Bayesian equilibrium (MPBE) *is a PBE in which the decision maker's action in any period depends only on his beliefs about the expert's type, and the expert's advice depends on this common belief, and on the true state.*

In a MPBE, the decision maker's actions have no effect on the expert's future advice, and therefore no effect on the DM's continuation payoff. Therefore, in a MPBE, the uniquely optimal strategy for the DM is to choose the myopically best action in every period: that is,

$$y_t(m^t) \in \arg\max E[u^{DM}(y_t, \theta)|\mu_t] \tag{9}$$

where μ_t denotes the DM's period-t belief about expert's type. In the case of quadratic utility, this implies that the DM will always choose an action equal to his expectation of the state, conditional on the messages he has received. Our next theorem states that full information revelation is impossible when we restrict to MPBE (or, more generally, when the DM's actions depend only on his beliefs about the state[5]):

Theorem 4 *For any finite T and $\delta > 0$, there exists no fully revealing MPBE when utility is quadratic.*

Proof: in appendix.

6.1 Proof Sketch

The basic idea is as follows: if the DM always sets his action equal to his expectation of the type, then his first-period action $a_t(\theta)$ in state θ depends only on the set of types who (at equilibrium) send the same initial message as type θ; and in subsequent periods, the action $a_t(\theta)$ depends on which of these initial partners have not yet dropped out. (Note that a type who does not initially pool with θ cannot subsequently join the pool, as he has already separated from type θ). Our proof shows that for any "partnering structure" (specifying which types pool together in the initial round, and the pattern in which they subsequently separate), the resulting myopically optimal actions are incompatible with the incentive compatipbility constraints required to induce full information revelation by the expert.

To illustrate the incompatibility, consider the following simple example with $T = 3$ and $\delta = 1$: suppose that each type θb sends the same 1st-period message as exactly two other types, call these $g_1(\theta)b, g_2(\theta)b$. And moreover, suppose that these "partner functions" $g_1(\cdot), g_2(\cdot)$ are continuous and differentiable around some point θ, and that types separate according to the following structure: in period 2, $\{\theta b, g_1(\theta)b\}$ again pool together (for all θ), while $g_2(\theta)b$ splits off by sending a different message; finally, in period 3, θb and $g_1(\theta)b$ separate. Then the local IC constraints for nearby expert types to not mimic each other (analogous to equation (*) in Section 5.1) around types $\theta, g_1(\theta), g_2(\theta)$ are:

$$a_1'(\theta)\left(a_1(\theta)-\theta-1\right)+a_2'(\theta)\left(a_2(\theta)-\theta-1\right)=1 \qquad (\theta)$$

$$a_1'(\theta)\left(a_1(\theta)-g_1(\theta)-1\right)+a_2'(\theta)\left(a_2(\theta)-g_1(\theta)-1\right)=g_1'(\theta) \qquad (g_1)$$

$$a_1'(\theta)\left(a_1(\theta)-g_2(\theta)-1\right)=2g_2'(\theta) \qquad (g_2)$$

Now, denote the DM's probability distribution (belief) over the types $(\theta, g_1(\theta), g_2(\theta))$ in his information set by $\left(\frac{p_0}{p_0+p_1+p_2}, \frac{p_1}{p_0+p_1+p_2}, \frac{p_2}{p_0+p_1+p_2}\right)$. Multiply the IC equations for types $\theta, g_1(\theta), g_2(\theta)$ by (respectively) $\frac{p_0}{p_0+p_1+p_2}, \frac{p_1}{p_0+p_1+p_2}, \frac{p_2}{p_0+p_1+p_2}$, and take the weighted sum, to obtain

$$a_1'(\theta)\left(a_1(\theta)-E[\theta|\{\theta,g_1(\theta),g_2(\theta)\}]-1\right)+\frac{(p_0+p_1)}{(p_0+p_1+p_2)}a_2'(\theta)\left(a_2(\theta)-E[\theta|\{\theta,g_1(\theta)\}]-1\right)=\frac{p_0+p_1g_1'+2p_2g_2'}{(p_0+p_1+p_2)}$$

Now, since the DM (in a MPBE) always sets his action equal to the expected state, the LHS of this equation reduces to

$$a_1'(\theta)(-1)+\frac{(p_0+p_1)}{(p_0+p_1+p_2)}a_2'(\theta)(-1)$$

[5] For example, this restriction would also apply if the expert is unable to observe the actions chosen by the DM.

For the RHS, note that our equations $a_1(\theta) = E[\theta|\{\theta, g_1(\theta), g_2(\theta)\}]$ and $a_2(\theta) = E[\theta|\{\theta, g_1(\theta)\}]$ imply further that

$$\frac{p_1 g_1'(\theta)}{(p_0+p_1+p_2)} = \frac{(p_0+p_1)}{(p_0+p_1+p_2)} a_2'(\theta) - \frac{p_0}{(p_0+p_1+p_2)}$$
$$\frac{p_2 g_2'(\theta)}{(p_0+p_1+p_2)} = a_1'(\theta) - \frac{(p_0+p_1)}{(p_0+p_1+p_2)} a_2'(\theta)$$

Substituting in, our weighted IC equation becomes

$$a_1'(\theta)(-1) + \tfrac{(p_0+p_1)}{(p_0+p_1+p_2)} a_2'(\theta)(-1) = 2a_1'(\theta) - \tfrac{(p_0+p_1)}{(p_0+p_1+p_2)} a_2'(\theta)$$
$$\Rightarrow a_1'(\theta)(-1) = 2a_1'(\theta), \text{ or } a_1'(\theta) = 0$$

Now plug this back into the equation (g_2), obtaining $0 = 2g_2'(\theta)$, which (using $\frac{p_2 g_2'(\theta)}{(p_0+p_1+p_2)} = a_1'(\theta) - \frac{(p_0+p_1)}{(p_0+p_1+p_2)} a_2'(\theta)$ and $a_1'(\theta) = 0$) further implies $a_2'(\theta) = 0$. Finally, substituting $a_1'(\theta) = a_2'(\theta) = 0$ into the IC equation for θ (equation (θ)) yields the contradiction $0 = 1$.

The general proof in Appendix B follows exactly this idea. Steps 1 and 2 prove that while expert types of course need not, in general, be partnered with equal numbers of other types, all drawn according to continuous differentiable functions like $g_1(\cdot), g_2(\cdot)$, it is always possible to find some point θ around which the partner functions are "approximately" continuous and differentiable (this notion is made more precise in the appendix); then, in Step 3, we obtain a contradiction analogous to the one in this sketch, establishing that no matter how the partner functions are chosen, the resulting myopically optimal action functions are inconsistent with the IC constraints required to induce full information revelation by the expert. ■

The previous theorem showed that as long as both agents are patient, no fully revealing MPBE exists. In subsection 4.1 we constructed an example that this conclusion fails if the expert is myopic ($\delta = 0$): With an impatient expert it is possible to have a fully revealing MPBE.

To get a better sense of why we can get full a fully revealing MPBE with $\delta = 0$ but not when $\delta > 0$, even if δ is arbitrarily small, it may helpful to step back and examine the conditions that need to be satisfied in order for a fully revealing equilibrium to exist. Two sets of conditions have to be satisfied. The first set of conditions comes from the best response constraints for the expert. The second one arises from the best response constraints of the decision maker. Now, recall that full information revelation requires the division of states into separable groups. For the decision maker, the set of feasible separable groups together with the restrictions imposed by MPBE on the decision maker's behavior imply a set of implementable actions. For the expert, for any given δ, the induced action has to give him incentive to join a separable group. For all $\delta > 0$, these two conditions cannot be satisfied simultaneously: for the actions that are equilibrium feasible for the decision maker, the expert has an incentive to deviate. The only situation where these two conditions can be satisfied simultaneously is when $\delta = 0$ for a certain range of biases.

7. Pareto comparisons of dynamic cheap talk equilibria

In this section we show another important difference between the dynamic and the static cheap talk games in terms of comparing welfare. In static case Crawford and Sobel (1982) show that under their Assumption M equilibria can be easily Pareto ranked: both the expert and the decision maker ex-ante (before the state is realized) prefer the equilibrium with the highest number of partitions and actions induced. We show that Pareto comparisons in dynamic cases are less straightfoward.

Another stark prediction of a static cheap talk model is that ex-ante welfare of both the expert and the decision maker is strictly higher in equilibrium with more partitions. The following examples demonstrate that this is not true in dynamic settings.

Example 4 *Equilibria with more partitions in each period can be Pareto inferior to the equilibria with fewer partitions*

Take $\delta_E = \delta_{DM} = 1$ and $b = 0.08$, and consider the most informative static partition equilibrium where the number of partitions is $p = 3$. At this equilibrium the state space is divided into $[0, 0.013]$, $[0.013, 0.347]$ and $[0.347, 1]$. The corresponding optimal actions of the decision maker are given by

$$y_1 = 0.0067 \quad y_2 = 0.18 \quad y_3 = 0.673,$$

from which we can calculate the ex-ante expected utility levels for the expert -0.032 and for the decision maker -0.0263. This implies that at the equilibrium of the dynamic game that induces the same partition as this most informative equilibrium by playing the most informative at $t = 1$ and babbling thereafter, the total expected utility is -0.065 for the expert, and -0.053 for the decision maker.

Now we construct a dynamic equilibrium where the type space is subdivided into more subintervals, but both players' ex-ante expected payoffs are lower. We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0, \theta_1] \text{ send message sequence } (m_{1(1)}, m_{2(1)})\\
&\text{types in } [\theta_1, \theta_2] \text{ send message sequence } (m_{1(2)}, m_{2(2)})\\
&\text{types in } [\theta_2, \theta_3] \text{ send message sequence } (m_{1(2)}, m_{2(3)})\\
&\text{types in } [\theta_3, 1] \text{ send message sequence } (m_{1(3)}, m_{2(4)}).
\end{aligned}$$

So types are partitioned into four intervals in stage 2, but in stage 1, the types in $[\theta_1, \theta_2]$ and $[\theta_2, \theta_3]$ pool together to send the same message $m_{1(2)}$. Since the signaling rule does not depend on the decision maker's action at stage 1, the decision maker will choose the following myopically optimal actions:

$$
\begin{aligned}
y_{1(1)} &= y_{2(1)} = \frac{\theta_1}{2} \\
y_{1(2)} &= \frac{\theta_1+\theta_3}{2}, \ y_{2(2)} = \frac{\theta_1+\theta_2}{2}, \ y_{2(3)} = \frac{\theta_2+\theta_3}{2} \\
y_{1(3)} &= y_{2(4)} = \frac{1+\theta_3}{2}.
\end{aligned}
$$

After any out-of-equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of-equilibrium beliefs it is immediate to see that no type has an incentive to send an out-of equilibrium message.

This is an equilibrium provided that type θ_1 is indifferent between action sequences $\{y_{1(1)}, y_{2(1)}\}$ and $\{y_{1(2)}, y_{2(2)}\}$, type θ_2 is indifferent between 2nd-period actions $y_{2(2)}$ and $y_{2(3)}$, and type θ_3 is indifferent between action sequences $\{y_{1(2)}, y_{2(3)}\}$ and $\{y_{1(3)}, y_{2(4)}\}$. Therefore, equilibrium cutoffs are the solution to the following system of equations:[6]

$$
\begin{aligned}
2\left(\frac{\theta_1}{2}-\theta_1-b\right)^2-\left(\frac{\theta_1+\theta_3}{2}-b-\theta_1\right)^2-\left(\frac{\theta_1+\theta_2}{2}-b-\theta_1\right)^2 &= 0 \\
\left(\frac{\theta_1+\theta_2}{2}-b-\theta_2\right)^2-\left(\frac{\theta_2+\theta_3}{2}-b-\theta_2\right)^2 &= 0 \\
2\left(\frac{1+\theta_3}{2}-b-\theta_3\right)^2-\left(\frac{\theta_1+\theta_3}{2}-b-\theta_3\right)^2-\left(\frac{\theta_2+\theta_3}{2}-b-\theta_3\right)^2 &= 0
\end{aligned}
$$

At $b = 0.08$, the only solution that gives numbers in $[0,1]$ is $\theta_1 = 0.0056, \theta_2 = 0.015, \theta_3 = 0.345$, and the actions induced are given by

$$
\begin{aligned}
&y_{1(1)} = y_{2(1)} = 0.00278, \quad y_{1(2)} = 0.175\,, \quad y_{2(2)} = 0.0105 \\
&y_{2(3)} = 0.18 \quad y_{1(3)} = y_{2(4)} = 0.673\ .
\end{aligned}
$$

This implies the following total ex-ante expected utility for the expert -0.066, which is lower than $2(-0.033) = -0.0656$. The utility for the decision-maker is -0.053 which is gain lower than $2(-0.026) = 0.052$.

This example illustrates that although the interval is divided into more subintervals here, *both* players strictly lose compared to the one where the most informative static equilibrium is played in the first period and babbling thereafter. The feature that less partitions lead to higher ex-ante welfare for both players also appears in example 1 of Blume, Board and Kawamura (2007).

In the previous sections we showed that dynamic games allow for the equilibria that do not exist in static cheap talk game - non-monotone equilibria. The following example shows that such

[6] It is trivial to check exactly as we did in previous examples that these indifference conditions suffice for global incentive compatibility.

equilibria can be Pareto superior to the best monotone partition equilibrium.

Example 5 *Non-monotonic equilibria can be Pareto superior to all monotonic equilibria*

Consider the construction in Example 2**.** Direct calculations show that the expert's and the decision maker's ex-ante expected total welfare at this non-monotonic equilibrium for bias $b = 0.256$ are respectively given by

$$\text{Expert} = -0.275\,, \text{ and Decision Maker} = -0.144\,.$$

For this bias, the only static equilibrium is the babbling equilibrium, which implies by Proposition 2 that the only dynamic monotonic partition equilibrium is the babbling equilibrium. In this equilibrium the decision maker takes the action $y^B = 0.5$ and the expert's and the decision maker's welfare are respectively given by:

$$\text{Expert} = -0.298 \text{ and Decision Maker} = -0.167.$$

Both the expert and the decision maker are strictly better-off at our non-monotonic equilibrium compared to the most informative static equilibrium.

Example 6 *Fully revealing equilibria can be Pareto inferior to monotonic partition equilibria*

In appendix A, following the proof that fully revealing equilibria are possible, we provide the details for an equilibrium at bias $b = \frac{1}{22.558}$. We now note that full information revelation may be welfare-decreasing for *both* players, by comparing the welfare properties of our equilibrium with the best static equilibrium.

At bias $b = \frac{1}{22.558}$, the best static equilibrium has 3 partitions, with cutoffs $\{0, \frac{1}{3} - \frac{4}{22.558}, \frac{2}{3} - \frac{4}{22.558}, 1\}$. If we repeat this equilibrium 4 times, the DM's expected disutility is

$$4\left(\int_0^{\frac{1}{3}-\left(\frac{4}{22.558}\right)}\left(\frac{\frac{1}{3}-4\left(\frac{1}{22.558}\right)}{2}-x\right)^2 dx+\int_{\frac{1}{3}-\frac{4}{22.558}}^{\frac{2}{3}-\frac{4}{22.558}}\left(\frac{1-\frac{8}{22.558}}{2}-x\right)^2 dx+\int_{\frac{2}{3}-\frac{4}{22.558}}^{1}\left(\frac{\frac{5}{3}-\frac{4}{22.558}}{2}-x\right)^2 dx\right)$$
$$= 0.058$$

and the expert's expected disutility is 0.066 (obtained by replacing x with $\left(x + \frac{1}{22.558}\right)$ in the above expression).

