EQUIVALENCE OF THE INFORMATION STRUCTURE WITH UNAWARENESS TO THE LOGIC OF AWARENESS

SANDER HEINSALU

Abstract. Here it is shown that the unawareness structure in Li (2009) is equivalent to a single-agent version of the canonical Kripke structure for the logic of awareness in Fagin and Halpern (1988). The results of Li (2009), e.g. the possibility of nontrivial unawareness in the unawareness structure, follow from the properties of the model in Fagin and Halpern (1988).

1. Introduction

The standard approach to modelling knowledge in economics uses information partitions on the state space and is shown to be incompatible with nontrivial unawareness by Dekel, Lipman, and Rustichini (1998) Theorem 1. There are a number of ways the literature has circumvented this impossibility theorem. The unawareness structure constructed in Li (2009) bypasses the impossibility result by having the agent use a (possibly different) subjective state space at each state. The logic of awareness in Fagin and Halpern (1988) captures nontrivial unawareness by making the agent aware of only a subset of the propositions in the model. The agent can believe or disbelieve only formulas constructed from these propositions. A similar approach is taken in the logic defined in Modica and Rustichini (1999)—the agent can reason only about the atomic sentences of which he is aware. Heifetz, Meier, and Schipper (2006) introduce a lattice of state spaces with different expressive power. The agent's possibility set at one state is contained in a possibly less expressive state.

Some of the models capturing nontrivial unawareness have been shown to be equivalent. Halpern (2001) proved that the propositional model in Modica and Rustichini (1999) is a special case of Fagin and Halpern's logic of awareness. In addition, Halpern and Rêgo (2008) Theorem 3.2 shows the lattice of state spaces in Heifetz, Meier, and Schipper (2006) is equivalent to the logic of awareness.

According to Heifetz, Meier, and Schipper (2008), it is still an open question whether the unawareness structure of Li (2009) is equivalent to those of Modica and Rustichini (1999) or Heifetz, Meier, and Schipper (2006). This paper answers the question by showing the equivalence of Li's unawareness structure and a single-agent version of the Fagin and Halpern (1988) logic of awareness. Since Halpern (2001) and Halpern and Rêgo (2008) proved that the models of Modica and Rustichini and Heifetz et al. are special cases of the logic of awareness, Li's model turns out to be equivalent to both of them.

2. Overview of Li (2009)

The primitives of the model in Li (2009) are a set of questions Q^*, a state space Ω^* consisting of vectors of answers to all the questions, an awareness function W^*, giving for each state the set of questions of which the agent is aware, and a possibility correspondence P^*, giving for each state the set of states that the agent thinks possible. Formally,

$$\Omega^* = \prod_{q\in Q^*} \{1_q, 0_q\} \times \{\Delta\}, \quad W^* : \Omega^* \to 2^{Q^*}, \quad P^* : \Omega^* \to 2^{\Omega^*}$$

Date: September 5, 2010.

The author thanks Benjamin Polak for a series of very helpful discussions on this paper. The remaining errors and omissions are the author's.

Li (2009) interprets the symbol Δ as "cogito ergo sum" and no comment is offered as to why it is included.

We can view the objective state space as being the unit cube in $|Q^*|$-dimensional space, where the dimensions are labelled with the questions in Q^*. Each state is a vertex of the cube, i.e. a $|Q^*|$-dimensional vector with coordinate values in $\{0,1\}$. At state ω^*, the awareness function W^* partitions the set of questions into the aware set $W^*(\omega^*)$ and the unaware set $W^*(\omega^*)^c$. Li (2009) defines the subjective state space at ω^* as the projection of the objective state space on the subspace generated by the dimensions corresponding to questions in $W^*(\omega^*)$.

$$\Omega(\omega^*) = \prod_{q\in W^*(\omega^*)} \{1_q, 0_q\} \times \{\Delta\}$$

Each subjective state space is a standard partitional information structure. The collection of events of a subjective state space is closed under union, intersection and complementation.

For any ω_0^* and any $\omega \in \Omega(\omega_0^*)$, denote by $\omega(q)$ the q-th coordinate of ω and by $\omega(F) = (\omega(q))_{q\in F}$ the vector created by picking the coordinates corresponding to questions $F \subset Q^*$. Similarly for ω^* we denote by $\omega^*(F)$ the vector of coordinates corresponding to F.

The function q is defined for all states ω_0^* and for all subjective events $E \subset \Omega(\omega_0^*)$ as the questions defining the subjective state space of E, formally $q : \cup_{\omega_0^*\in\Omega^*} 2^{\Omega(\omega_0^*)} \to 2^{Q^*}$ and $E \subset \Omega(\omega_0^*)$ maps to $q(E) = W^*(\omega_0^*)$. Li (2009) interprets $q(E)$ as the awareness information of E.

For any ω_0^* and for any $E \subset \Omega(\omega_0^*)$, the notation E^* means the set of objective states that project into E, formally $E^* = \{\omega^* : \omega^*(W^*(\omega_0^*)) \in E\}$. Li interprets E^* as the factual information of E.

Events $E_1, E_2 \subset \Omega(\omega_1^*)$ in the same subjective state space are distinguished by the objective states they contain, $E_1^* \neq E_2^*$, but their awareness information is the same, $q(E_1) = q(E_2) = q(\Omega(\omega_1^*)) = W^*(\omega_1^*)$. Events from different state spaces $E_1 \subset \Omega(\omega_1^*)$, $E_3 \subset \Omega(\omega_3^*)$ may or may not contain the same objective states, but their awareness information is different, $q(E_1) \neq q(E_3)$. An example may clarify the distinctions.