Now, consider the fully revealing equilibrium: in our construction, each pair of types$\{\frac{x}{22.558}, \frac{x}{22.558} + \frac{13.09}{22.558}\}$ with $x \in [0, 8]$ pool together for two periods, with the DM choosing action $\frac{u_1(x)}{22.558}$ in period 1, $\frac{u_2(x)}{22.558}$ in period 2, then learning the truth in period 3, where $u_1(x) = 12.833 - \sqrt{2x + 2.4669}$, $u_2(x) = 12.833 + \sqrt{2x + 2.4669}$. This contributes the following amount to the DM's total expected disutility:

$$\int_0^8 \left(\begin{array}{c} \left(\frac{12.833-\sqrt{2x+2.4669}-x}{22.558}\right)^2 + \left(\frac{12.833+\sqrt{2x+2.4669}-x}{22.558}\right)^2 \\ + \left(\frac{12.833-\sqrt{2x+2.4669}-x-13.09}{22.558}\right)^2 + \left(\frac{12.833+\sqrt{2x+2.4669}-x-13.09}{22.558}\right)^2 \end{array} \right) \frac{dx}{22.558} = 0.17806$$

Note that this is a *lower bound* on the DM's disutility, obtained by only counting the disutility over the intervals $[0, \frac{8}{22.558}] \cup [\frac{13.09}{22.558}, \frac{21.09}{22.558}]$, and setting disutility to zero if the true state lies outside of these intervals (the disutility over the omitted intervals is in fact approximately 0.05).

Similarly, a lower bound on the expert's ex ante expected disutility (lower bound again obtained by setting disutility over the excluded intervals $[\frac{8}{22.558}, \frac{13.09}{22.558}] \cup [\frac{21.09}{22.558}, 1]$ to zero, then replacing x above with $\left(x + \frac{1}{22.558}\right)$, and noting that in the final 2 periods, the fully informed DM's disutility is zero, while the expert's disutility is $2\left(\frac{1}{22.558}\right)^2$) is

$$2\left(\frac{1}{22.558}\right)^2 + \int_0^8 \left(\begin{array}{c} \left(\frac{12.833-\sqrt{2x+2.4669}-x-1}{22.558}\right)^2 + \left(\frac{12.833+\sqrt{2x+2.4669}-x-1}{22.558}\right)^2 \\ + \left(\frac{12.833-\sqrt{2x+2.4669}-x-13.09-1}{22.558}\right)^2 + \left(\frac{12.833+\sqrt{2x+2.4669}-x-1-13.09}{22.558}\right)^2 \end{array} \right) \frac{dx}{22.558} = 0.172\,02$$

So, both players are almost four times worse off in the fully revealing equilibrium than in the best static equilibrium.

This is not that surprising: in order to induce eventual full information revelation, each expert type initially pools with some other far-away type (in our above construction, type $\frac{x}{22.558}$ initially pools together with $\frac{x+13.09}{22.558}$). This information is actually much less useful to the DM than what he learns in the best static equilibrium: even if the DM were to choose the myopically optimal action in the pre-separation periods (and in fact, he is forced to choose a non-myopically optimal action in order to provide the expert with appropriate truth-telling incentives), the myopically optimal action at information set $\{\frac{x}{22.558}, \frac{x+13.09}{22.558}\}$ is quite far away from both types in the information set; and, in particular, he DM would do better if he instead knew that the true state was contained in some small ε-neighborhood around $\frac{x}{22.558}$. For small biases, the best static equilibrium (in which the DM always learns a small ε-interval containing the truth) therefore provides more useful information than our fully revealing equilibrium, and also results in an action choice which, on average, is closer to the expert's bliss point. Therefore, for small biases, fully revealing equilibria tend to be Pareto-inferior to the best static equilibrium.

8. Concluding Remarks

In this paper we argue that dynamic cheap talk games are fundamentally different from their static counterparts. Despite the difficulty of the analysis in the dynamic setting we presented general results on constructing fully revealing equilibria and on the properties of Markov and monotonic equilibria. We also presented a series of examples that show how the canonical results of the static cheap talk games may not hold in the dynamic settings.

The main novel ingredient of our model is that there are many actions chosen, one at a time. Prior to each decision made there is a new round of communication. The dynamic nature of actions makes the expert's best response constraints dynamic. The dynamic considerations of the expert allow us to group together types that are far apart, forming "separable" groups which is the key ingredient behind the construction of fully revealing equilibria and of non-monotonic equilibria. They also allow us to use trigger strategies. The intertemporal considerations thus enrich the equilibrium set quite dramatically in quite complex ways. This complexity is what makes general Pareto rankings impossible in contrast to the static case.

Our analysis allows us to draw some conclusion about the nature of incentive compatibility in dynamic incentive settings. One important characteristic is that dynamic incentive constraints can bind for a disconnected set of types. Also the fact that posteriors are endogenous, implies that earlier rounds in the game can be used to induce posteriors that eventually allow the decision maker to learn the truth.

9. Appendix A: Proof of Theorem 3

In this section, we construct a fully revealing equilibrium for $T = 4$ which works for any bias $b < \frac{1}{19.718}$. We will follow the notation and strategies described in Section 5.1.

9.1 Expert Incentive Compatibility

Here, we find exact expressions for all four action recommendation functions u_1, u_2, v_1, v_2 described in Section 5.1, defining $h(x) \equiv \gamma + x_1 - 1 + \sqrt{x + 1 - x_1}$. Outline: our construction partitions the type space $[0, 1]$ into 4 intervals:

$$[0, x_1 b] \cup [x_1 b, \gamma b] \cup [\gamma b, (x_1 + \gamma)b] \cup [(x_1 + \gamma)b, 1]$$

Each type θ in the first interval pairs with type $\theta + \gamma b$ from the 3rd interval, to recommend the action functions $u_1(\frac{\theta}{b})b$ in period 1, then $u_2(\frac{\theta}{b})b$ in period 2, then separate in period 3: to make sure that truth-telling is better than mimicking any other types from the 1st, 3rd intervals, we choose u_1, u_2 to satisfy the differential equations in (A1),(A2). The solution (up to two constants k_1, C_1) is given by (A3), and we explain in the paragraph following (A5) why these (necessary) differential equations are also sufficient. Each type θ in the 2nd interval initially pairs up with the type $h\left(\frac{\theta}{b}\right) b$ from the 4th interval, to recommend $v_1(\frac{\theta}{b})b$ for one period, then $v_2\left(\frac{\theta}{b}\right) b$ for two periods, then separate in period 4. The conditions for optimality of truth-telling, rather than mimicking any other types from the 2nd, 4th intervals, reduce to differential equations (A6), (A7). To make sure that no type wishes to mimic a type from another interval, we need to make types $x_1 b, \gamma b$, and $(x_1 + \gamma)b$ indifferent between the two action sequences that they can induce: the resulting three indifference equations imply a relationship that must hold between x_1, γ (equation (A21)), as well as initial conditions for the functions v_1, v_2 (equations (A8),(A9)). Using the initial conditions from (A8),(A9), the solutions $v_1(\cdot), v_2(\cdot)$ to the differential equations in (A6),(A7) are given by (A12), (A13). Choosing constants k_1, C_1 to satisfy (A17),(A18) guarantee that the action functions in (A3),(A12),(A13) are real-valued, thus completing the construction of an equilibrium which makes truth-telling optimal for the expert in the pre-separation phase. Finally, for optimality of truth-telling in the separation period, it is sufficient to choose $x_1 \geq 2$ and $\gamma \geq x_1$ (equation (A22).

To calculate $\mathbf{u}_1$,$\mathbf{u}_2$:

The disutility to type $xb \in [0, x_1 b]$ from mimicking type $zb \in [0, x_1 b]$ is

$$b^2 \cdot \left[(u_1(z) - x - 1)^2 + (u_2(z) - x - 1)^2 + 2\,(z - x - 1)^2 \right] \tag{A0}$$

Therefore, in order for type xb with $x \in [0, x_1]$ to find it optimal to send the prescribed recommendations, rather than mimicking the advice of some other type $zb \in [0, x_1 b]$, the derivative of (A0) w.r.t. z must equal zero when evaluated at $z = x$. This yields the following differential equation:

$$u_1'(x)\,(u_1(x) - x - 1) + u_2'(x)\,(u_2(x) - x - 1) = 2 \tag{A1}$$

Similarly, in order for type $(x+\gamma)b$ to follow the prescribed strategy, rather than mimicking some other type $(z+\gamma)b \in [\gamma b, x_1 b + \gamma b]$, we need

$$u_1'(x)\left(u_1(x) - x - \gamma - 1\right) + u_2'(x)(u_2(x) - x - \gamma - 1) = 2 \tag{A2}$$

Substitute (A1) into (A2), and it becomes

$$\begin{aligned}
&2 - \gamma\left(u_1'(x) + u_2'(x)\right) = 2\\
&\Rightarrow \left(u_1'(x) + u_2'(x)\right) = 0, \text{ so } u_1(x) + u_2(x) \text{ must be constant}
\end{aligned}$$

Define $k_1 \equiv \frac{u_1(0)+u_2(0)}{2}$ (initial values to be determined from the DM's incentive compatibility constraints), so that $u_2(x) = 2k_1 - u_1(x)$. Substitute this back into (A1), and it becomes

$$u_1'(x)\left(u_1(x) - k_1\right) = 1$$

This is solved by $u_1(x) = k_1 \pm \sqrt{2x + C_1}$, C_1 a constant. For given initial values $u_1(0) < u_2(0)$, we need $C_1 = (u_1(0) - k_1)^2 = \left(\frac{u_2(0)-u_1(0)}{2}\right)^2$. Substituting back into (A2), we find that expert incentive compatibility within the intervals $[0, x_1] \cup [\gamma, x_1 + \gamma]$ requires the following action functions:

$$\begin{aligned}
u_1(x) &= k_1 - \sqrt{2x + C_1}\\
u_2(x) &= k_1 + \sqrt{2x + C_1}\\
\text{where } k_1 &= \frac{u_1(0) + u_2(0)}{2}, \text{ and } C_1 = \left(\frac{u_2(0) - u_1(0)}{2}\right)^2
\end{aligned} \tag{A3}$$

With these functions, the expected disutility to the expert of type xb is $b^2 \cdot D_{\text{expert}}(x)$, where

$$\begin{aligned}
D_{\text{expert}}(x) &= \left(k_1 - \sqrt{2x + C_1} - x - 1\right)^2 + \left(k_1 + \sqrt{2x + C_1} - x - 1\right)^2 + 2\\
&= 2x^2 + 4(2 - k_1)x + 2\left(k_1 - 1\right)^2 + 2C_1 + 2 && \text{(A4)}\\
D_{\text{expert}}(x+\gamma) &= 2x^2 + 4(2 - k_1)x + 2\left(k_1 - 1\right)^2 + 2C_1 + 2 + 2\gamma\left(2x + \gamma - 2k_1 + 2\right) && \text{(A5)}
\end{aligned}$$

Finally, to verify that our F.O.C.'s (A2),(A3) are indeed implying that disutility is *minimized* by truth-telling, we verify the 2nd-order conditions: the 2nd derivative of (A0) w.r.t. z is

$$2\left[u_1''(z)\left(u_1(z) - x - 1\right) + u_2''(z)\left(u_2(z) - x - 1\right) + \left(u_1'(z)\right)^2 + \left(u_2'(z)\right)^2 + 2\right]$$

Fully differentiating (A1) w.r.t. x implies

$$u_1''(x)\left(u_1(x) - x - 1\right) + u_2''(x)\left(u_2(x) - x - 1\right) + u_1'(x)\left(u_1(x) - 1\right) + u_2'(x)\left(u_2(x) - 1\right) = 0$$

Substituting this into the previous expression, we obtain that the 2nd derivative of (A0) w.r.t. z, evaluated at $z = x$, reduces to

$$2\left[u_1'(z) + u_2'(z) + 2\right]$$

Then $u_1'(z) + u_2'(z) = 0$ implies that this is strictly positive, hence the disutility to type $x \in [0, x_1 b]$ from mimicking another type z in this interval is indeed *minimized* by truth-telling. Similarly, (A2) implies that the second derivative of the disutility (w.r.t. z) to type $(x+\gamma)b$ from mimicking type $(z+\gamma)b$ is 4, so that truth-telling is better than mimicking any other type in the interval $[\gamma b, x_1 b + \gamma b]$.

To calculate $\mathbf{v}_1$,$\mathbf{v}_2$:

The IC constraints for types $\theta \in [x_1 b, \gamma b] \cup [x_1 b + \gamma b, 1]$ to prefer truth-telling, rather than mimicking any other types from these intervals, reduce to the following pair of differential equations (these are analogous to the differential equations (A1) and (A2), noting that types in this range do not separate until period 4):

$$\frac{dv_1}{dx}(v_1 - x - 1) + 2\frac{dv_2}{dx}(v_2 - x - 1) = 1 \tag{A6}$$
$$\frac{dv_1}{dx}(v_1 - h - 1) + 2\frac{dv_2}{dx}(v_2 - h - 1) = \frac{dh}{dx} \tag{A7}$$

Define $v_{10} \equiv v_1(x_1), v_{20} \equiv v_2(x_1)$ as the initial values for functions v_1, v_2. To make sure that types in the 1st, 3rd intervals do not want to mimic types from the 2nd, 4th intervals, and vice versa, we need to make types $x_1 b$ and $(x_1+\gamma)b$ indifferent between the two action sequences that they can induce (right endpoint of the interval to their left, vs left endpoint of the interval to their right): this requires (with $D(x_1), D(x_1+\gamma)$ as given by (A4) and (A5))

$$D(x_1) - 1 = (v_{10} - x_1 - 1)^2 + 2(v_{20} - x_1 - 1)^2$$
$$D(x_1+\gamma) - 1 = (v_{10} - x_1 - \gamma - 1)^2 + 2(v_{20} - x_1 - \gamma - 1)^2$$

After some tedious algebra, these can be solved as follows:

$$v_{10} = \frac{\left(\frac{1}{2}\gamma + 2k_1 + x_1 + 1\right)}{3} - \sqrt{\left(\frac{\left(\frac{1}{2}\gamma + 2k_1 + x_1 + 1\right)}{3}\right)^2 - \frac{2}{3}\left(\frac{\left(\frac{1}{2}\gamma + 2k_1 + x_1 + 1\right)^2}{2} - x_1^2 - 6x_1 - 2 - (x_1+1)\gamma - 2k_1^2 - 2C_1\right)}$$
$$v_{20} = \frac{\frac{\gamma}{2} + 2k_1 + x_1 + 1 - v_{10}}{2} \tag{A8, A9}$$

Now, to solve our differential equations: substitute (A6) into (A7), to obtain

$$\frac{dv_1}{dx} + 2\frac{dv_2}{dx} = -\frac{\frac{dh}{dx} - 1}{h(x) - x} \tag{A7a}$$
$$\Rightarrow v_1(x) + 2v_2(x) = \text{constant} - \ln(h(x) - x)$$

By (A8) and (A9), we have

$$v_1(x_1) + 2v_2(x_1) = \frac{\gamma}{2} + 2k_1 + x_1 + 1$$

So, the constant in (A7a) must equal $\frac{\gamma}{2} + 2k_1 + x_1 + 1 + \ln(h(x_1) - x_1)$. Substitue the constraint $h(x_1) = x_1 + \gamma$ into (A7a) and solve for $v_2(x)$, to obtain:

$$\begin{aligned} v_2(x) &= \frac{y(x) - v_1(x)}{2} \\ \text{where } y(x) &\equiv \frac{\gamma}{2} + 2k_1 + x_1 + 1 - \ln\left(\frac{h(x) - x}{\gamma}\right) \end{aligned} \tag{A7b}$$

Substituting $v_2(x) = \frac{y(x)-v_1(x)}{2}$ (and hence $2v_2'(x) = y'(x) - v_1'(x)$) back into (A6), we find that $v_1(x)$ is the solution to the following differential equation:[7]

$$\begin{aligned}
&\tfrac{dv_1}{dx}(v_1 - x - 1) + 2\tfrac{dv_2}{dx}(v_2 - x - 1) = 1 \\
&\Leftrightarrow \tfrac{dv_1}{dx}(v_1 - x - 1) + \left(\tfrac{dy}{dx} - \tfrac{dv_1}{dx}\right)(v_2 - x - 1) = 1 \\
&\Leftrightarrow \tfrac{dv_1}{dx}(v_1 - v_2) + \tfrac{dy}{dx}(v_2 - x - 1) = 1 \\
&\Leftrightarrow \tfrac{dv_1}{dx}\left(\tfrac{3v_1 - y}{2}\right) + \tfrac{dy}{dx}\left(\tfrac{y - v_1}{2} - x - 1\right) = 1 \\
&\Leftrightarrow \tfrac{dv_1}{dx}\left(\tfrac{3v_1 - y}{2}\right) + \tfrac{dy}{dx}\left(\tfrac{\frac{2}{3}y - \frac{1}{3}(3v_1 - y)}{2} - x - 1\right) = 1 \\
&\Leftrightarrow \tfrac{1}{3}\left(3\tfrac{dv_1}{dx} - \tfrac{dy}{dx}\right)\left(\tfrac{3v_1 - y}{2}\right) + \tfrac{dy}{dx}\left(\tfrac{1}{3}y - x - 1\right) = 1
\end{aligned}$$

Defining a new variable $Z(x) = 3v_1(x) - y(x)$, and multiplying through by 3, this becomes

$$\frac{1}{2}Z\frac{dZ}{dx} + y\frac{dy}{dx} = 3\left[1 + (x + 1)\frac{dy}{dx}\right]$$

Integrating with respect to x, and substituting in $\frac{dy}{dx} = -\frac{\frac{dh}{dx} - 1}{h(x) - x}$, we obtain:

$$\frac{1}{4}Z^2 + \frac{1}{2}y^2 = 3\left(x - \int (x + 1)\left(\frac{\frac{dh}{dx} - 1}{h(x) - x}\right)dx\right)$$

Substituting in the definition of $Z(x)$, solving for $v_1(x)$, and then substituting this back into $v_2(x) =$

[7] Note that this analysis is more complicated than that for the functions u_1, u_2 only because we are allowing for an arbitrary "partner" function $h(x)$ (necessary because there is no equilibrium in which $h(x) - x$ is a constant).

$\frac{y(x)-v_1(x)}{2}$, we obtain:

$$v_1(x) = \frac{\frac{\gamma}{2}+2k_1+x_1+1-\ln\left(\frac{h(x)-x}{\gamma}\right)}{3} - \sqrt{2\left(\frac{2}{3}x - \left(\frac{y(x)}{3}\right)^2 - \frac{2}{3}\int\left(\frac{\left(\frac{dh}{dx}-1\right)(x+1)}{h(x)-x}\right)dx + C_2\right)}$$

$$v_2(x) = \frac{\frac{\gamma}{2}+2k_1+x_1+1-\ln\left(\frac{h(x)-x}{\gamma}\right)-v_1(x)}{2} \tag{A10, A11}$$

where C_2 is a constant, and $\int(x+1)\left(\frac{\frac{dh}{dx}-1}{h(x)-x}\right)dx$ is the antiderivative with no constant. We determine C_2 from the equation $v_{10} = v_1(x_1)$, using (A10) and (A8) and noting that $h(x_1) = x_1+\gamma \Rightarrow \ln\left(\frac{h(x_1)-x_1}{\gamma}\right) = 0$. Finally substituting the resulting value for C_2 back into (A10), (A11), we obtain:

$$v_1(x) = \frac{y}{3} - \sqrt{2\left(\frac{2}{3}x + k_2(x) - \left(\frac{y(x)}{3}\right)^2\right)} \tag{A12}$$

$$v_2(x) = \frac{y}{3} + \sqrt{\frac{1}{2}\left(\frac{2}{3}x + k_2(x) - \left(\frac{y(x)}{3}\right)^2\right)} \tag{A13}$$

$$\text{where } y(x) \equiv \frac{\gamma}{2} + 2k_1 + x_1 + 1 - \ln\left(\frac{h(x)-x}{\gamma}\right) \tag{A14}$$

$$\text{and } k_2(x) \equiv \frac{\gamma(1+x_1)+2C_1-2+2k_1^2+(x_1+2)^2}{3} - \frac{2}{3}\int_{x_1}^{x}(s+1)\left(\frac{\frac{dh}{dx}-1}{h(s)-s}\right)ds \tag{A15}$$

The second-order condition for truth-telling to be optimal (by a similar calculation to the one above for functions u_1, u_2) reduces to

$$v_1'(x) + 2v_2'(x) + 1 \geq 0 \text{ for all } x \in [x_1, \gamma]$$

Since we have (by (A7b)) $v_1'(x)+2v_2'(x) = y'(x) = \frac{1-h'(x)}{h(x)-x} = \frac{1-\frac{1}{2\sqrt{x+1-x_1}}}{\gamma+\sqrt{x+1-x_1}-(x+1-x_1)} > 0$ for $x \in [x_1, \gamma]$, this is clearly satisfied.

Next, we need to make sure that our equations generate real-valued solutions, requiring

$$\frac{2}{3}x + k_2(x) - \left(\frac{y(x)}{3}\right)^2 \geq 0 \text{ for all } x \in [x_1, \gamma] \tag{A16}$$

The derivative of the LHS of (A16) w.r.t. x (noting that $k_2'(x) = \frac{2}{3}(x+1)\left(\frac{h'(x)-1}{h(x)-x}\right) = \frac{2}{3}(x+1)y'(x)$) is $\frac{2}{3}\left(1 - y'\left(\frac{y}{3} - x - 1\right)\right)$. We will impose the pair of constraints that this is derivative is strictly positive (this is (A17), implying that (A16) is most difficult to satisfy at the left endpoint x_1), and that (A16) holds at x_1 (this is (A18)):

$$\left(\frac{1-h'(x)}{h(x)-x}\right)\left(\frac{\frac{\gamma}{2}+2k_1+x_1+1-\ln\left(\frac{h(x)-x}{\gamma}\right)}{3}-x-1\right)<1 \text{ for all } x\in[x_1,\gamma] \tag{A17}$$

$$2C_1=\frac{\left(\frac{\gamma}{2}+2k_1+x_1+1\right)^2}{3}-\left(\gamma+(\gamma+6)\,x_1+x_1^2+2k_1^2+2\right) \tag{A18}$$

Our last constraint for expert incentive compatibility in the pre-separation phase: type γ must be indifferent between sending 1st-period messages $u_1(0)$, $v_1(\gamma)$, requiring (LHS is the disutility from recommending the $u_t(0)$ sequence, RHS from sending the $v_t(\gamma)$ sequence):

$$2\left(k_1-1\right)^2+2C_1+2+2\gamma\left(\gamma-2k_1+2\right)=\left(v_1(\gamma)-\gamma-1\right)^2+2\left(v_2(\gamma)-\gamma-1\right)^2+1 \tag{A19}$$

Evaluating (A12),(A13) at $x=\gamma$ to obtain expressions $v_1(\gamma), v_2(\gamma)$, this becomes

$$0=\int_{x_1}^{\gamma}\ln\left(\frac{h(x)-x}{\gamma}\right)dx+\frac{1}{2}x_1^2+\left(1-\frac{\gamma}{2}\right)x_1+\gamma \tag{A20}$$

Finally, we need to make sure that all paired expert types indeed prefer to separate in the prescribed separation period. Expert type x prefers the action x to the action x' iff $(x-x-1)^2\leq(x'-x-1)^2\Leftrightarrow|x-x'|\geq 2$. A sufficient condition:

$$x_1\geq 2 \text{ and } \gamma\geq 2 \tag{A22}$$

9.2 Incentive Compatibility for the Decision-Maker

For any two-type information set $\{\theta,p(\theta)\}$, let $S(\{\theta,p(\theta)\})$ denote the DM's maximum expected gain to deviating at information set $\{\theta,p(\theta)\}$. We need to show that for the recommendation functions u_1,u_2,v_1,v_2 chosen above to satisfy the expert's IC constraints, and with partner function

$$p(\theta)=\begin{cases}\theta+\gamma b & \text{if } \theta\leq x_1 b\\ h\left(\frac{\theta}{b}\right)b & \text{if } \theta\in[x_1 b,\gamma b]\end{cases}$$

this gain $S(\{\theta,p(\theta)\})$ is weakly negative for all $\theta\in[0,\gamma b]$. As explained in Section 5.1, the most profitable time to deviate is in period 1, and we assume that at the information set $\{\theta,p(\theta)\}$, the DM's beliefs assign probability $\frac{1}{1+p'(\theta)}$ to type θ, and the residual probability to type $p(\theta)$. Therefore, $S(\{\theta,p(\theta)\})$ is equal to the DM's equilibrium payoff conditional on this information set, minus his expected payoff from choosing expected type $\frac{\theta+p'(\theta)p(\theta)}{1+p'(\theta)}$ in all four periods.

At information sets $\{xb,h(x)\,b\}$ for $x\in[x_1,\gamma]$: if the DM follows the prescribed strategy, his expected disutility (suppressing dependence of action and partner functions v_1,v_2,h on x) is b^2 times

$$\begin{aligned}
&= \frac{(v_1-x)^2+2(v_2-x)^2}{1+\frac{dh}{dx}} + \frac{dh}{dx}\frac{(v_1-h)^2+2(v_2-h)^2}{1+\frac{dh}{dx}} \\
&= \frac{(v_1-x)^2+2\left(\frac{y-v_1}{2}-x\right)^2}{1+\frac{dh}{dx}} + \frac{dh}{dx}\frac{(v_1-h)^2+2\left(\frac{y-v_1}{2}-h\right)^2}{1+\frac{dh}{dx}} \text{ by (A7b)} \\
&= v_1^2+\frac{(y-v_1)^2}{2}+3\left(\frac{x^2+h^2\delta}{1+\delta}\right)-\left(\frac{x+\delta h}{1+\delta}\right)2y, \text{ with } \delta\equiv h'(x)
\end{aligned}$$

If he deviates and instead chooses the myopically optimal action $\frac{x+h'(x)h(x)}{1+h'(x)}b$ in all four periods, disutility is b^2 times

$$4\cdot\left[\frac{1}{1+\delta}\left(\frac{x+\delta h}{1+\delta}-x\right)^2+\frac{\delta}{1+\delta}\left(\frac{x+\delta h}{1+\delta}-h\right)^2\right]=\frac{4\delta}{(1+\delta)^2}(h-x)^2$$

So, the expected gain to deviating in period 1 is b^2 times

$$v_1^2+\frac{(y-v_1)^2}{2}+3\left(\frac{x^2+h^2\delta}{1+\delta}\right)-\left(\frac{x+\delta h}{1+\delta}\right)2y-\frac{4\delta}{(1+\delta)^2}(h-x)^2$$

Now substitute in $v_1(x)=\frac{y(x)}{3}-\sqrt{2\left(\frac{2x}{3}+k_2(x)-\left(\frac{y}{3}\right)^2\right)}$ from (A12), and we obtain

$$\frac{S(\{xb,h(x)b\})}{b^2}\equiv 2x+3k_2(x)+3\left(\frac{x^2+h^2\delta}{1+\delta}\right)-\left(\frac{x+\delta h}{1+\delta}\right)2y(x)-\frac{4\delta}{(1+\delta)^2}(h-x)^2 \tag{A23}$$

Using (A14) and (A15), rewrite the RHS as

$$\begin{aligned}
&2x+\left[\gamma(1+x_1)+2C_1-2+2k_1^2+(x_1+2)^2-2\int_{x_1}^{x}(s+1)\left(\frac{\frac{dh}{dx}-1}{h(s)-s}\right)ds\right] \\
&+3\left(\frac{x^2+h^2\delta}{1+\delta}\right)-2\left(\frac{x+\delta h}{1+\delta}\right)\left(\frac{\gamma}{2}+2k_1+x_1+1-\ln\left(\frac{h(x)-x}{\gamma}\right)\right)-\frac{4\delta}{(1+\delta)^2}(h-x)^2
\end{aligned}$$

Now substitute in the expression for C_1 from (A18), and the inequality $\frac{S(\{xb,h(x)b\})}{b^2}\leq 0$ becomes

$$\begin{aligned}
&\left(\frac{\gamma}{2}+2k_1+x_1+1\right)^2-6\left(\frac{x+\delta h}{1+\delta}\right)\left(\frac{\gamma}{2}+2k_1+x_1+1\right) \\
\leq\ &\left[\frac{12\delta}{(1+\delta)^2}(h-x)^2-9\left(\frac{x^2+h^2\delta}{1+\delta}\right)-6\ln\left(\frac{h(x)-x}{\gamma}\right)+6(x_1-x)+6\int_{x_1}^{x}(s+1)\left(\frac{\frac{dh}{dx}-1}{h(s)-s}\right)ds\right]
\end{aligned}$$

Solving, we need

$$\left(\frac{\gamma}{2}+2k_1+x_1+1\right) \leq 3\left(\frac{x+\delta h}{1+\delta}\right)+\sqrt{3}\sqrt{Z(x)} \text{ for all } x\in[x_1,\gamma] \tag{A24}$$

$$\text{where } Z(x) \equiv \delta\left(\frac{h-x}{1+\delta}\right)^2-2\left(\frac{x+\delta h}{1+\delta}\right)\ln\left(\frac{h-x}{\gamma}\right)+2\left(x_1-x\right)+2\int_{x_1}^{x}\left(\frac{\left(\frac{dh}{dx}-1\right)(s+1)}{h(s)-s}\right) \tag{A25}$$