Example 1. *The questions are* $Q^* = \{q_1, q_2, q_3\}$*, so the states are* $\Omega^* = \{000, 001, 010, 011, 100, 101, 110, 111\}$ *where* 010 *is interpreted as an affirmative answer to question* q_2 *and negative answers to* q_1 *and* q_3*. Awareness is given by* $W^*(000) = \emptyset$, $W^*(001) = \{q_1\}$, $W^*(010) = \{q_1, q_2\}$, $W^*(\omega^*) = Q^*\ \forall\omega^* \notin \{000, 001, 010\}$*. The subjective state spaces based on this awareness information are* $\Omega(000) = \{\Delta\}$, $\Omega(001) = \{0_1, 1_1\}$, $\Omega(010) = \{0_1 0_2, 0_1 1_2, 1_1 0_2, 1_1 1_2\}$, $\Omega(\omega^*) = \Omega^*\ \forall\omega^* \notin \{000, 001, 010\}$*. Another notation for these spaces is* $\Omega(000) = \{xyz\}$, $\Omega(001) = \{0yz, 1yz\}$, $\Omega(010) = \{00z, 01z, 10z, 11z\}$, $\Omega(\omega^*) = \Omega^*\ \forall\omega^* \notin \{000, 001, 010\}$.

Take $E_1 = \{1_1 0_2, 1_1 1_2\} \subseteq \Omega(010)$, $E_2 = \{0_1 0_2, 1_1 0_2\} \subseteq \Omega(010)$, $E_3 = \{1_1\} \subseteq \Omega(001)$*. The events* E_1, E_2 *are from the same state space and both have the awareness information* $q(E_1) = q(E_2) = \{q_1, q_2\}$*, but they are distinguished by their factual information:* $E_1^* = \{100, 101, 110, 111\}$ *and* $E_2^* = \{000, 001, 100, 101\}$*. The events* E_1, E_3 *have the same factual information,* $E_1^* = \{100, 101, 110, 111\} = E_3^*$*, but their awareness information is different,* $q(E_1) = \{q_1, q_2\}$ *but* $q(E_3) = \{q_1\}$.

Since subjective states correspond to events in the objective state space, another representation is

$$E_1 = \{\{100, 101\}, \{110, 111\}\}, \quad E_3 = \{\{100, 101, 110, 111\}\}$$

Looking only at the objective states that an event contains deletes the awareness information (the partition the event came from), resulting in $E_1^* = \{100, 101, 110, 111\} = E_3^*$*. This alternative representation is the same as for a knowledge partition. Unlike in a knowledge partition, the agent does not know that his partition cells consist of multiple states. To the unaware agent, each partition cell looks like a single subjective state.*

The traditional definition of knowledge in epistemic logic is truth in all possible states. For any event $F \subset \Omega^*$, the agent knows F in the states $K_{\text{trad}}(F) = P^{*+}(F) = \{\omega^* : P^*(\omega^*) \subseteq F\}$, where P^{*+} denotes the lower inverse of the possibility correspondence. By Bergé (1963), the lower inverse of a correspondence coincides with the inverse if the correspondence is a function, so knowledge can be thought of as a generalized inverse of possibility.

Since the unawareness structure contains both objective and subjective events, the traditional definition of knowledge is modified slightly and the resulting operator K^* is called objective knowledge. K^* is the extension of traditional knowledge to subjective events and presupposes full awareness. For any subjective event E in any state space $\Omega(\omega_0^*)$, objective knowledge of E is defined as

$$K^*(E) = K_{\text{trad}}(E^*) = \{\omega^* : P^*(\omega^*) \subseteq E^*\}$$

Fix ω_0^*, let the projection of the possibility correspondence at ω_0^* onto the subjective space at ω_0^* be

$$P_{\omega_0^*} = \{\omega \in \Omega(\omega_0^*) : \exists \omega^* \in P^*(\omega_0^*) \text{ s.t. } \omega^*(W^*(\omega_0^*)) = \omega\}$$

Li (2009) defines the subjective possibility correspondence at the objective state ω_0^* as $P_{\omega_0^*}(\omega) = P_{\omega_0^*}$ if $\omega \in P_{\omega_0^*}$, and leaves $P_{\omega_0^*}(\omega)$ undefined for $\omega \notin P_{\omega_0^*}$. The subjective possibility correspondence is quite trivial, since the set of possible states is the same for every subjective state where the possibility correspondence is defined. Based on subsequent definitions in Li (2009), implicitly the definition of the subjective possibility correspondence is extended to the undefined part by $P_{\omega_0^*}(\omega) = \Omega(\omega^*)\ \forall \omega \notin P_{\omega_0^*}$.

Li (2009) distinguishes two concepts of knowledge in the unawareness structure: subjective knowledge from the modeller's perspective, denoted K, and subjective knowledge from the agent's perspective at state ω^*, denoted $\tilde{K}_{\omega^*}$. We start by defining the agent's subjective knowledge of event $E \subset \Omega(\omega_1^*)$ from his own perspective as

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \{\omega \in \Omega(\omega^*) : (P_{\omega^*}(\omega))^* \subseteq E^*\} & \text{if } q(E) \subseteq W^*(\omega^*) \\ \emptyset_E & \text{if } q(E) \nsubseteq W^*(\omega^*) \end{cases}$$

where $\emptyset_E^* = \emptyset$ and $q(\emptyset_E) = q(E)$. The above definition is equivalent to Li's definition[1]. Note that the event $\tilde{K}_{\omega^*}(E)$ may be in different state spaces, depending on whether $q(E) \subseteq W^*(\omega^*)$ or not, because the event $\emptyset_E$ is in the same state space as E, which may be different from $\Omega(\omega^*)$. This property is used to simplify the definition of iterated subjective knowledge from the modeller's perspective K^n later.

For any ω^*, ω_1^* and any subjective event $E \subseteq \Omega(\omega_1^*)$, Li defines iterated subjective knowledge from the agent's perspective as $\tilde{K}^n_{\omega^*}(E) = \tilde{K}_{\omega^*}(\tilde{K}^{n-1}_{\omega^*}(E))$ and uses this to define (iterated) subjective knowledge from the modeller's perspective as

$$K^n(E) = \left\{\omega^* \in \Omega^* : \omega^*(W^*(\omega^*)) \in \tilde{K}^n_{\omega^*}(E)\right\} \tag{1}$$

This is the expansion of the agent's knowledge to the objective states. According to Li (2009) equation (2.8), first order subjective knowledge from the modeller's perspective can alternatively be expressed as

$$K(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\}$$

In words, knowledge of E means that the factual content of the subjective states that are considered possible implies the factual content of event E and the agent has sufficient awareness to recognize E. The derivation of the

[1] Li (2009) definition of subjective knowledge from the agent's perspective is in the notation of the present paper

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \{\omega \in \Omega(\omega^*) : P_{\omega^*}(\omega) \subseteq \{\omega' \in \Omega(\omega^*) : \omega'(W^*(\omega_1^*)) \in E\}\} & \text{if } q(E) \subseteq W^*(\omega^*) \\ \emptyset_E & \text{if } q(E) \nsubseteq W^*(\omega^*) \end{cases}$$

alternative expression of first order subjective knowledge from the modeller's perspective is

$$K(E) = \left\{\omega^* : \omega^*(W^*(\omega^*)) \in \tilde{K}_{\omega^*}(E)\right\} = \tag{2}$$
$$= \{\omega^* : \omega^*(W^*(\omega^*)) \in \{\omega \in \Omega(\omega^*) : (P_{\omega^*}(\omega))^* \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\}\} =$$
$$= \{\omega^* : (P_{\omega^*}(\omega^*(W^*(\omega^*))))^* \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\} = \{\omega^* : P^*(\omega^*) \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\}$$

We only need to use the first line of the definition of $\tilde{K}_{\omega^*}$ in this derivation, because no state can belong to the empty set in any state space.

Define iterated agent's knowledge from the modeller's perspective (for any subjective event E in any space) as

$$K^n(E) = \left\{\omega^* : P^*(\omega^*) \subseteq K^{n-1}(E),\ q(E) \subseteq W^*(\omega^*)\right\} \tag{3}$$

Note that $q(E)$, not $q(K^{n-1}(E))$ is used in the definition. Since the event $K(E)$ is objective and thus has full awareness attached, iterating knowledge by $K^n(E) = K(K^{n-1}(E))$ would force the agent to have full awareness if he had any higher order knowledge. By an inductive argument, this definition is equivalent to

$$K^n(E) = (K^*)^n(E) \cap (K^*)^{n-1}(\{\omega^* : q(E) \subseteq W^*(\omega^*)\}) \cap ... \cap K^*(\{\omega^* : q(E) \subseteq W^*(\omega^*)\}) \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\}$$

Denoting $\{\omega^* : q(E) \subset W^*(\omega^*)\}$ by $A(E)$, the derivation of this is

$$K^n(E) = \left\{\omega^* : P^*(\omega^*) \subseteq K^{n-1}(E)\right\} \cap A(E) = K^*(K^{n-1}(E)) \cap A(E) =$$
$$= K^*\left(\left\{\omega^* : P^*(\omega^*) \subseteq K^{n-2}(E)\right\} \cap A(E)\right) \cap A(E) =$$
$$= (K^*)^2\left(K^{n-2}(E)\right) \cap K^*A(E) \cap A(E) = ... = (K^*)^n(E) \cap (K^*)^{n-1}A(E) \cap ... \cap K^*A(E) \cap A(E)$$

We show by induction on n that the definition of K^n in equation (3) is equivalent to Li's definition in equation (1). The base case K is proved in equation (2). For the inductive step, assume that the result has been shown for K^{n-1}.

$$K^n(E) = \left\{\omega^* : \omega^*(W^*(\omega^*)) \in \tilde{K}_{\omega^*}(\tilde{K}^{n-1}_{\omega^*}(E))\right\} =$$
$$= \begin{cases} \left\{\omega^* : \omega^*(W^*(\omega^*)) \in \left\{\omega \in \Omega(\omega^*) : (P_{\omega^*}(\omega))^* \subseteq (\tilde{K}^{n-1}_{\omega^*}(E))^*\right\}\right\} & \text{if } q(\tilde{K}^{n-1}_{\omega^*}(E)) \subseteq W^*(\omega^*) \\ \left\{\omega^* : \omega^*(W^*(\omega^*)) \in \emptyset_{\tilde{K}^{n-1}_{\omega^*}(E)}\right\} & \text{if } q(\tilde{K}^{n-1}_{\omega^*}(E)) \not\subseteq W^*(\omega^*) \end{cases} =$$
$$= \begin{cases} \left\{\omega^* : (P_{\omega^*}(\omega^*(W^*(\omega^*))))^* \subseteq (\tilde{K}^{n-1}_{\omega^*}(E))^*\right\} & \text{if } q(E) \subseteq W^*(\omega^*) \\ \emptyset & \text{if } q(E) \not\subseteq W^*(\omega^*) \end{cases} =$$
$$= \left\{\omega^* : P^*(\omega^*) \subseteq (\tilde{K}^{n-1}_{\omega^*}(E))^*,\ q(E) \subseteq W^*(\omega^*)\right\} = K^*\left((\tilde{K}^{n-1}_{\omega^*}(E))^*\right) \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\} =$$
$$= K^*\left(\left\{\omega^* : P^*(\omega^*) \subseteq (\tilde{K}^{n-2}_{\omega^*}(E))^*\right\} \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\}\right) \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\} =$$
$$= (K^*)^n(E) \cap (K^*)^{n-1}A(E) \cap ... \cap K^*A(E) \cap A(E)$$

For the third step in the derivation we use the fact that, based on the definition of $\tilde{K}_{\omega^*}$,

$$q(\tilde{K}_{\omega^*}(E)) = \begin{cases} W^*(\omega^*) & \text{if } q(E) \subseteq W^*(\omega^*) \\ q(E) & \text{if } q(E) \not\subseteq W^*(\omega^*) \end{cases}$$

from which it follows that $q(\tilde{K}_{\omega^*}(E)) \subseteq W^*(\omega^*)$ iff $q(E) \subseteq W^*(\omega^*)$, and by an inductive argument $q(\tilde{K}^{n-1}_{\omega^*}(E)) \subseteq W^*(\omega^*)$ iff $q(E) \subseteq W^*(\omega^*)$. Note that for iterated subjective knowledge $K^n(E) \neq K^{n-k}K^k(E)$, since the definition of K^n contains $q(E)$, not $q(K^{n-1}(E))$.