At information sets $\{xb, xb+\gamma b\}$ with $x\in[0,x_1]$, the DM's gain to deviating (noting that the partner function $p(\theta)=\theta+\gamma b$ in this range implies beliefs which assign probability $\frac{1}{2}$ to each type) is b^2 times

$$\begin{aligned}
&\frac{(u_1(x)-x)^2+(u_2(x)-x)^2+(u_1(x)-x-\gamma)^2+(u_2(x)-x-\gamma)^2}{2}-4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(x+\gamma-x)^2\\
=\;&\frac{1}{2}\left(k_1-\sqrt{2x+C_1}-x\right)^2+\frac{1}{2}\left(k_1+\sqrt{2x+C_1}-x\right)^2\\
&+\frac{1}{2}\left(k_1-\sqrt{2x+C_1}-x-\gamma\right)^2+\frac{1}{2}\left(k_1+\sqrt{2x+C_1}-x-\gamma\right)^2-\gamma^2 \text{ by (A3)}
\end{aligned}$$

Simplifying,

$$\frac{S(\{xb,(x+\gamma)\,b\})}{b^2}=2x^2+4\left(1+\frac{\gamma}{2}-k_1\right)x+2C_1+2k_1^2-2\gamma k_1 \tag{A26}$$

Substituting in expression (A18) for C_1, the RHS becomes

$$2x^2-4\left(k_1-1-\frac{\gamma}{2}\right)x+\left(2k_1^2-2\gamma k_1+\frac{\left(\frac{\gamma}{2}+2k_1+x_1+1\right)^2}{3}-\left(\gamma+(\gamma+6)\,x_1+x_1^2+2k_1^2+2\right)\right)$$

Clearly this expression is strictly convex (2nd derivative w.r.t. x is 4), therefore *maximized* (over $[0,x_1]$) at the endpoints. Therefore it is sufficient to check that it is weakly negative at both 0 and x_1, yielding the following final constraints:

$$0 \;:\; k_1\leq\frac{\gamma-x_1-1}{2}+\frac{\sqrt{3}}{2}\sqrt{\left(\frac{\gamma}{2}\right)^2+x_1^2+6x_1+2} \tag{A27}$$

$$x_1 \;:\; k_1\geq\frac{\gamma+2x_1-1}{2}-\frac{\sqrt{3}}{2}\sqrt{\left(\frac{\gamma}{2}\right)^2+2} \tag{A28}$$

(These are compatible (for the relationship between x_1,γ implied by (A21)) for biases b above $\underline{b}\cong\frac{1}{200}$; for smaller biases, equilibrium requires decreasing the gap between the length of the full revelation phase for types $x\in[0,x_1]$, and that for types $x\in[x_1,\gamma]$, details to be added).

9.3 Completing the Proof for $h(x)=\gamma+x_1-1+\sqrt{x+1-x_1}$

Claim: For partner function $h(x)=\gamma+x_1-1+\sqrt{x+1-x_1}$, a fully revealing equilibrium exists iff $b<\frac{1}{19.718}$. In this case, (A17),(A24) do not bind. Start by choosing a value $x_1\geq 6.4$, and then choose γ to satisfy

$$0=\left(\frac{\sqrt{4\gamma+1}+2x_1-1}{2\sqrt{4\gamma+1}}\right)\ln\left(\frac{x_1-1+\sqrt{\gamma-x_1+1}}{\gamma}\right)-\ln\left(1+\frac{2\sqrt{\gamma-x_1+1}-2}{\sqrt{4\gamma+1}+1}\right)+\frac{\sqrt{\gamma-x_1+1}-1}{\sqrt{4\gamma+1}}+\frac{x(\gamma-x_1-4)}{2\sqrt{4\gamma+1}} \quad (21a)$$

The action functions described in (A3),(A12),(A13) then generate a fully revealing equilibrium at bias $b=\frac{1}{\gamma+x_1-1+\sqrt{\gamma+1-x_1}}$, for any constants C_1 satisfying (A18), and k_1 satisfying

$$\frac{\gamma+2x_1-1}{2}-\frac{\sqrt{3}}{2}\sqrt{\left(\frac{\gamma}{2}\right)^2+2}\leq k_1\leq\frac{\gamma-x_1-1}{2}+\frac{\sqrt{3}}{2}\sqrt{\left(\frac{\gamma}{2}\right)^2+x_1^2+6x_1+2} \quad \text{(A27,28)}$$

Proof: We prove the claim via a sequence of lemmas. To simplify the algebra in Lemma A, also define $z_x\equiv\sqrt{x+1-x_1}$, $a_x\equiv\frac{\gamma+z_x-z_x^2}{2z_x+1}, b_x=\frac{\gamma+z_x-z_x^2}{\gamma}$. Note that $\frac{da_x}{dz_x}=\frac{1-2z_x-2a_x}{1+2z_x}$, and that z_x is strictly increasing in x, with a minimum of $z_{x_1}=1$, maximum of $z_\gamma=\sqrt{\gamma+1-x_1}$.

With $h(x)=\gamma+x_1-1+\sqrt{x+1-x_1}$, the expert's constraint (A21) relating x_1,γ becomes (evaluating the integral and simplifying)

Lemma A: For all $x\in[x_1,\gamma]$, (i) $a_x>0$; (ii) (A21) implies $\gamma\leq\frac{x_1(x_1+2)}{(x_1-2)}$; (iii) (A21) implies $\gamma\geq x_1+4$; (iv) for all $x_1\geq 6$, $b_x\in[\frac{1}{2},1]$..

Proof of Lemma A:

For part (i): since $z_x\geq 1$, the numerator of a_x is strictly decreasing in z_x (hence x), with a minimum value of $\gamma+z_\gamma-z_\gamma^2=x_1-1+\sqrt{\gamma+1-x_1}>x_1$, clearly positive. For (ii): since $h''<0$ and $h'(x_1)=\frac{1}{2\sqrt{x_1+1-x_1}}=\frac{1}{2}$, the function $h(x)-x$ is strictly decreasing in x, with a maximum value (over $[x_1,\gamma]$) of $h(x_1)-x_1=\gamma$. This implies that $\frac{h(x)-x}{\gamma}<1$ for all $x\in(x_1,\gamma]$, hence (A21) implies

$$\begin{aligned}\frac{1}{2}x_1^2+\left(1-\frac{\gamma}{2}\right)x_1+\gamma &= -\int_{x_1}^{\gamma}\ln\left(\frac{h(x)-x}{\gamma}\right)dx\\ &\Rightarrow\ \gamma\left(1-\frac{x_1}{2}\right)+x_1\left(1+\frac{x_1}{2}\right)>0\end{aligned}$$

Solving for γ, we obtain that $\frac{\gamma}{x_1}<\frac{x_1+2}{x_1-2}$, as desired. For (iii): fully differentiate (A21), to obtain (using $\ln(\frac{h(x_1)-x_1}{\gamma})=1$)

$$\begin{aligned}0 &= \left(\ln\left(\frac{h(\gamma)-\gamma}{\gamma}\right)+\frac{x_1}{\gamma}-\frac{x_1}{2}\right)d\gamma+\left(x_1+1-\frac{\gamma}{2}\right)dx_1\\ \Rightarrow\ \frac{d\gamma}{dx_1} &= \frac{\left(x_1+1-\frac{\gamma}{2}\right)}{\frac{x_1}{2}-\frac{x_1}{\gamma}-\ln\left(\frac{h(\gamma)-\gamma}{\gamma}\right)}\end{aligned}$$

Since $\frac{h(\gamma)-\gamma}{\gamma}<1$ (as noted in the proof of Lemma A (ii)), the final term in the denominator is positive, while $x_1<\gamma$ implies $-\frac{x_1}{\gamma}>-1$; therefore, the denominator of the expression for $\frac{d\gamma}{dx_1}$ is greater than $\frac{x_1}{2} - 1$, and therefore

$$\frac{d\gamma}{dx_1} < \frac{2x_1 + 2 - \gamma}{x_1 - 2}$$

This is less than 1 whenever $\gamma > x_1 + 4$, implying that $(\gamma - x_1)$ is decreasing in x_1 whenever it lies above 4. Now consider (A21a): the fact that $(\gamma - x_1)$ is decreasing, hence bounded (maximum value for $x_1 \geq 6.4$ is $11.79 - 6.4$, and the expression cannot be negative), implies that as $x_1 \to \infty$, the LHS of (A21a) goes to

$$\frac{x_1}{\sqrt{4\gamma + 1}} \ln\left(\tfrac{x_1}{\gamma}\right) - \ln(1+0) + 0 + \frac{x_1}{2\sqrt{4\gamma+1}}\left(\gamma - x_1 - 4\right)$$

Therefore, as $x_1 \to \infty$, (A21a) implies $2\ln\left(\frac{x_1}{\gamma}\right) + (\gamma - x_1 - 4) \to 0$; since $\gamma - x_1 \in [0,6]$ implies that $\ln\left(\frac{x_1}{\gamma}\right) \to \infty$ as $x_1 \to \infty$, this requires $\gamma \to x_1 + 4$ as $x_1 \to \infty$. Combined with our previous observation that $(\gamma - x_1)$ is decreasing whenever $\gamma > x_1 + 4$, this implies the desired result. For (iv): as noted in the proof of (i), the numerator of b_x is decreasing in z_x; this implies,

$$\begin{aligned}
b_x &\leq \left[\frac{\gamma + z_x - z_x^2}{\gamma}\right]_{x=x_1} = \frac{\gamma + 1 - 1}{\gamma} = 1 \\
b_x &\geq \left[\frac{\gamma + z_x - z_x^2}{\gamma}\right]_{x=\gamma} \geq \frac{x_1 - 1 + \sqrt{\gamma + 1 - x_1}}{\gamma} > \frac{x_1}{\gamma}
\end{aligned}$$

By (ii), $\frac{x_1}{\gamma} > \frac{x_1 - 2}{x_1 + 2}$, which is $\geq \frac{1}{2}$ for $x_1 \geq 6$, so the lower bound on b_x is $\frac{x_1}{\gamma} > \frac{1}{2}$, as desired. $\blacksquare$

Lemma B: For all $x_1 \geq 6$, $Z(x)$ (defined in (A25)) is strictly decreasing in x, therefore minimized (over $[x_1, \gamma]$) at $x = \gamma$; (vi) for all x, the expression in (A17) is most difficult to satisfy at $x = \gamma$.

Proof of Lemma B:

For (i): first, integrate by parts to rewrite (A25) as

$$\begin{aligned}
Z(x) &\equiv \delta\left(\tfrac{h-x}{1+\delta}\right)^2 - 2\left(\frac{x + \delta h}{1+\delta}\right)\ln\left(\frac{h(x) - x}{\gamma}\right) + 2\left(x_1 - x\right) + 2(x+1)\ln\left(\frac{h(x)-x}{\gamma}\right) - 2\int_{x_1}^{x} \ln\left(\frac{h(s)-s}{\gamma}\right) ds \\
&= \delta\left(\tfrac{h-x}{1+\delta}\right)^2 + 2\left(1 - \delta\left(\frac{h-x}{1+\delta}\right)\right)\ln\left(\tfrac{h-x}{\gamma}\right) - 2\int_{x_1}^{x}\left(1 + \ln\left(\frac{h(s)-s}{\gamma}\right)\right) ds
\end{aligned}$$

Substituting in $z_x \equiv \sqrt{x + 1 - x_1}$, $a_x \equiv \frac{\gamma + z_x - z_x^2}{2z_x + 1}$, and $\delta \equiv h'(x) = \frac{1}{2z_x}$, this is

$$2z_x\left(a_x\right)^2 + 2\left(1 - a_x\right)\ln\left(\frac{\gamma + z_x - z_x^2}{\gamma}\right) - 2\int_{x_1}^{x}\left(1 + \ln\left(\frac{\gamma + z_s - z_s^2}{\gamma}\right)\right) ds$$

The derivative of this expression w.r.t. z_x, noting that $\frac{da_x}{dz_x} = \dfrac{1 - 2z_x - 2a_x}{1 + 2z_x}$, is

$$\left(\frac{4a_x}{1+2z_x}\right)\ln\left(b_x\right) + \left(\frac{1-2z_x}{1+2z_x}\right)\left[2a_x^2 + 4z_x a_x - 2(1-a_x)\left(\frac{1}{a_x}\right) - 2\ln\left(b_x\right)\right] - 4z_x\left(1 + \ln b_x\right) \quad \text{(A29)}$$

Since (A29) is the derivative of $Z(x)$ w.r.t. z_x, and since z_x is strictly increasing in x, to show that $Z(\cdot)$ is strictly decreasing in x, we just need to show that (A29) is strictly negative for all $x \in [x_1, \gamma]$ (with $x_1 \geq 6$ and γ given by (A21)). Since we showed (Lemma A (i)) that $a_x > 0$ and $z_x \geq 1$, the first term in (A29) is negative by the upper bound $b_x \leq 1$ in Lemma A (iv), and the last term is negative by the lower bound $b_x \geq \frac{1}{2}$ in (iv). Since $z_x \geq 1 \Rightarrow \left(\frac{1-2z_x}{1+2z_x}\right) < 0$, it therefore suffices to show that the term in square brackets in (A29) is strictly positive. Again, the upper bound in (iv) implies that the last term $-2\ln(b_x)$ is positive, and the first term in square brackets is clearly positive. Therefore, it is sufficient to show that $4z_x a_x - 2(1-a_x)\left(\frac{1}{a_x}\right)$, and for this in turn, it is sufficient to show that $2z_x a_x \geq 1$; by the lower bound in (iv), together with $z_x \geq 1$ (and the observation that $\frac{2z_x}{1+2z_x}$ is strictly increasing in z_x), we have

$$2z_x a_x = \frac{2z_x}{1+2z_x}\left(\gamma + z_x - z_x^2\right) \geq \left[\frac{2z_x}{1+2z_x}\right]_{x=x_1}\left[\left(\gamma + z_x - z_x^2\right)\right]_{x=\gamma} = \frac{2}{3}\gamma b_\gamma = \frac{2}{3}x_1$$

Clearly, exceeds 1 for $x_1 \geq 6$, so the square bracketed term in (A29) is indeed strictly positive, as needed to complete the proof of (i). Finally, for (ii), rewrite (A17) as

$$\frac{\frac{\gamma}{2} + 2k_1 + x_1 + 1}{3} < x + 1 + \frac{h(x) - x}{1 - h'(x)} + = x_1 + \frac{\gamma + \frac{1}{2}z_x}{1 - \frac{1}{2z_x}} + \frac{\ln\left(\frac{\gamma + z_x - z_x^2}{\gamma}\right)}{3} \tag{A17a}$$

The derivative w.r.t. z_x is $\left(-\frac{2\left(\gamma + z_x - z_x^2\right)}{(2z_x - 1)^2} - \frac{2z_x - 1}{3\left(\gamma + z_x - z_x^2\right)}\right)$.Since $z_x \geq 1 \Rightarrow 2z_x - 1 > 0$, and the lower bound in Lemma A (iv) gives $\gamma + z_x - z_x^2 \geq x_1 > 0$, this is strictly negative; this implies that the RHS of (A17) is strictly decreasing in z_x (hence x), and therefore the lower bound on the RHS, at which (A17a) is most difficult to satisfy, is attained at $x = \gamma$, as desired. ■

Lemma C: With $h(x) = \gamma + x_1 - 1 + \sqrt{x + 1 - x_1}$, (i) a fully revealing equilibrium exists iff $b < \frac{1}{19.718}$; (ii) (A17) does not bind; (iii) (A24) does not bind.