Unawareness of an event E in state ω^* means that the awareness information of E is not in the agent's awareness set at ω^*.

$$U(E) = \{\omega^* : q(E) \nsubseteq W^*(\omega^*)\}$$

Awareness is the complement of unawareness, it is the states in which all questions of E are in the awareness set $W^*(\omega^*)$, formally $A(E) = \{\omega^* : q(E) \subseteq W^*(\omega^*)\} = \neg U(E)$.

3. Overview of Fagin and Halpern (1988) logic of awareness

Part 4 of Fagin and Halpern (1988) describes two intertwined models with the same content—a set-based model called the Kripke structure for awareness and a propositional model called the logic of awareness. Both models use the same set of primitive propositions Φ. The main connecting element is the truth assignment to primitives. Unlike in traditional modal logic, the two models are not defined separately in Fagin and Halpern (1988). The definition of the logic of awareness uses elements of the Kripke structure.

A Kripke structure for awareness is the tuple $\langle S, \pi, \mathcal{A}_1, ..., \mathcal{A}_n, \mathcal{B}_1, ..., \mathcal{B}_n \rangle$. S is the set of states. The truth function $\pi : S \times \Phi \to \{0, 1\}$ assigns for each primitive proposition $p \in \Phi$ and for each state s a truth value with the interpretation that $\pi(s, p) = 1$ iff p is true in s. We have two special primitive propositions **true** and **false** with the property that $\pi(s, \mathbf{true}) = 1\ \forall s$ and $\pi(s, \mathbf{false}) = 0\ \forall s$. If a formula ϕ is true in all states of the model, we call ϕ valid. Based on the truth assignment, we can define the truth set $\|p\|$ of each primitive proposition p as the states in which the proposition is true, formally $\|p\| = \{s \in S : \pi(s, p) = 1\}$.

For each agent i and for each state s, Fagin and Halpern (1988) add an awareness set $\mathcal{A}_i(s)$ collecting the primitive propositions of which the agent is aware, so $\mathcal{A}_i : S \to 2^{\Phi}$. Unlike most papers on logic where the set-based and propositional models are initially kept separate and only connected after they are fully constructed, Fagin and Halpern (1988) connect states and propositions with the operators $\mathcal{A}_i$ while defining their models.

We can define the Kripke structure for awareness without referring to the logic of awareness by using truth sets to confine the $\mathcal{A}_i$ operators to the set-based model. The modified definition of awareness operators is $\mathbb{A}_i : S \to 2^{2^S}$, where $\mathbb{A}_i(s)$ is the collection of truth sets of the propositions in $\mathcal{A}_i(s)$, formally $\mathbb{A}_i(s) = \{\|p\| : p \in \mathcal{A}_i(s)\}$.

For each agent, Fagin and Halpern (1988) also define a possibility correspondence $\mathcal{B}_i : S \to 2^S$, which gives for each state the set of states that a logically omniscient agent would think possible.

The logic of awareness consists of primitive propositions $p \in \Phi$, including **true** and **false**, to which logical and modal operators are applied to produce formulas $\phi \in \Psi$. The primitive logical operators are $\wedge$ and $\neg$. The logical operators $\vee, \Rightarrow, \Leftrightarrow$, are derived from $\wedge, \neg$ as in propositional logic:

$$\phi \vee \psi \equiv \neg(\neg\phi \wedge \neg\psi), \quad \phi \Rightarrow \psi \equiv \neg\phi \vee \psi, \quad \phi \Leftrightarrow \psi \equiv (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$$

The modal operators are (for each agent i) the operator for implicit belief L_i, the operator for explicit belief B_i and the awareness operator A_i.

The operators are combined with primitive propositions in an inductive way to construct well-formed formulas (wff) denoted by ϕ, ψ. The syntactic construction of well-formed formulas is

(1) All primitive propositions $p \in \Phi$ are wff
(2) If ϕ and ψ are wff, then $\neg\phi$ and $\phi \wedge \psi$ are wff
(3) If ϕ is a wff, then $L_i\phi$, $B_i\phi$ and $A_i\phi$ are wff

The logical and modal operators map formulas to formulas. The primitive propositions are a subset of the set of well-formed formulas. An example of a wff is $\phi = \neg B_i L_j(\neg p)$, which is interpreted as 'agent i does not explicitly believe that agent j implicitly believes that p is false'. For any formula ϕ, denote the set of primitive propositions found in ϕ by $\text{Prim}(\phi)$.

Certain formulas of the logic, called theorems, are later used to connect the propositional and set-based models. Any formula valid in the Kripke structure (true in all its states) is proved to be a theorem of the logic and vice versa. All logical tautologies are identified with the proposition **true** and the negations of tautologies with **false**. All formulas of the form

- **(3):** $\neg L_i\mathbf{false}$
- **(K):** $L_i\phi \wedge L_i(\phi \Rightarrow \psi) \Rightarrow L_i\psi$
- **(4):** $L_i\phi \Rightarrow L_iL_i\phi$
- **(5):** $\neg L_i\phi \Rightarrow L_i\neg L_i\phi$

are axioms of the logic of awareness. In modal logic, all axioms of a model are theorems of the model by assumption. All tautologies of propositional logic are also axioms of the logic of awareness. Additional theorems can be derived from the axioms and previous theorems using rules of inference.

The rules of inference in the logic of awareness are the same as in traditional modal logic, but the rule (RN) is applied to the implicit belief operator.

- **(RN):** If ϕ is a theorem, then $L_i\phi$ is a theorem for all i.
- **(MP):** If ϕ and $\phi \Rightarrow \psi$ are theorems, then ψ is a theorem.

Consistency is defined as in Chellas (1980): a set of formulas is consistent if the negation of the conjunction of these formulas is not a theorem of the logic. Maximal consistent sets are those that become inconsistent with the addition of any formula outside the set.

The implicit belief operator L_i of Fagin and Halpern (1988) is similar to a standard knowledge operator in logic. Implicit belief does not satisfy the truth axiom (T) $L_i\phi \Rightarrow \phi$, but it satisfies the weaker (3), as well as (K), (4) and (5) given above. The condition (K) ensures that implicit belief is closed under implication. Since all tautologies are implied by any formula, if the agent implicitly believes anything, he implicitly believes all tautologies. By condition (4), the agent implicitly knows what he knows and by (5) he implicitly knows what he does not know.

The explicit belief operator B_i is defined by setting $B_i\phi$ equivalent to the condition $\text{Prim}(\phi) \in \mathcal{A}_i(s)$ together with $L_i\phi$. To define this operator in the logic, we refer to the Kripke structure operator $\mathcal{A}_i$, so the logic is not constructed independently of the Kripke structure. The interpretation of the explicit belief operator B_i is that the agent believes ϕ explicitly if it is implicitly known and the agent is aware of all primitive propositions in it. Explicit belief implies implicit belief: $B_i\phi \Rightarrow L_i\phi$ is a theorem of the logic.

The awareness operator is defined via explicit belief, $A_ip = B_i(p \vee \neg p)$ and $A_i\phi = \wedge_{p\in\text{Prim}(\phi)}A_ip$, so the agent is aware of a formula iff he is aware of all its primitive propositions. The property $A_i\phi = \wedge_{p\in\text{Prim}(\phi)}A_ip$ is called awareness generated by primitive propositions (AGPP) in Halpern (2001) and Halpern and Rêgo (2008).

The logic of awareness and the Kripke structure for awareness are proved to be equivalent in Fagin and Halpern (1988) using the canonical structure for awareness consisting of a set of states S, a truth assignment π and modal operators $\mathcal{A}_1, ..., \mathcal{A}_n$, $\mathcal{B}_1, ..., \mathcal{B}_n$ that are constructed as follows.

$$S = \{s_V : V \text{ is a maximal consistent set of formulas}\}, \quad \pi(s_V, p) = \begin{cases} 1, & p \in V \\ 0, & p \notin V \end{cases}$$

$$\mathcal{A}_i(s_V) = \{p : B_i(p \vee \neg p) \in V\}, \quad \mathcal{B}_i(s_V) = \{s_W : \{\phi : L_i\phi \in V\} \subseteq W\}$$

The possibility operators $\mathcal{B}_i$ express the idea that at a state s_V, the states considered possible are those at which the formulas implicitly believed in s_V are true. The formulas valid in the canonical Kripke structure are exactly those that are contained in every maximally consistent set.

Since every state in the structure corresponds to a maximal consistent set of formulas, we can define for each formula ϕ the corresponding event $\|\phi\| = \{s_V : \phi \in V\}$, as in Chellas (1980) Definition 2.9. We thus extend the definition of a truth set from primitive propositions to formulas. For every formula there is a truth set by Theorem 2.10 of Chellas (1980). It is the empty set for the formula **false**. Logically equivalent formulas belong to the same maximal consistent sets, so they correspond to the same event: $\phi \Leftrightarrow \psi$ iff $[\phi \in V \Leftrightarrow \psi \in V]$ iff $\|\phi\| = \|\psi\|$.

4. Equivalence of structures

We have the following natural correspondence between the primitives of Li's unawareness structure and a single-agent version of the Kripke structure for the logic of awareness:

Fagin and Halpern (1988)		Li (2009)	
S	state space	Ω^*	state space
Φ	propositions (dimensions of) S	Q^*	questions (dimensions of) Ω^*
$p, \neg p$	proposition p, its negation	$1_q, 0_q$	yes to question q, no to q
$\pi(s_j, p_i) = 1$	truth function	$\omega_j^*(q_i) = 1$	answer to question
$\mathcal{A} : S \to 2^\Phi$	awareness set	$W^* : \Omega^* \to 2^{Q^*}$	awareness operator
$\mathcal{B} : S \to 2^S$	possibility correspondence	$P^* : \Omega^* \to 2^{\Omega^*}$	possibility correspondence

We can show by induction that if we set the primitives of the unawareness structure and of the logic to correspond as above, then the full structures are equivalent. The induction is on the complexity of formulas in the logic and the corresponding events in the Kripke structure, as in some proofs in Chellas (1980) and Fagin and Halpern (1988). The induction starts from the equivalence of the primitive propositions, their truth sets and the corresponding events in the unawareness structure (this is the base case below). Next we show that this equivalence is preserved by the logical and modal operators (steps 1-5 below). Finally we show that the awareness operators and higher order knowledge have the same structure in the two models (steps 6, 7).

The base case Denote the equivalence of events in the Li (2009) unawareness structure to events in the Kripke structure for awareness by $\cong$ and the equivalence of formulas in the logic of awareness to their truth sets in the Kripke structure by $\leftrightarrow$. Thus $\{\omega^* : g(\Omega^*, Q^*, \Delta, 1_q, 0_q, W^*, P^*)\} \cong \{s_V : g(S, \Phi, \mathbf{true}, p, \neg p, \mathbb{A}, \mathcal{B})\}$, where $g()$ is an arbitrary condition relating its arguments to each other, and $\phi \leftrightarrow \|\phi\| = \{s_V : \phi \in V\}$.

Step 1 Complement in the unawareness structure corresponds to negation in the logic. If E corresponds to ϕ, then $E^c \cong \|\phi\|^c$. In the Kripke structure $\|\phi\|^c$ is expressed as $\{s_V : \phi \notin V\}$. We also have $\neg\phi \leftrightarrow \|\neg\phi\| = \{s_V : \neg\phi \in V\}$. Since V is a maximal consistent set, $\neg\phi \in V$ iff $\phi \notin V$.

Step 2 Intersection in the unawareness structure corresponds to conjunction in the logic. If E corresponds to ϕ and F to ψ, then $E \cap F \cong \|\phi\| \cap \|\psi\| = \{s_V : \phi \in V\} \cap \{s_V : \psi \in V\} = \{s_V : \phi \in V,\ \psi \in V\}$. We also have $\phi \wedge \psi \leftrightarrow \|\phi \wedge \psi\| = \{s_V : \phi \wedge \psi \in V\}$.