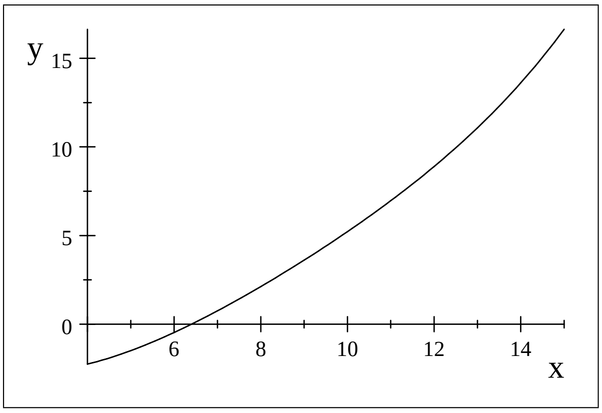
Proof of Lemma C:

For part (i): clearly (A24) is possible only if $Z(\gamma) \geq 0$; evaluating (A25) at $x = \gamma$ and substituting in (A20) to replace the integral, this requires

$$\left[z\left(\frac{\gamma + z - z^2}{2z+1}\right)^2 + \left(1 - \frac{\gamma + z - z^2}{2z+1}\right)\ln\left(\frac{\gamma + z - z^2}{\gamma}\right) + \frac{1}{2}(5 - z^2)\left(\gamma + 1 - z^2\right)\right]_{z=\sqrt{\gamma + 1 - x_1}} \geq 0$$

Substituting in (A21a), the graph of the LHS of the above inequality is as follows, as a function of

x_1 :



Positive iff $x_1 \geq 6.4$, which corresponds to (by (A21a) $\gamma \geq 11.791$, or $b = \frac{1}{\gamma + x_1 - 1 + \sqrt{\gamma+1-x_1}} \leq \frac{1}{19.719}$.

So, our construction can only yield an equilibrium if $b \leq \frac{1}{19.719}$. In this case, by Lemma A part (iv), a sufficient condition for (A24) to hold (replacing $\sqrt{Z(x)}$ with lower bound zero, and evaluating the remaining expression at $x = \gamma$) is

$$\frac{\left(\frac{\gamma}{2} + 2k_1 + x_1 + 1\right)}{3} \leq \left[\left(\frac{x + \delta h}{1+\delta}\right)\right]_{x=\gamma} = \gamma + \frac{x_1 - 1 + \sqrt{\gamma + 1 - x_1}}{1 + 2\sqrt{\gamma + 1 - x_1}} \quad \text{(A24a)}$$

By Lemma A part (v), a sufficient condition for (A17) to hold (evaluating at $x = \gamma$ and $h(\gamma) = \gamma + x_1 - 1 + \sqrt{\gamma + 1 - x_1}$ and simplilfying) is

$$\begin{aligned}\left(\frac{\frac{\gamma}{2} + 2k_1 + x_1 + 1}{3}\right) &< \left[\gamma + 1 + \frac{h(x) - x}{1 - h'(x)} + \frac{\ln\left(\frac{x_1 - 1 + \sqrt{\gamma+1-x_1}}{\gamma}\right)}{3}\right]_{x=\gamma} \\ &= \gamma + \frac{x_1 - 1 + \sqrt{\gamma + 1 - x_1}}{1 - \frac{1}{2\sqrt{\gamma+1-x_1}}} + \left(1 + \frac{\ln\left(\frac{x_1 - 1 + \sqrt{\gamma+1-x_1}}{\gamma}\right)}{3}\right) \quad \text{(A17a)}\end{aligned}$$

We noted in the proof of Lemma A (iv) that $\ln\left(\frac{x_1-1+\sqrt{\gamma+1-x_1}}{\gamma}\right) > \ln\left(\frac{1}{2}\right)$ (for γ,x_1 solving (A21)), which implies that the final RHS term in (A17a) is positive; since the middle term in (A17a) is larger than the final term in (A24a) (numerators are equal and positive (by $\gamma > x_1$)), while $\frac{1}{1-\frac{1}{2\sqrt{\gamma+1-x_1}}} > \frac{1}{1+2\sqrt{\gamma+1-x_1}}$), the RHS of (A17a) is therefore strictly larger than the RHS of (A24a). Hence, (A17a) is implied by (A24a), and therefore the constraint (A17) does not bind for our h-function.

Finally, rewrite (A29) and (A24a) as

$$\frac{\frac{\gamma}{2}+2k_1+x_1+1}{3} \leq \left[\frac{1}{2}\gamma+\sqrt{\frac{\left(\frac{\gamma}{2}\right)^2+x_1^2+6x_1+2}{3}}\right]_{x_1=8,\gamma=13.09} \tag{A27a}$$

$$\frac{\frac{\gamma}{2}+2k_1+x_1+1}{3} \leq \gamma+\frac{1}{2}+\frac{x_1-\frac{3}{2}}{1+2\sqrt{\gamma+1-x_1}} \tag{A24b}$$

Note that (A21) and (A21a) imply that $(\gamma - x_1)$ is strictly decreasing in x, and lies everywhere above 4.[8] By $(\gamma - x_1)$ strictly decreasing in x_1 and $x_1 \geq 6.4$, we have (recalling that at $x_1 = 6.4$, (21a) implies $\gamma = 11.791$)

$$\frac{x_1-\frac{3}{2}}{1+2\sqrt{\gamma+1-x_1}} > \frac{x_1-\frac{3}{2}}{\left[1+2\sqrt{\gamma+1-x_1}\right]_{x_1=6.4,\gamma=11.791}} = \frac{x_1-\frac{3}{2}}{6.0561}$$

So, a sufficient condition for the RHS of (A24b) to exceed the RHS of (A27a) is

$$\left(\frac{\gamma+1}{2}+\frac{x_1-\frac{3}{2}}{6.06}\right)^2-\left(\frac{\left(\frac{\gamma}{2}\right)^2+x_1^2+6x_1+2}{3}\right)>0$$

Clearly the LHS of this expression is increasing in γ (derivative w.r.t. γ exceeds $\frac{\gamma}{2}-\frac{\gamma}{6}$), so since (A21a) implies that $\gamma - x_1$ lies everywhere above 4, a lower bound on the LHS is obtained by setting $\gamma = x_1 + 4$, resulting in an expression which is strictly positive for all x_1.Therefore the RHS of (A24b) is strictly larger than the RHS of (A27a), and hence (A24) is implied by (A27)+(A21) (for $x_1 \geq 6.4$ and $h(x) = \gamma + x_1 - 1 + \sqrt{x+1-x_1}$), therefore does not bind. ■

This completes the proof that constraints (A17), (A24) do not bind, and hence that the only remaining constraints for full revelation (as stated in the Claim) are (A18) for C_1, (A27), (A28) for k_1. This completes the construction of a fully revealing equilibrium. ■

9.4 Example of Equilibrium Construction

Set $x_1 = 8$. Then by (A21), we need $\gamma = 13.09$, so we are constructing an example for bias $\frac{1}{h(13.09)} = \frac{1}{13.09+7+\sqrt{13.09-7}} = \frac{1}{22.558}$. At $x_1 = 8$ and $\gamma = 13.09$, the bounds in (A27),(A28) require $k_1 \in [8.2461, 12.891]$, and the bound in (A24) (which we showed in Lemma C to be non-binding) is $k_1 \leq 19.844$. Here we choose $k_1 = 12.833$, and substitute this into (A18) to solve for C_1 :

$$k_1 = 12.833,\ C_1 = 2.4669$$

With these, (A3) yields the following recommendations u_1, u_2 for pairs $\{x, x+13.09\}$ with

[8] Fully differentiate (A21) to obtain $\frac{d\gamma}{dx_1} = \frac{2x_1+2-\gamma}{-2\ln\left(\frac{h(\gamma)-\gamma}{\gamma}\right)+x_1-\frac{2x_1}{\gamma}} < \frac{2x_1+2-\gamma}{0+x_1-2}$; this is less than 1 whenever $\gamma - x_1 > 4$; by inpection of (A21a), $\lim_{x_1\to\infty}(\gamma - x_1) = 4$, implying the desired result.

$x \in [0,8]$ (where $u_1(x)b$ is the recommendation for types $\{xb, xb + 13.09b\}$):

$$u_1(x) = 12.833 - \sqrt{2x + 2.4669},\ u_2(x) = 12.833 + \sqrt{2x + 2.4669}$$

With these, the disutility to type z from sending initial recommendation $u_1(x)$ is

$$\begin{aligned}&\left(12.833 - \sqrt{2x + 2.4669} - z - 1\right)^2 + \left(12.833 + \sqrt{2x + 2.4669} - z - 1\right)^2 + 2(x - z - 1)^2 \\ &= 2x^2 - 4xz + 4z^2 - 43.332z + 286.97\end{aligned}$$

Clearly this is indeed minimized by truth-telling ($x = z$), and similarly for type $x + \gamma$. The resulting disutility expression is given by (A31) for type x, and (A32) for type $x + \gamma$, for any $x \in [0,8]$:

$$(u_1(x) - x - 1)^2 + (u_2(x) - x - 1)^2 + 2 = 2x^2 - 43.332x + 286.97 \tag{A31}$$

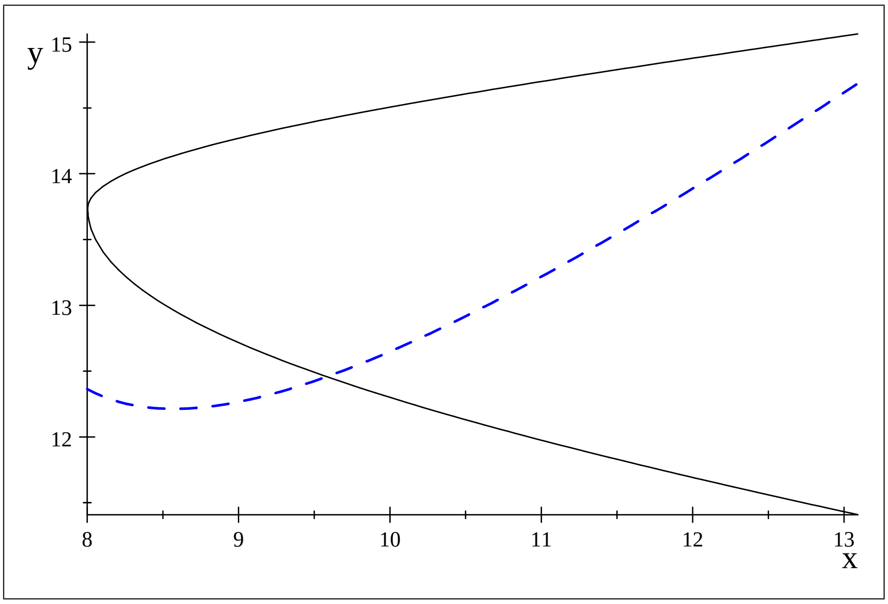
$$(u_1(x) - x - 13.09 - 1)^2 + (u_2(x) - x - 13.09 - 1)^2 + 2 = 2x^2 + 9.028x + 10.094 \tag{A32}$$

By (A12), (A13) at $\alpha = 1$, the recommendation functions v_1, v_2 for types $\{x, h(x)\}$ with $x \in [8, 13.09]$ (where types $\{xb, h(x)b\}$ recommend $v_1(x)b$, then $v_2(x)b$ for two periods) are given by

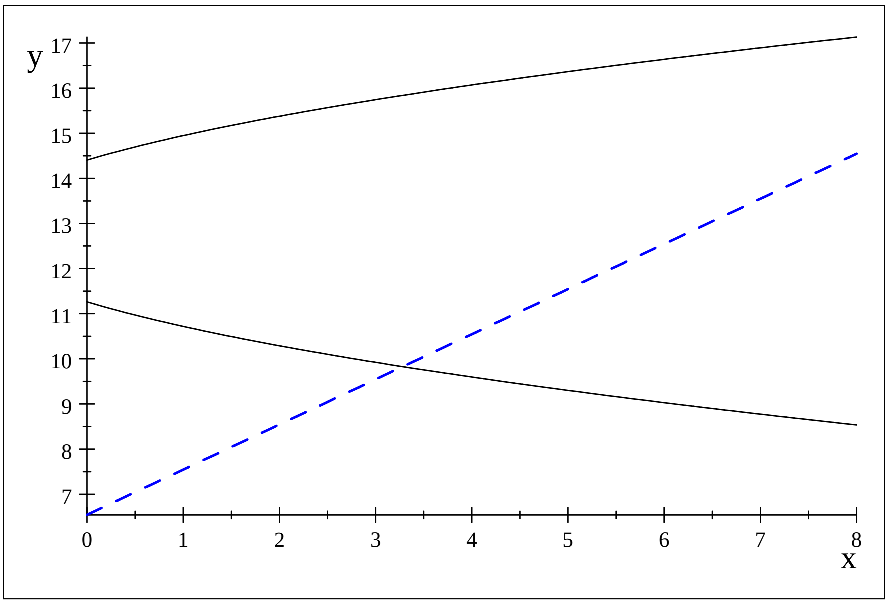
$$v_1(x) = \frac{41.211 - \ln\left(\frac{13.09+7-x+\sqrt{x-7}}{13.09}\right)}{3} - \sqrt{2}\sqrt{\begin{array}{c}189.37 - \frac{2\sqrt{x-7}}{3} - 16.828\ln\left(1.3172 - 0.3172\sqrt{x-7}\right) \\ -11.959\ln\left(0.2408\sqrt{x-7} + 0.7592\right) - \left(\frac{41.211 - \ln\left(\frac{13.09+7-x+\sqrt{x-7}}{13.09}\right)}{3}\right)^2\end{array}}$$

$$v_2(x) = \frac{41.211 - \ln\left(\frac{13.09+7-x+\sqrt{x-7}}{13.09}\right)}{3} + \frac{1}{\sqrt{2}}\sqrt{\begin{array}{c}189.37 - \frac{2\sqrt{x-7}}{3} - 16.828\ln\left(1.3172 - 0.3172\sqrt{x-7}\right) \\ -11.959\ln\left(0.2408\sqrt{x-7} + 0.7592\right) - \left(\frac{41.211 - \ln\left(\frac{13.09+7-x+\sqrt{x-7}}{13.09}\right)}{3}\right)^2\end{array}}$$

Here is a graph of the action functions v_1 (bottom) and v_2 (top); the blue dotted line shows, for each $x \in [8, 13.09]$, the myopically optimal action (divided by b) at information set $\{xb, h(x)b\}$, i.e. $\frac{x + h'(x)h(x)}{1 + h'(x)} = \frac{x + \frac{20.09 + \sqrt{x-7}}{2\sqrt{x-7}}}{1 + \frac{1}{2\sqrt{x-7}}}$:



Here is a graph of u_1 (botttom), u_2 (top), and (blue dotted line) for each $x \in [0, 8]$, the myopically optimal action (divided by b), $\frac{x+(x+\gamma)}{2}$ at information set $\{xb, (x+\gamma)b\} : 12.833 + \sqrt{2x + 2.466\,9}$



With these functions: the disutility to type 8 from recommending $v_1(8)b$ in period 1, then $v_2(8)b$ in periods 2,3, then 8 in period 4 is b^2 times

$$(v_1(8) - 9)^2 + 2\,(v_2(8) - 9)^2 + 1 = 68.319$$

while, by (A31), his disutility from recommending $u_1(8)b$ in period 1, then $u_2(8)b$ in period 2, then

8 in periods 3 and 4 is b^2 times

$$\left[2x^2 - 43.332x + 286.97\right]_{x=8} = 68.314$$

So indeed (subject to rounding error) type 8 is indifferent.