Defining $\vee, \Rightarrow$ via $\wedge, \neg$ and $\cup, \subseteq$ via $\cap,\ ^c$ as is standard in propositional logic, we get that $\vee$ is equivalent to $\cup$ and $\Rightarrow$ to $\subseteq$.

Step 3 Assume event E in the unawareness structure corresponds to the formula ϕ in the logic of awareness. Then the application of the awareness operators to E and ϕ gives $A(E)$ and $A\phi$ that correspond to each other.

$$A(E) = \{\omega^* : q \in W^*(\omega^*)\ \forall q \in q(E)\} \cong \{s_V : \|p\| \in \mathbb{A}(s_V)\ \forall p \in \mathrm{Prim}(\phi)\} = \left\{s_V : \wedge_{p \in \mathrm{Prim}(\phi)} Ap \in V\right\} =$$
$$= \{s_V : A\phi \in V\} = \|A\phi\| \leftrightarrow A\phi$$

Step 4 The objective knowledge operator for Li's structure corresponds to the implicit belief operator in Fagin and Halpern (1988). Both are defined as truth in all possible states. The knowledge operator is closed under supersets, union and intersection, while the implicit belief operator is closed under the corresponding operations implication, conjunction and disjunction. Applying K^* to E and L to ϕ preserves equivalence.

$$K^*(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*\} \cong \{s_V : \mathcal{B}(s_V) \subseteq \|\phi\|\} = \{s_V : \{s_W : \{\phi : L\phi \in V\} \subseteq W\} \subseteq \{s_T : \phi \in T\}\} =$$
$$= \{s_V : (L\phi \in V \Rightarrow \phi \in W) \Rightarrow \phi \in W\} = \{s_V : L\phi \in V \vee \phi \in W\} = \|L\phi\| \leftrightarrow L\phi$$

Step 5 The agent's subjective knowledge from the modeller's perspective in Li (2009) corresponds to explicit belief in the logic of awareness. Subjective knowledge becomes objective knowledge under full awareness; explicit belief is implicit belief relativized to awareness.

$$K(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\} \cong \{s_V : \mathcal{B}(s_V) \subseteq \|\phi\|,\ \text{Prim}(\phi) \subseteq \mathcal{A}(s_V)\} \leftrightarrow L\phi \wedge A(\phi)$$

Step 6 In the logic of awareness, $A\phi = \wedge_{p \in \text{Prim}(\phi)} Ap = \wedge_{p \in \text{Prim}(\phi)} B(p \vee \neg p)$ by Proposition 4.2 in Fagin and Halpern (1988), so awareness is generated by primitive propositions. A similar decomposition of the awareness operator can be derived in Li (2009). Since E and E^c belong to the same subjective state space, the same set of questions $q(E) = q(E^c)$ determines the agent's awareness of both events. Corresponding to $A\phi = B(\phi \vee \neg\phi)$, we can write

$$A(E) = \{\omega^* : q(E) \subseteq W^*(\omega^*)\} = \{\omega^* : [P^*(\omega^*) \subseteq E^* \vee P^*(\omega^*) \nsubseteq E^*] \wedge [q(E) \subseteq W^*(\omega^*) \wedge q(E^c) \subseteq W^*(\omega^*)]\} =$$
$$= \{\omega^* : [P^*(\omega^*) \subseteq E^* \cup E^{*c}] \vee [q(E \cup E^c) \subseteq W^*(\omega^*)]\} = K(E \cup E^c)$$

We can define $E_q = \{\omega^* : \omega^*(q) = 1\} \cong \|p\| \leftrightarrow p$, where p is the proposition corresponding to question q. Then corresponding to the equation $A\phi = \wedge_{p \in \text{Prim}(\phi)} Ap$, we have

$$\cap_{q \in q(E)} A(E_q) = \cap_{q \in q(E)} \{\omega^* : q \in W^*(\omega^*)\} = \{\omega^* : q(E) \subseteq W^*(\omega^*)\} = A(E)$$

Awareness is determined by primitive propositions in the logic of awareness and by questions in Li's unawareness structure. In both cases, awareness of a proposition or question is equivalent to explicitly believing or knowing a tautology formed from it.

Step 7 Higher order subjective knowledge is equivalent to higher order explicit belief by an inductive argument. The base case is Step 5. For the inductive step, assume that $K^{n-1}(E)$ is equivalent to $B^{n-1}\phi$. Based on AGPP we have $\text{Prim}(B^{n-1}\phi) \subseteq \mathcal{A}(s_V) \Leftrightarrow \text{Prim}(\phi) \subseteq \mathcal{A}(s_V)$, and we can write

$$K^n(E) = \left\{\omega^* : P^*(\omega^*) \subseteq K^{n-1}(E),\ q(E) \subseteq W^*(\omega^*)\right\} \cong \left\{s_V : \mathcal{B}(s_V) \subseteq \|B^{n-1}\phi\|,\ \text{Prim}(\phi) \subseteq \mathcal{A}(s_V)\right\} =$$
$$= \left\{s_V : LB^{n-1}\phi \in V,\ \text{Prim}(B^{n-1}\phi) \subseteq \mathcal{A}(s_V)\right\} = \left\{s_V : LB^{n-1}\phi \wedge AB^{n-1}\phi \in V\right\} = \|B^n\phi\| \leftrightarrow B^n\phi$$

The primitives of the unawareness structure of Li (2009) correspond to the primitives of the Kripke structure for awareness in Fagin and Halpern (1988) part 4. Because this correspondence is preserved by the logical, knowledge and awareness operators, the resulting structures are equivalent by induction. Since Halpern (2001) proves the equivalence of the logic of awareness to the generalized standard model of Modica and Rustichini (1999), Li's unawareness structure is also equivalent to Modica and Rustichini's model and since Halpern and Rêgo (2008) prove the equivalence of the logic of awareness to Heifetz, Meier, and Schipper (2006), Li's structure is equivalent to the lattice of state spaces of Heifetz and coauthors.