Next, the disutility to type $h(8)b = (13.09 + 8)\,b$ from recommending $v_1(8)b$ in period 1, then $v_2(8)b$ in periods 2,3, then 21.09 in period 4 is b^2 times

$$(v_1(8) - 13.09 - 9)^2 + 2\,(v_2(8) - 13.09 - 9)^2 + 1 = 210.32$$

while (by (A32)) if he recommends $u_1(8)b$, then $u_2(8)b$, then 21.09 in periods 3,4 he gets b^2 times

$$\left[2x^2 + 9.028x + 10.094\right]_{x=8} = 210.32$$

Again, indifferent as desired.

Finally for expert type $\gamma b = 13.09b$: recommending $v_1(13.09)b$, then $v_2(13.09)b$ for two periods, then $13.09b$ yields disutility b^2 times

$$(v_1(13.09) - 14.09)^2 + 2\,(v_2(13.09) - 14.09)^2 + 1 = 10.088$$

while if he recommends $u_1(0)b, u_2(0)b$, then $13.09b$ in periods 3 and 4, he gets (by (A32) evaluated at $x = 0$) b^2 times

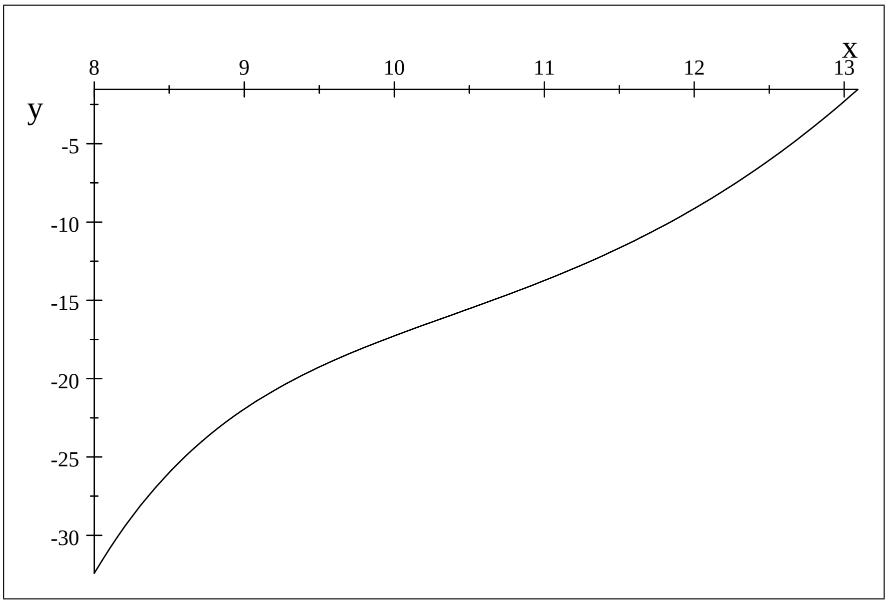
$$\left[2x^2 + 9.028x + 10.094\right]_{x=0} = 10.094$$

Again, correct subject to slight rounding error in our solution for γ.

Now, to verify that the solution is also incentive compatible for the DM: at the information set $\{xb, \left(20.09 + \sqrt{x-7}\right) b\}$, the DM's expected gain to deviating is b^2 times

$$\frac{(v_1(x)-x)^2+2(v_2(x)-x)^2}{1+\frac{1}{2\sqrt{x-7}}} + \frac{1}{2\sqrt{x-7}}\left(\frac{\left(v_1(x)-20.09-\sqrt{x-7}\right)^2+2\left(v_2(x)-20.09-\sqrt{x-7}\right)^2}{1+\frac{1}{2\sqrt{x-7}}}\right) - \frac{4\left(\frac{1}{2\sqrt{x-7}}\right)}{\left(1+\frac{1}{2\sqrt{x-7}}\right)^2}\left(20.09+\sqrt{x-7}-x\right)$$

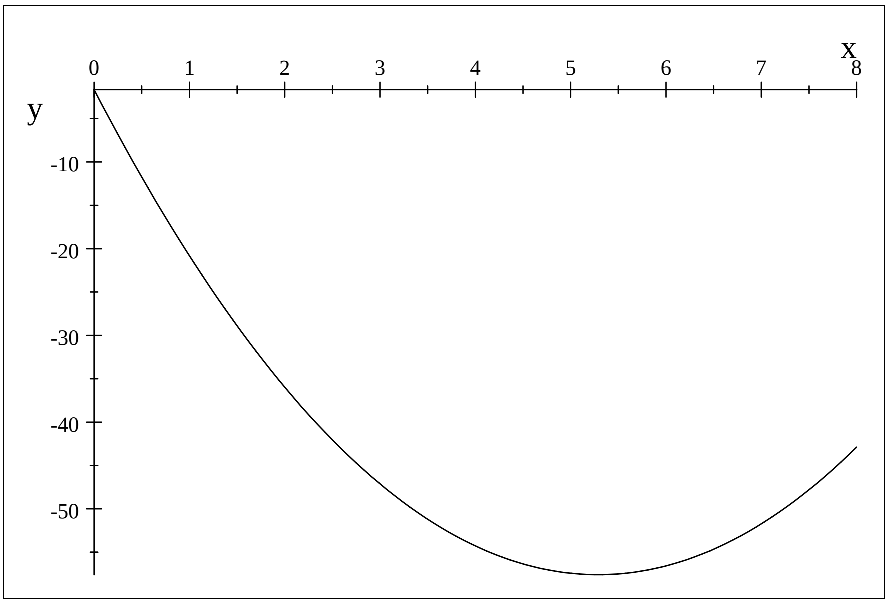
Here is a graph of the expression over the interval $[8, 13.09]$, indeed negative as desired:



Finally, the DM's gain to deviating at information sets $\{x, x + 13.09\}$ with $x \in [0, 8]$ is

$$\frac{\left(12.833-\sqrt{2x+2.4669}-x\right)^2+\left(12.833+\sqrt{2x+2.4669}-x\right)^2+\left(12.833-\sqrt{2x+2.4669}-x-13.09\right)^2+\left(12.833+\sqrt{2x+2.4669}-x-13.09\right)^2}{2}-(13.09)^2$$

Here is the graph of this expression over $[0, 8]$, again clearly negative as desired:



Therefore, our construction is indeed incentive compatible for both players, providing an example with full information revelation for bias $b = \frac{1}{22.558}$.

10. Appendix B: Proof of Theorem 4

Set-up/Notation: Fix an equilibrium. For each $\theta \in [0, \frac{1}{b}]$, define $y_t(\theta b)$ as the action induced by type θb in period $t \in \{1, 2, ..., T\}$, and define $a_t(\theta)$ by $a_t(\theta) = \frac{y_t(\theta b)}{b}$. So if the true state is θb, the expert wishes to minimize

$$\frac{1}{b^2}\sum_{t=1}^{T}\delta^{t-1}\left(y_t(\theta b) - \theta b - b\right)^2 = \sum_{t=1}^{T}\delta^{t-1}\left(a_t(\theta) - \theta - 1\right)^2$$

In a MPBE with quadratic utility, the DM sets $y_t(\theta b)$ equal to his expectation of the state, conditional on the equilibrium message sequence sent by type θb; equivalently,

$$a_t(\theta) = E\left[\theta \mid \text{type } \theta b\text{'s equilibrium message sequence by } t\right].$$

In a fully revealing T-period equilibrium, we have $a_T(\theta) = \theta$ for all $\theta \in [0, \frac{1}{b}]$, and the following incentive compatibility constraint must be satisfied for all expert types $\theta b, \theta' b$:

$$\sum_{t=1}^{T}\delta^{t-1}\left(a_t(\theta) - \theta - 1\right)^2 \leq \sum_{t=1}^{T}\delta^{t-1}\left(a_t(\theta') - \theta - 1\right)^2 \Leftrightarrow \sum_{t=1}^{T}\delta^{t-1}\left(a_t(\theta) - a_t(\theta')\right)\left(\frac{a_t(\theta) + a_t(\theta')}{2} - \theta - 1\right) \leq 0 \tag{B1}$$

Step 1 of Proof: "Approximate Continuity"

We say that types θ, θ' are *pooled in period* t if, in equilibrium σ^*, they send the message in every period $t' \leq t$. Suppose, for the moment, that every type θb with $\theta \in [0, \frac{1}{b}]$ pools with only finitely many types in period 1. In this step, we prove existence of a type θb, pooled with a set of types $\{\gamma_1(\theta b), ..., \gamma_K(\theta b)\}$ in period 1, such that infinitely many types $\theta' b$ near θb are pooled with partners $(\gamma_1(\theta' b), .., \gamma_K(\theta' b))$ arbitrarily close to $\{\gamma_1(\theta b), ..., \gamma_K(\theta b)\}$.Step 4 will then use this result to show that whenever equilibrium conditions are satisfied for types near θb, they are violated for types near $\gamma_j(\theta b)$, for some j.

Proof: Define $K(\theta)$ as the # types pooled with θb in period 1, and $K^* = \max_{\theta \in [0, \frac{1}{b}]} K(\theta)$. For every $\theta \in [0, \frac{1}{b}]$, enumerate the types pooled with θb as an increasing sequence $\gamma_1(\theta b) < \gamma_2(\theta b) <$

... $< \gamma_{K(\theta)}(\theta b)$. Next, for all $\theta \in [0, \frac{1}{b}]$ and $j \in \{1, 2, ..., K^*\}$, define $g_j(\theta)$ by[9]

$$bg_j(\theta) = z_j(\theta b) \text{ for } j \in \{1, 2, ..., K(\theta)\}$$
$$bg_j(\theta) = \frac{\theta + \sum_{i=1}^{K(\theta)} g_i(\theta)}{K(\theta) + 1} \text{ for } j \in \{K(\theta) + 1, ..., K^*\}$$

By optimality for the myopic decision-maker, we have, for all θ and $j \leq K(\theta)$, that

$$a_1(\theta) = a_1(g_j(\theta)) = \frac{\theta + \sum_{i=1}^{K^*} g_i(\theta)}{K^* + 1}$$

Now, for each $\theta \in [0, \frac{1}{b}]$, define

$$\mathbf{z}(\theta) = (\theta, g_1(\theta), ..., g_{K^*}(\theta), a_1(\theta), a_2(\theta), ..., a_T(\theta))$$

Since $g_j(\theta) \in [0, \frac{1}{b}]$ for all θ, and $a_t(\theta) \in [0, \frac{1}{b}]$ (by optimality for the receiver) for all θ, the set $\{\mathbf{z}(\theta) \mid \theta \in [0, \frac{1}{b}]\}$ is an infinite bounded subset of $\mathbb{R}^{K^*+T+1}$. Therefore, by the Bolzano-Weierstrass theorem, this set contains a limit point z^* and subsequence $\mathbf{z}(\theta_n) \to \mathbf{z}^*$. In other words, we can find a point $\theta^* \in [0, \frac{1}{b}]$ and sequence $\theta_n \to \theta^*$ such that

$$g_j(\theta_n) \to g_j(\theta^*) \text{ for all } j \in \{1, ..., K^*\}$$
$$\text{and } a_t(\theta_n) \to a_t(\theta^*) \text{ for all } t \in \{1, ..., T\}$$

Note that points near θ^* necessarily have the same number of "artificial partners" as θ^*.[10] Therefore, eliminate all elements $\mathbf{z}(\theta_n)$ with $K(\theta_n) \neq K(\theta^*)$, and for all remaining elements $\mathbf{z}(\theta_m)$, together with $\mathbf{z}(\theta^*)$, delete coordinates $K(\theta^*)+1 \leq i \leq K^*$. For notational convenience, relabel this modified sequence as simply $\mathbf{z}(\theta_n)$, and redefine $K^* \equiv K(\theta^*)$. Clearly we have $\mathbf{z}(\theta_n) \to \mathbf{z}(\theta^*)$, and for each $\mathbf{z}(\theta_n) = (\theta_n, (g_j(\theta_n))_{j=1}^{K^*}, (a_t(\theta_n))_{t=1}^{T})$, the set of partners $(g_j(\theta_n))_{j=1}^{K^*}$ to θ_n does not contain any artificial types.

Step 2: "Approximate Differentiability"

In this step, we show that there must exist some j^* such that *all* partner functions g_j are "approximately differentiable" with respect to g_{j^*}.

[9]For types θb already pooled with the max. number of partners K^*, $g_j(\theta)$ is simply the jth partner to θb, divided by b. For types initially pooled with fewer than K^* "partners", we add an artificial set of partners, each equal to the average of types actually pooled with θ. (This would not affect the decision-maker's 1st-period inference when types are uniformly distributed (i.e., "true type lies in {0,1} " results in the same action choice as the message "true type lies in $\{0, \frac{1}{2}, 1\}$), but in any case the artificial types will be eliminated. The purpose of adding them here is to faciliate construction of a sequence, which we will use in Step 4 to prove that any fully revealing equilibrium violates incentive constraints for some types).

[10]This follows from convergence and the fact that, by construction, the coordinates of $\mathbf{z}(\theta)$ are strictly increasing as long as they represent real partners (max at coordinate $K(\theta)$), while coordinates between $K(\theta)$ and K^* are equal to each other, and strictly less than coordinate $K(\theta)$.

Take the sequence $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$ from Step 2, and construct a "partner" j^* as follows: first, set $j_1^* = 1$, and check whether

$$\left(\left|\frac{g_2(\theta_{n+1}) - g_2(\theta_n)}{g_1(\theta_{n+1}) - g_1(\theta_n)}\right|\right)$$

forms a bounded sequence. If yes, set $j_2^* = 1$; if not, extract a subsequence of $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$ along which $\left(\left|\frac{g_2(\theta_{n+1})-g_2(\theta_n)}{g_1(\theta_{n+1})-g_1(\theta_n)}\right|\right)$ is strictly increasing, relabelling the new sequence as simply $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$, and set $j_2^* = 2$. Continue iteratively: in step $m \in \{1, 2, ..., K^*\}$, determine first whether

$$\left(\left|\frac{g_m(\theta_{n+1}) - g_m(\theta_n)}{g_{j^*_{m-1}}(\theta_{n+1}) - g_{j^*_{m-1}}(\theta_n)}\right|\right)$$

forms a bounded sequence. If yes, set $j_m^* = j_{m-1}^*$. If no, extract a subsequence of (the current version of) $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$ along which $\left(\left|\frac{g_m(\theta_{n+1})-g_m(\theta_n)}{g_{j^*_{m-1}}(\theta_{n+1})-g_{j^*_{m-1}}(\theta_n)}\right|\right)$ is strictly increasing, relabel this as the original sequence, and set $j_m^* = m$.