5. Results for the unawareness structure from the logic of awareness

The results in section 3 of Li (2009) have natural counterparts in the logic of awareness or are derivable from the properties of the logic. The unawareness structure has the following properties by Lemma 2 in Li (2009).

U0* (symmetry):	$U(E) = U(\neg E)$
U1' (strong plausibility):	$U(E) \subseteq \cap_{n=1}^{\infty}(\neg K)^n(E)$
U2* (AU introspection):	$U(E) \subseteq \cap_{n=1}^{\infty}(\neg K)^n U(E)$
U3' (weak KU introspection):	$U(E) \cap KU(E) = \emptyset_{\Omega^*}$
K1* (subjective necessitation):	$\omega^* \in K(\Omega(\omega^*))$
K2* (generalized monotonicity):	$E \subseteq F,\ q(E) \supset q(F) \Rightarrow K(E) \subseteq K(F)$
K3* (conjunction):	$K(E) \cap K(F) = K(E \cap F)$

Lemma 2 of Li (2009) can be proved based on the axioms of the logic of awareness in Fagin and Halpern (1988) as follows.

U0* follows from AGPP: $\mathrm{Prim}(\phi) = \mathrm{Prim}(\neg\phi) \Leftrightarrow A\phi = A\neg\phi$. Alternatively this can be derived from awareness being symmetric in $\phi, \neg\phi$ by definition, $A\phi = \wedge_{p\in\mathrm{Prim}(\phi)} B(p \vee \neg p) = A\neg\phi$.

U1' We get the contrapositive result $\vee_{n=1}^{\infty} B(\neg B)^{n-1}\phi \Rightarrow A\phi$ from

$$\bigvee_{n=1}^{\infty} B(\neg B)^{n-1}\phi = \left[\bigvee_{n=1}^{\infty} L(\neg B)^{n-1}\phi\right] \bigwedge \left[\bigvee_{n=1}^{\infty} A(\neg B)^{n-1}\phi\right] = \left[\bigvee_{n=1}^{\infty} L(\neg B)^{n-1}\phi\right] \bigwedge \left[\bigvee_{n=1}^{\infty} \bigwedge_{p\in\mathrm{Prim}((\neg B)^{n-1}\phi)} Ap\right] =$$

$$= \left[\bigvee_{n=1}^{\infty} L(\neg B)^{n-1}\phi\right] \bigwedge \left[\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right] \Rightarrow A\phi$$

U2* Similarly to U1', we use AGPP (in the form $\mathrm{Prim}(B(\neg B)^{n-1}\neg A\phi) = \mathrm{Prim}(\phi)$ here) to show

$$\vee_{n=1}^{\infty} B(\neg B)^{n-1}\neg A\phi \Rightarrow A\phi$$

U3' The empty set corresponds to the proposition **false**. We get weak KU introspection from

$$\neg A\phi \bigwedge B\neg A\phi = \neg\left(\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right) \bigwedge A(\neg A\phi) \bigwedge L\neg A\phi = \neg\left(\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right) \bigwedge \left(\bigwedge_{p\in\mathrm{Prim}(\neg A\phi)} Ap\right) \bigwedge L\neg A\phi =$$

$$= \neg\left(\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right) \bigwedge \left(\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right) \bigwedge L\neg A\phi = \mathbf{false}$$

K1* By the facts $K(\Omega(\omega_0^*)) = \{\omega^* : P^*(\omega^*) \subseteq \Omega^*,\ q(\Omega(\omega_0^*)) \subseteq W^*(\omega^*)\}$ and $q(\Omega(\omega_0^*)) = W^*(\omega_0^*)$ and setting $s_V \cong \omega_0^*$, we can write

$$K(\Omega(\omega_0^*)) = \{\omega^* : P^*(\omega^*) \subseteq \Omega^*,\ W^*(\omega_0^*) \subseteq W^*(\omega^*)\} \cong \{s_W : \mathcal{B}(s_W) \subseteq S,\ \mathcal{A}(s_V) \subseteq \mathcal{A}(s_W)\} \ni s_V \cong \omega_0^*$$

K2* If $\phi \Rightarrow \psi$ and $\mathrm{Prim}(\phi) \supset \mathrm{Prim}(\psi)$, then by closure of implicit belief under logical operations, $L\phi \Rightarrow L\psi$ and by AGPP, $A\phi \Rightarrow A\psi$. From this we have $B\phi \Rightarrow B\psi$, which for $E \cong \|\phi\|, F \cong \|\psi\|$ is equivalent to $K(E) \subseteq K(F)$.

K3*

$$B\phi \bigwedge B\psi = L\phi \bigwedge L\psi \bigwedge \left[\bigwedge_{p\in\mathrm{Prim}(\phi)} Ap\right] \bigwedge \left[\bigwedge_{p\in\mathrm{Prim}(\psi)} Ap\right] =$$

$$= L(\phi \wedge \psi) \bigwedge \left[\bigwedge_{p\in\mathrm{Prim}(\phi)\cup\mathrm{Prim}(\psi)} Ap\right] = L(\phi \wedge \psi) \bigwedge \left[\bigwedge_{p\in\mathrm{Prim}(\phi\wedge\psi)} Ap\right] = B(\phi \wedge \psi)$$

Li (2009) calls an information structure rational if $\omega^* \in P^*(\omega^*)$ and $\omega_1^* \in P^*(\omega_2^*) \Rightarrow (W^*(\omega_1^*), P^*(\omega_1^*)) = (W^*(\omega_2^*), P^*(\omega_2^*))$. By definition, a rational information structure has a reflexive, transitive and Euclidean possibility correspondence (P^* is an equivalence relation) and awareness constant within elements of the partition generated by P^*. The following properties hold in rational unawareness structures by Lemma 3 of Li (2009).

U3* (KU introspection):	$KU(E) = \emptyset_{\Omega^*}$
K4*a (axiom of knowledge I):	$K(E) \subseteq E$
K4*b (axiom of knowledge II):	$K^n(E) \subseteq K^{n-1}(E)$
K5* (transparency):	$K(E) \subseteq KK(E)$
K6* (limited wisdom):	$\neg K(E) \cap \neg U(E) \subseteq K\neg K(E)$
U1* (unawareness as unknown unknowns):	$U(E) = \cap_{n=1}^{\infty} (\neg K)^n(E)$

In the logic of awareness of Fagin and Halpern (1988), the rational information structure condition of Li (2009) is equivalent to the axioms

(T): $L\phi \Rightarrow \phi$

(4): $L\phi \Rightarrow LL\phi$

(5): $\neg L\phi \Rightarrow L\neg L\phi$

(KA): $A\phi \Rightarrow LA\phi$

(KU): $\neg A\phi \Rightarrow L\neg A\phi$

By Theorem 3.5 of Chellas (1980), a possibility correspondence satisfies $s_1 \in \mathcal{B}(s_1)\ \forall s_1$ and $s_1 \in \mathcal{B}(s_2) \Rightarrow \mathcal{B}(s_1) = \mathcal{B}(s_2)$ iff the logic satisfies the truth axiom (T) and the introspection axioms (4) and (5). By Halpern (2001), awareness is constant on partition elements, formally $s_1 \in \mathcal{B}(s_2) \Rightarrow \mathcal{A}(s_1) = \mathcal{A}(s_2)$, iff the axioms of knowledge of awareness (KA) and knowledge of unawareness (KU) are satisfied. This is the condition called 'the agent knows what he is aware of' in Halpern and Rêgo (2008).

The following shows that Lemma 3 of Li (2009) follows from the logic of awareness under the additional axioms that correspond to Li's definition of a rational information structure.

U3* $B\neg A\phi = A\neg A\phi \wedge L\neg A\phi$. Using (T), $L\neg A\phi \Rightarrow \neg A\phi$ and using AGPP, $A\neg A\phi = A\phi$. From this we get $B\neg A\phi = A\phi \wedge \neg A\phi =$ **false**, which corresponds to $KU(E) = \emptyset_{\Omega^*}$.

K4a* $B\phi \Rightarrow L\phi \Rightarrow \phi$ by the definition of B and the truth axiom (T).

K4b* $B^n\phi = LB^{n-1}\phi \wedge AB^{n-1}\phi \Rightarrow B^{n-1}\phi \wedge A\phi \Rightarrow B^{n-1}\phi$ by (T) and AGPP.

K5* By AGPP and axioms (4) and (KA),

$$B\phi = L\phi \wedge A\phi = LL\phi \wedge LA\phi \wedge AB\phi = LB\phi \wedge AB\phi = BB\phi$$

which corresponds to $K(E) = KK(E)$.

K6*

$$\neg B\phi \wedge A\phi = (\neg L\phi \vee \neg A\phi) \wedge A\phi = \neg L\phi \wedge A\phi = L\neg L\phi \wedge LA\phi \wedge AA\phi \wedge A\neg L\phi \Rightarrow L\neg B\phi \wedge A\neg B\phi = B\neg B\phi$$

where we use AGPP, (5) and (KA) in the third step.

U1* From K6*, $\neg B\phi \wedge A\phi \Rightarrow B\neg B\phi$, which is logically equivalent to $B\phi \vee \neg A\phi \vee B\neg B\phi$. Also,

$$[B\phi \vee \neg A\phi \vee B\neg B\phi] \equiv [A\phi \Rightarrow B\phi \vee B\neg B\phi] \equiv [\neg B\phi \wedge \neg B\neg B\phi \Rightarrow \neg A\phi],$$

therefore $\wedge_{n=1}^{\infty} (\neg B)^n\phi \Rightarrow \neg A\phi$. The converse of this implication is U1', so the implication is correct in both directions.

Since in Li (2009) Theorem 1 follows from Lemma 3, all the results for unawareness structures can be obtained from Fagin and Halpern's logic of awareness. This is not surprising, as the models are equivalent.

6. Conclusion

There is a natural equivalence between the primitive objects of the unawareness structure of Li (2009) and the primitives of the logic of awareness of Fagin and Halpern (1988). This equivalence is preserved by the operators that are used to build up these models inductively, so it extends to the full structures. In light of this it is not surprising that the results derived by Li for the unawareness structure have direct counterparts in the logic of awareness. Since the logic of awareness is able to bypass the impossibility result of Dekel, Lipman, and Rustichini (1998) and capture nontrivial unawareness by its construction, the same thing is possible in the unawareness structure.

Through its equivalence with the logic of awareness, Li (2009) is also equivalent to the models of Modica and Rustichini (1999) and Heifetz, Meier, and Schipper (2006), which similarly model nontrivial unawareness. This equivalence answers the question posed in Heifetz, Meier, and Schipper (2008) about the possible connection between Li's, Heifetz and coauthors' and Modica and Rustichini's models.

References

Bergé, C. (1963): *Topological spaces*. Dover Publications, 31 East 2nd Street, Mineola, NY 11501.

Chellas, B. F. (1980): *Modal logic*. Cambridge University Press, The Edinburgh Building, Shaftesbury Road, Cambridge CB2 8RU, United Kingdom.

Dekel, E., B. L. Lipman, and A. Rustichini (1998): "Standard State-Space Models Preclude Unawareness," *Econometrica*, 66(1), 159–173.

Fagin, R., and J. Y. Halpern (1988): "Belief, awareness and limited reasoning," *Artificial Intelligence*, 34, 39–76.

Halpern, J. Y. (2001): "Alternative semantics for unawareness," *Games and Economic Behavior*, 37, 321–339.

Halpern, J. Y., and L. C. Rêgo (2008): "Interactive unawareness revisited," *Games and Economic Behavior*, 62, 232–262.

Heifetz, A., M. Meier, and B. K. Schipper (2006): "Interactive unawareness," *Journal of Economic Theory*, 130, 78–94.

——— (2008): "A canonical model for interactive unawareness," *Games and Economic Behavior*, 62, 304–324.

Li, J. (2009): "Information structures with unawareness," *Journal of Economic Theory*, 144, 977–993.

Modica, S., and A. Rustichini (1999): "Unawareness and partitional information structures," *Games and economic behavior*, 27, 265–298.