With this construction, a straightforward inductive argument implies that after step m,

$$\left(\left|\frac{g_{m'}(\theta_{n+1}) - g_{m'}(\theta_n)}{g_{j^*_m(\theta_{n+1})} - g_{j^*_m}(\theta_n)}\right|\right) \text{ is a bounded sequence for all } m' \leq m \tag{B2}$$

To see this, assume (B2) holds for $m-1$. In step m, the construction first checks whether

$$\left(\left|\frac{g_m(\theta_{n+1}) - g_m(\theta_n)}{g_{j^*_{m-1}}(\theta_{n+1}) - g_{j^*_{m-1}}(\theta_n)}\right|\right)$$

is bounded. If yes, then we set $j_m^* = j_{m-1}^*$, so it is immediate that (B2) holds also for m. If not, then the construction extracts a strictly increasing subsequence, along which $\left(\left|\frac{g_m(\theta_{n+1})-g_m(\theta_n)}{g_{j^*_{m-1}}(\theta_{n+1})-g_{j^*_{m-1}}(\theta_n)}\right|\right) \to \infty$, implying that the reciprocal subsequence converges to zero: that is,

$$\lim_{n\to\infty}\left(\left|\frac{g_{j^*_{m-1}}(\theta_{n+1}) - g_{j^*_{m-1}}(\theta_n)}{g_m(\theta_{n+1}) - g_m(\theta_n)}\right|\right) = 0$$

Setting $j_m^* = m$, and recalling that the product of two bounded sequences is itself bounded, this, together with (B2) at $m-1$, implies that

$$\frac{g_{m'}(\theta_{n+1}) - g_{m'}(\theta_n)}{g_{j^*_m}(\theta_{n+1}) - g_{j^*_m}(\theta_n)} = \left(\left|\frac{g_{m'}(\theta_{n+1}) - g_{m'}(\theta_n)}{g_{j^*_{m-1}(\theta_{n+1})} - g_{j^*_{m-1}}(\theta_n)}\right|\left|\frac{g_{j^*_{m-1}}(\theta_{n+1}) - g_{j^*_{m-1}}(\theta_n)}{g_m(\theta_{n+1}) - g_m(\theta_n)}\right|\right)$$

is bounded, and therefore (2) again holds also for m. Finally, after step K^*, determine whether

$$\left(\left|\frac{\theta_{n+1}-\theta_n}{g_{j^*_{K^*}}(\theta_{n+1})-g_{j^*_{K^*}}(\theta_n)}\right|\right)$$

is bounded. If yes, set $j^* = j^*_{K^*}$; if no, extract a final subsequence along which $\left(\left|\frac{\theta_{n+1}-\theta_n}{g_{j^*_{K^*}}(\theta_{n+1})-g_{j^*_{K^*}}(\theta_n)}\right|\right)$ is monotone, and set $j^* = 0$.

Letting $g_0(\cdot)$ denote the identity function, we have now created a subsequence $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$ (which converges, since it is a subsequence of a convergent sequence) such that for some $j^* \in \{0, 1, ..., K^*\}$,

$$\left(\left|\frac{g_j(\theta_{n+1})-g_j(\theta_n)}{g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)}\right|\right)_{n\in\mathbb{N}} \text{ is bounded for all } j \in \{0, 1, ..., K^*\}$$

A second application of Bolzano-Weierstrass then implies that we can find a convergent sequence $(\mathbf{z}(\theta_n))_{n\in\mathbb{N}}$ along which

$$\lim_{n\to\infty}\left|\frac{g_j(\theta_{n+1})-g_j(\theta_n)}{g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)}\right| \text{ exists for all } j \in \{0, .., K^*\}$$

That is, g_j is "approximately differentiable" with respect to g_{j^*}. ■

Step 3: Completing the Proof

Take the sequence $(\mathbf{z}(\theta_n))$ constructed in Step 3. For each n, with $g_0(\cdot)$ denoting the identify function, let $(g_j(\theta_n)_{n=0}^{K^*}$ be the initial type pool associated with point $\mathbf{z}(\theta_n)$. Re-order the $g_j's$ such that (i) $\forall j \in \{1, .., K^*\}$, $g_j(\theta_n)$ reveals his type no later than $g_{j-1}(\theta_n)$, and (ii) whenever $g_{j-1}(\theta_n), g_j(\theta_n), g_{j+1}(\theta_n)$ are pooled together in any period t, $g_{j+1}(\theta_n)$ separates from $g_{j-1}(\theta_n)$ no later than $g_j(\theta_n)$.[11] Also redefine T as the first period in which $(\theta_n)_{n\in\mathbb{N}}$ fully reveals his type.[12] So, we have that all types $\{\theta_n b\}_{n\in N}$ remain pooled with at least one other type prior to period T, and then fully separate in period T (thus $a_T(\theta_n) = \theta_n$); all types $\{g_j(\theta_n)b\}_{n\in\mathbb{N}, 2\le j\le K^*}$ fully separate in or prior to period T. Now, in each period t, let k_t denote the number of groups into which $\{\theta^*, g_1(\theta^*), ..., g_{K^*}(\theta^*)\}$ is partitioned (so $k(1) = 1$, $k(T) = K^* + 1$, and $1 \le k(t) < K^* + 1$ for all $2 \le t \le T - 1$), and label the k_t elements of the period-t partition such that higher groups contain higher-indexed types $g_j(\theta^*)$. Recall (again by convergence of our sequence) that whenever $g_j(\theta^*), g_{j'}(\theta^*)$ are pooled together, so must be $g_j(\theta_n), g_{j'}(\theta_n)$ for all θ_n.Therefore, for all t and $i \in \{1, ..., k_t\}$, we can define $a_{t,i}(\theta_n)$ as the period-t action induced by all types in the ith lowest element of the period$-t$ partition of $\{g_j(\theta_n)\}_{j\in\{0,..,K^*\}}$. By optimality for the receiver, $a_{t,i}(\theta_n)$ is equal to a weighted average of these types, with weights proportional to the weight that the DM's beliefs (conditional on his current information set) place on each type. For simplicity, we assume

[11] Note that both conditions are independent of n, by convergence of the $y_t's$.

[12] If this is shorter than the actual horizon, all IC equations in this section require adding duplicates of the final term. This affects them in a symmetric manner, and does not affect the contradiction obtained.

that all weights are equal, so that $a_{t,i}(\theta_n)$ is simply the average type; it should be clear from the sketch in Step 1 that the modification for weighted averages is straightforward, simply requiring that whenever we sum IC equations, we instead use a correspondingly weighted sum. Note also that by construction, $a_{t,i}(\theta_n) \to a_{t,i}(\theta^*)$ for all t, i, and θ_n induces action $a_{t,1}(\theta_n)$ in period t (as our ordering from this paragraph implies that θ_n is always in the lowest element of the partition).[13]

Now: the expert incentive compatibility constraint for type θ_{n+1} to not mimic θ_n is

$$\sum_{t=1}^{T-1} \delta^{t-1} \left(a_{t,1}(\theta_{n+1}) - a_{t,1}(\theta_n)\right) \left(\frac{(a_{t,1}(\theta_{n+1}) + a_{t,1}(\theta_n))}{2} - \theta_{n+1} - 1 \right) \leq \delta^{T-1} \left(\theta_{n+1} - \theta_n\right) \left(1 + \frac{\theta_{n+1} - \theta_n}{2}\right) \tag{B3}$$

and the expert incentive compatibility constraint for type $g_j(\theta_{n+1})$ to not mimic $g_j(\theta_n)$ (for $2 \leq j \leq K^*\}$, if fully separated for the final $\tau_j \geq 1$ periods, and letting $a_t(g_j(\theta_n))$ denote the period-t action induced by type $a_t(g_j(\theta_n))$, is

$$\begin{aligned} &\sum_{t=1}^{T-\tau_j} \delta^{t-1} \left(a_t(g_j(\theta_{n+1})) - a_t(g_j(\theta_n))\right) \left(\frac{(a_t(g_j(\theta_{n+1})) + a_t(g_j(\theta_n)))}{2} - g_j(\theta_{n+1}) - 1 \right) \\ &\leq \left(\sum_{t=T-\tau_j+1}^{T} \delta^t \right) \left(g_j(\theta_{n+1}) - g_j(\theta_n)\right) \left(1 + \frac{(g_j(\theta_{n+1}) - g_j(\theta_n))}{2}\right) \end{aligned}$$

Now, add up this set of $(K^* + 1)$ equations (for $\theta, g_1(\theta), ..., g_{K^*}(\theta)$), denoting the resulting equation by $(\text{IC})_{n+1,n}$.

For the LHS of $(\text{IC})_{n+1,n}$: note first that

$$a_{1,1}(\theta_n) = a_{1,1}(g_1(\theta_n)) = ... = a_{1,1}(g_{K^*}(\theta_n)) = \frac{\theta_{n+1} + g_1(\theta_{n+1}) + ... + g_{K^*}(\theta_{n+1})}{K^* + 1}$$

Using this, the LHS of $(\text{IC})_{n+1,n}$ contains $(K^* + 1)$ appearances of the term $(a_{1,1}(\theta_{n+1}) - a_{1,1}(\theta_n))$, and the total LHS coefficient on this term is

$$\begin{aligned} &(K^* + 1) \left(\frac{(a_{1,1}(\theta_{n+1}) + a_{1,1}(\theta_n))}{2} - \frac{\theta_{n+1} + g_1(\theta_{n+1}) + ... + g_{K^*}(\theta_{n+1})}{K^* + 1} - 1 \right) \\ &\equiv (K^* + 1) \left(-1 - \frac{(a_{1,1}(\theta_{n+1}) - a_{1,1}(\theta_n))}{2} \right) \end{aligned}$$

Similarly, for any subsequent period t, there are k_t distinct action functions $a_{t,i}(\cdot)$: each such function is involved in a number of equations equal to the number of types inducing period-t action $a_{t,i}(\theta^*)$, denote this by $\#(t, i)$, and and contributes the following amount to the LHS of $(\text{IC})_{n+1,n}$

[13] Illustrative example: suppose that initially θ is pooled with 3 other types, then this pool separates into two groups of two in period 2. Our ordering implies that the 2nd-period groups are $\{\theta, g_1(\theta)\}$ (group 1, inducing 2nd-period action $a_{21}(\theta) = \frac{\theta + g_1(\theta)}{2}$), and $\{g_2(\theta), g_3(\theta)\}$ (group 2, inducing $a_{22}(\theta) = \frac{g_2(\theta) + g_3(\theta)}{2}$).

whenever $\#(t,i) \neq 1$ (if $\#(t,i) = 1$, the corresponding term appears on the RHS of $(\mathrm{IC})_{n+1,n}$) :

$$\delta^{t-1} \cdot \#\left(t,i\right) \cdot \left(a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)\right) \left(-1 - \frac{a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)}{2}\right)$$

Therefore, setting $Z \equiv \{(t,i) | t \in \{2,..,T-1\}, i \in \{1,..,k_t\}$, and $\#(t,i) \neq 1\}$, the LHS is of $(\mathrm{IC})_{n+1,n}$ is

$$\begin{aligned} &(K^*+1)\left(a_{1,1}(\theta_{n+1}) - a_{1,1}(\theta_n)\right)\left(-1 - \frac{\left(a_{1,1}(\theta_{n+1}) - a_{1,1}(\theta_n)\right)}{2}\right) \\ &+ \sum_{(t,i)\in Z} \delta^{t-1} \cdot \#\left(t,i\right) \cdot \left(a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)\right)\left(-1 - \frac{a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)}{2}\right) \end{aligned}$$

The RHS of $(\mathrm{IC})_{n+1,n}$ is simply

$$\sum_{j=0}^{K^*}\left(\sum_{t=T-\tau_j+1}^{T}\delta^{t-1}\right)\left[\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right) + \frac{\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right)^2}{2}\right]$$

The relationship between the $a_{t,i}$'s and $g_j's$ implies that this simplifies to

$$\begin{aligned} &\left(\sum_{t=1}^{T-1}\delta^{t-1}\right)(K^*+1)\left(a_1(\theta_{n+1}) - a_1(\theta_n)\right) + \sum_{j=0}^{K^*}\left(\sum_{t=T-\tau_j+1}^{T}\delta^{t-1}\right)\frac{\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right)^2}{2} \\ &- \sum_{(t,i)\in Z}\delta^{t-1} \cdot \#\left(t,i\right) \cdot \left(a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)\right) \end{aligned}$$

Therefore, $(\mathrm{IC})_{n+1,n}$ reduces to

$$\left(a_1(\theta_{n+1}) - a_1(\theta_n)\right) \geq -\frac{\left[\sum_{(t,i)\in Z}\delta^{t-1} \cdot \#\left(t,i\right) \cdot \frac{\left(a_{t,i}(\theta_{n+1}) - a_{t,i}(\theta_n)\right)^2}{2} + \sum_{j=0}^{K^*}\left(\sum_{t=T-\tau_j+1}^{T}\delta^{t-1}\right)\frac{\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right)^2}{2}\right]}{(K^*+1)\left(\sum_{t=0}^{T-1}\delta^t + \left(\frac{\left(a_1(\theta_{n+1}) - a_1(\theta_n)\right)}{2}\right)\right)}$$

Now, divide both sides of this equation by $|g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n)|$. Since $\left(\frac{\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right)}{|g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n)|}\right)$ is bounded (by construction), while convergence of our sequence $(\mathbf{z}(\theta_n))$ implies that $\lim_{n\to\infty}\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right) = 0$, it follows that

$$\lim_{n\to\infty}\sum_{j=0}^{K^*}\left(\sum_{t=T-\tau_j+1}^{T}\delta^{t-1}\right)\frac{\left(g_j(\theta_{n+1}) - g_j(\theta_n)\right)^2}{2\left|g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n)\right|} = 0$$

Since each $a_{t,i}(\cdot)$ is a weighted average of $\{g_j(\cdot)\}_{j=0}^{K^*}$, and the $a'_{t,i}s$ converge, the identical argument implies

$$\lim_{n\to\infty}\sum_{(t,i)\in Z}\delta^{t-1}\cdot\#(t,i)\cdot\frac{(a_{t,i}(\theta_{n+1})-a_{t,i}(\theta_n))^2}{2\,|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)|}=0$$

Therefore, as $n\to\infty$, $(\mathrm{IC})_{n+1,n}$ becomes

$$\lim_{n\to\infty}\frac{(a_1(\theta_{n+1})-a_1(\theta_n))}{|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)|}\geq-\lim_{n\to\infty}\frac{\left[\sum_{(t,i)\in Z}\delta^{t-1}\cdot\frac{(a_{t,i}(\theta_{n+1})-a_{t,i}(\theta_n))^2}{2}+\sum_{j=0}^{K^*}\left(\sum_{t=T-\tau+1}^{T}\delta^{t-1}\right)\frac{(g_j(\theta_{n+1})-g_j(\theta_n))^2}{2}\right]}{(K^*+1)\left(\sum_{t=0}^{T-1}\delta^t+\left(\frac{(a_1(\theta_{n+1})-a_1(\theta_n))}{2}\right)\right)}=0$$

Similarly, the sum of equations for types $g_j(\theta_n)$ to not mimic $g_j(\theta_{n+1})$ (for $j\in\{0,..,K^*\}$), call this $(\mathrm{IC})_{n,n+1}$, is

$$\frac{(a_1(\theta_{n+1})-a_1(\theta_n))}{|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)|}\leq\frac{\left[\sum_{(t,i)\in Z}\delta^{t-1}\cdot\#(t,i)\cdot\frac{(a_{t,i}(\theta_{n+1})-a_{t,i}(\theta_n))^2}{2\left|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)\right|}+\sum_{j=0}^{K^*}\left(\sum_{t=T-\tau+1}^{T}\delta^{t-1}\right)\frac{(g_j(\theta_{n+1})-g_j(\theta_n))^2}{2\left|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)\right|}\right]}{(K^*+1)\left(\sum_{t=0}^{T-1}\delta^t-\left(\frac{(a_1(\theta_{n+1})-a_1(\theta_n))}{2}\right)\right)}$$
$$\to 0\text{ as }n\to\infty$$

So by the squeeze theorem, $(\mathrm{IC})_{n+1,n}$ together with $(\mathrm{IC})_{n,n+1}$ imply that we need

$$\lim_{n\to\infty}\frac{(a_1(\theta_{n+1})-a_1(\theta_n))}{|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)|}=0$$

This implies that as $n\to\infty$, the effect of the period-1 action rule on incentive constraints becomes negligible for *all* types, and therefore the IC constraints reduce to those starting in period 2.[14] Sum these, and apply an argument analogous to the one above, to establish that we need also

$$\lim_{n\to\infty}\frac{(a_{2,i}(\theta_{n+1})-a_{2,i}(\theta_n))}{|g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)|}=0$$

for all period-2 action functions $a_{2,i}(\cdot)$. Thus, the actions in period 2 also contribute nothing to the incentive compatibility constraints, so we can repeat the argument starting in period 3. Continue iteratively, establishing that if all of the IC equations hold, then we need

$$\lim_{n\to\infty}\frac{(a_{t,i}(\theta_{n+1})-a_{t,i}(\theta_n))}{|g_{j^*}(\theta_{n+1})-g_j(\theta_n)|}=0\text{ for all }t\in\{1,..,T\},i\in\{1,..,k_t\}\qquad\text{(B4)}$$

[14] This relies on δ being positive, so that the effect of subsequent periods on the incentive constraints is large relative to a term which becomes arbitrarily close to zero as $n\to\infty$.

Now, finally, recall that equation (B3) applied to the curve $g_{j^*}(\cdot)$ requires

$$\frac{\sum_{t=1}^{T-\tau_j} \delta^{t-1} \frac{(a_{t,1}(\theta_{n+1})-a_{t,1}(\theta_n))}{\left|g_{j^*}(\theta_{n+1})-g_j(\theta_n)\right|} \left(\frac{(a_{t,1}(\theta_{n+1})+a_{t,1}(\theta_n))}{2} - g_{j^*}(\theta_{n+1}) - 1\right)}{\left(1+\frac{g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)}{2}\right)} \leq \left(\sum_{t=T-\tau_j+1}^{T} \delta^{t-1}\right) \frac{(g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n))}{|g_{j^*}(\theta_{n+1}) - g_j(\theta_n)|}$$

And the incentive compatibility condition for $g_{j^*}(\theta_n)$ to not mimic $g_{j^*}(\theta_{n+1})$ is

$$\frac{\sum_{t=1}^{T-\tau_j} \delta^{t-1} \frac{(a_{t,1}(\theta_{n+1})-a_{t,1}(\theta_n))}{\left|g_{j^*}(\theta_{n+1})-g_j(\theta_n)\right|} \left(\frac{(a_{t,1}(\theta_{n+1})+a_{t,1}(\theta_n))}{2} - g_{j^*}(\theta_n) - 1\right)}{\left(1-\frac{g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)}{2}\right)} \geq \left(\sum_{t=T-\tau_j+1}^{T} \delta^{t-1}\right) \frac{(g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n))}{|g_{j^*}(\theta_{n+1}) - g_j(\theta_n)|}$$

By (B4), and recalling that convergence of $(\mathbf{z}(\theta_n))$ implies that $\lim_{n\to\infty} \frac{g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n)}{2} = 0$, the LHS of both inequalities go to zero, so by the squeeze theorem we need

$$\lim_{n\to\infty} \frac{(g_{j^*}(\theta_{n+1}) - g_{j^*}(\theta_n))}{|g_{j^*}(\theta_{n+1}) - g_j(\theta_n)|} = 0$$

A contradiction, since $\frac{(g_{j^*}(\theta_{n+1})-g_{j^*}(\theta_n))}{\left|g_{j^*}(\theta_{n+1})-g_j(\theta_n)\right|}$ is either 1, -1, or oscillating, and cannot tend to zero. ■

11. Appendix C: Illustration of monotonic partition equilibria

Here we study monotonic partition equilibria when the expert's and the decision maker's preferences are given by (1).

From Proposition **??** we know that there exists some period $\hat{T} \leq T$ where the dynamic indifference condition reduces to the static because no further subdivisions occur. At $\hat{T}$ the following must be true:

$$-\Sigma_{t=\hat{T}+1}^{T}\delta^{t-1}(y_i - \theta_i - b)^2 = -\Sigma_{t=\hat{T}+1}^{T}\delta^{t-1}(y_{i+1} - \theta_i - b)^2, \tag{10}$$

where y_i (respectively y_{i+1}) is the action chosen by the expert after message m_i (respectively m_{i+1}) is sent at $\hat{T}$. These actions remain *constant* until the end of the game T, as by the definition of $\hat{T}$, no further subdivisions occur from $\hat{T}+1$ on. This observation together with (10) imply that for each interval $[\theta_i, \theta_{i+1}]$ at $\hat{T}$, the further subdivisions are described by cutoffs $\theta_0^i, ..., \theta_{p(i)}^i$ that satisfy the conditions of a static partitional equilibrium. The only difference from the static case at T (or $\hat{T}$), is that the decision maker's beliefs about the state are uniform on some interval $[z_1, z_2]$, instead of uniform on $[0,1]$. Hence, even though the arbitrage condition (2) still holds, the solutions of the resulting difference equation (3) differ.

We now describe the solution of (3), when the support of the decision maker's beliefs is uniform on some interval $[z_1, z_2]$. The solution of (3) is of the form $\theta_i = \lambda i^2 + \mu i + v$. Substituting this expression into (3) we get that $\lambda = 2b$. Now the initial condition is $\theta_0 = z_1$, which implies that $v = z_1$ and the final condition $\theta_p = z_2$ implies that

$$\mu = \frac{1}{p}(z_2 - z_1) - 2bp,$$

where, as before, p is the number of subintervals that we divide the type space. Then, when we subdivide an interval of length $[z_1, z_2]$ in p subintervals then the cutoffs are given by:

$$\theta_i = \left(\frac{1}{p}(z_2 - z_1) - 2bp\right) i + 2bi^2 + z_1, \;\; i = 0, .., p. \tag{11}$$

For $i = 1$ we have that

$$\theta_1 = \frac{z_2}{p} + \frac{z_1\,(p-1)}{p} - 2b\,(p-1) \tag{12}$$

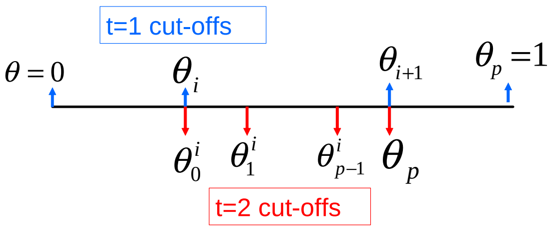
and for $i = p - 1$

$$\theta_{p-1} = \frac{z_1}{p} + \frac{(p-1)z_2}{p} - 2b(p-1). \tag{13}$$

With the help of these preliminary findings we move on to construct monotonic partition equilibria for the uniform-quadratic case when $T = 2$. This construction illustrates nicely the complications that arise from the intetemporal nature of the indifference equations at need to be satisfied at the cut-off states.

Constructing monotonic partition equilibria when $T = 2$

Suppose that $\delta_E = \delta_{DM} = 1$. The $t = 1$ cutoffs are denoted by $\theta_0, ..., \theta_p$. Then at $t = 2$ each interval i is divided into $p\,(i)$ subintervals. The $t = 2$ cutoffs are denoted by $\theta_0^i, ..., \theta_{p(i)}^i$.



Dynamic Monotonic Partitional Equilibria

If type θ_i is indifferent between sending message m_i and m_{i+1} at stage 1, the following must be true:

$$\left(\frac{\theta_{i-1} - \theta_i}{2} - b\right)^2 + \left(\frac{\theta_{p(i)-1}^i - \theta_i}{2} - b\right)^2 = \left(\frac{\theta_{i+1} - \theta_i}{2} - b\right)^2 + \left(\frac{\theta_1^{i+1} - \theta_i}{2} - b\right)^2, \tag{14}$$

where θ_1^{i+1} is the first cut-off of the $i + 1^{th}$ partition at $t = 2$,and $\theta_{p(i)-1}^i$ is the $p(i) - 1$ cut-off of the i^{th} partition at $t = 2$. At an equilibrium these $T = 2$ cutoffs have to satisfy the static arbitrage condition. Hence cut-off θ_1^{i+1} is given by (12), with $z_1 = \theta_i$ and $z_2 = \theta_{i+1}$ and $p = p(i + 1)$, where $p(i + 1)$ is the number of subintervals that the $i + 1$ interval is going to be divided in the second period:

$$\theta_1^{i+1} = \frac{\theta_{i+1}}{p(i+1)} + \frac{\theta_i\,(p(i+1) - 1)}{p(i+1)} - 2b\,(p(i+1) - 1)\,, \tag{15}$$

and cut-off $\theta_{p(i)-1}^i$ is given by (13) with p=$p(i)$; $\theta_{i-1} = z_1$ and $\theta_i = z_2$:

$$\theta_{p(i)-1}^i = -2b\,(p(i) - 1) + \frac{\theta_i\,(p(i) - 1)}{p(i)} + \frac{\theta_{i-1}}{p(i)}. \tag{16}$$

Subtracting θ_i from both sides of (15) and (16) we get that

$$\begin{aligned}\theta_1^{i+1} - \theta_i &= \frac{\theta_{i+1} - \theta_i}{p(i+1)} - 2b\left(p(i+1) - 1\right) \\ \theta_{p(i)-1}^i - \theta_i &= \frac{\theta_{i-1} - \theta_i}{p(i)} - 2b\left(p(i) - 1\right).\end{aligned}$$

Now by substituting these expressions into (14) and get that

$$\begin{aligned}&\left(\frac{\theta_{i-1} - \theta_i}{2} - b\right)^2 + \left(\frac{-2b\left(p(i) - 1\right) + \frac{\theta_{i-1} - \theta_i}{p(i)}}{2} - b\right)^2 \\ &= \left(\frac{\theta_{i+1} - \theta_i}{2} - b\right)^2 + \left(\frac{\frac{\theta_{i+1} - \theta_i}{p(i+1)} - 2b\left(p(i+1) - 1\right)}{2} - b\right)^2,\end{aligned}$$

which can be rewritten as

$$\left(\frac{\theta_{i-1} - \theta_i}{2} - b\right)^2 + \left(\frac{\theta_{i-1} - \theta_i}{2p(i)} - bp(i)\right)^2 = \left(\frac{\theta_{i+1} - \theta_i}{2} - b\right)^2 + \left(\frac{\theta_{i+1} - \theta_i}{2p(i+1)} - bp(i+1)\right)^2,$$

which can be further reduced to:

$$\begin{aligned}&\left[\frac{\theta_{i-1} - \theta_{i+1}}{2}\right]\left[\frac{\theta_{i-1} + \theta_{i+1} - 2\theta_i}{2} - 2b\right] \\ &= \left[\frac{p(i)\left(\theta_{i+1} - \theta_i\right) - p(i+1)\left(\theta_{i-1} - \theta_i\right)}{2p(i+1)p(i)} - b(p(i+1) - p)\right] \cdot \\ &\left[\frac{p\left(\theta_{i+1} - \theta_i\right) + p(i+1)\left(\theta_{i-1} - \theta_i\right)}{2p(i+1)p(i)} - b(p(i+1) + p(i))\right]\end{aligned} \tag{17}$$

Now by letting $p(i) = p(i+1)$ (17) reduces to

$$\left[\frac{\theta_{i-1} - \theta_{i+1}}{2}\right]\left[\frac{\theta_{i-1} + \theta_{i+1} - 2\theta_i}{2} - 2b\right] = \left[\frac{\theta_{i+1} - \theta_{i-1}}{2p}\right]\left[\frac{\theta_{i+1} + \theta_{i-1} - 2\theta_i}{2p} - 2bp\right],$$

which ultimately reduces to

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} + \frac{8bp^2}{1+p^2}. \tag{18}$$

By defining $\hat{b} = \frac{2p^2 b}{p^2+1}$, we see that (18) reduces to $\theta_{i+1} = 2\theta_i - \theta_{i-1} - 4\hat{b}$, which is the same as the static difference equation given by (3) but with higher bias factor $\hat{b} > b$.

Now, from (5) we get that the largest number of subintervals that the state space can be subdivided into in the first period is decreasing the number of subintervals that it is subdivided to in the second period, as $\frac{1}{2}\sqrt{1+\frac{2}{b}} = \frac{1}{2}\sqrt{1+\frac{2}{\frac{2bp^2}{p^2+1}}} = \frac{1}{2}\sqrt{1+\frac{2(p^2+1)}{2bp^2}}$ is decreasing in p, because $\frac{2(p^2+1)}{2bp^2}$ is decreasing in p (in fact its derivative is $\frac{8bp^3-2(p^2+1)4bp}{(2bp^2)^2} = \frac{8bp^3-8bp^3-8bp}{(2bp^2)^2} = \frac{-8bp}{(2bp^2)^2}$).

This completes our illustration of how monotonic partition equilibria look like in a two-stage game. For equilibria where each interval is divided in the same number of sub-intervals in the second stage, we showed that the location of first period cutoffs will be the same as in an equilibrium of the one shot game with a higher bias $\hat{b} = \frac{2bp^2}{p^2+1}$, where p is the number of subintervals each interval is divided in the second stage. As it became obvious from our construction, it is quite challenging to provide an explicit characterization of the cutoffs because of the intetemporal nature of the arbitrage conditions.

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