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CONTINUOUS TIME CONTESTS

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This paper presents an extension of the contest literature to a continuous time framework, in which players decide when to stop a privately observed Brownian motion with drift and incur costs depending on their stopping time. The player who stops his process at the highest value wins a prize.
Under mild assumptions on the cost function, we prove existence and uniqueness of the Nash equilibrium outcome, even if players have to choose bounded time stopping strategies. If the noise parameter goes to zero, the equilibrium converges to, and thus selects the symmetric equilibrium of an all-pay contest. For positive noise levels, results differ from those of all-pay contests—for instance, participants make positive profits and, for asymmetric endowments, more than two players can be active. Moreover, the profits of each participant increase if all participants have higher costs of research.

Keywords: Discontinuous Games, Contests, All-pay Contests.

1. INTRODUCTION

Two types of models are predominant in the literature on contests, races, and tournaments. In one of these, there is no feedback about the performance measure or standings throughout the competition at all, while the other one considers full feedback about the performance of each player at all points in time. The former category includes all-pay contests with complete information (Hillman and Samet, 1987; Siegel, 2009, 2010), Tullock contests (Tullock, 1980), silent timing games (Karlin, 1953; Park and Smith, 2008), and models with additive noise in the spirit of Lazear and Rosen (1981). The latter category consists of wars of attrition (Maynard Smith, 1974; Bulow and Klemperer, 1999), races (Aoki, 1991; Hörner, 2004; Anderson and Cabral, 2007), and contest models with full observability such as Harris and Vickers (1987) and Moscarini and Smith (2007).

In this paper, we want to analyze an intermediate case, in which there is partial feedback about the performance measure. More precisely, a player gets feedback about his own stochastic research progress over time, but he does not observe the progress of the other players or their effort decisions. A good example for this setting is an R&D contest. Each competitor is well-informed about his own progress, but often uninformed about the progress of his competitors.

Formally, our model is an n-player contest, in which each player decides when to stop a privately observed Brownian motion (X_t) with drift. As long as a player exerts effort, i.e., does not stop the process, he incurs flow costs of $c(X_t)$. The player who stops his process at the highest value wins a prize.

Under mild assumptions on the cost function—it has to be continuous and bounded away from zero—we show that the game has a unique Nash equilibrium

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outcome. This outcome is feasible in stopping strategies which stop almost surely in bounded time. Hence, provided the contest length is above a threshold, the equilibrium is independent of the contest length. Contrary to the predictions of an all-pay contest, each player makes positive expected profits, which are decreasing in the number of participants. As the noise level goes to zero, profits go to zero and the equilibrium converges to, and hence selects, the symmetric equilibrium of an all-pay contest. For asymmetries in the starting values, more than two players can be active and make positive profits.

The formal analysis proceeds as follows. Proposition 1 and Theorem 1 establish existence and uniqueness of the equilibrium distribution. The existence proof first characterizes the unique equilibrium distribution $F(x)$ of final values $X_\tau = x$ uniquely up to its endpoints. To establish existence, we then use a Skorokhod embedding approach to show that there exists a stopping strategy which induces this distribution. This technique from probability theory (e.g., Skorokhod, 1961, 1965; for a survey, see Obłój, 2004) was first introduced to game theory in Seel and Strack (2009).

Moreover, we verify a condition from a recent paper in probability theory by Ankirchner and Strack (2011) to show that there exists a bounded time stopping strategy—a strategy that stops almost surely before a fixed time $T < \infty$—which induces the equilibrium distribution. As most real-world contests have a fixed deadline, this result fortifies the economic interpretation of the model. It is also one of main technical contributions of the paper, since this new technique is also applicable to other models without observability.

As uncertainty vanishes, the equilibrium converges to the symmetric equilibrium of an all-pay contest—see Siegel (2009, 2010)—by Proposition 2. In the special case of constant costs, the equilibrium converges to the symmetric equilibrium of an all-pay auction. On the one hand, the model offers a microfoundation for the use of all-pay auctions to scrutinize environments, in which uncertainty is not a crucial ingredient; on the other hand, it gives an equilibrium selection result between the all-pay auction equilibria in Baye, Kovenock, and de Vries (1996). Moreover, we take this result serves as a starting point to discuss how the predictions of our model differ from all-pay models if $\sigma > 0$.

For any $\sigma > 0$, Proposition 3 shows that all players make positive expected profits in equilibrium. Intuitively, agents use their private information about their progress, which arrives continuously over time, to generate rents. The intuition is similar to an all-pay contest, in which players have incomplete information about the valuation of their rivals—see, e.g., Hillman and Riley (1989), Amann and Leininger (1996) or Moldovanu and Sela (2001).

The case of asymmetric initial endowments resembles a head start in an all-pay auction (see Siegel, 2011). Contrary to the predictions of the auction model, here, more than 2 participants can be active and each of them might receive a positive profit. Hence, some of the well-known results in the all-pay auction literature crucially depend on the assumption of a deterministic mapping from effort or bid into output.

There are a many possible applications for the model. For instance, most R&D competitions would suit the modeling framework, since firms are not informed about the research of their rivals; for concrete examples of such competitions, see Taylor (1995). Other examples include high-technology sports contests, such as the America's Cup for sailors or 24h car races, where the outcome of the contest hugely depends on the quality of the equipment which is produced before the contest at utmost secrecy.

1.1. *Related Literature*

In a companion paper, Seel and Strack (2009), we analyze a model in which players do not have any costs of research, but have a (usually negative) drift and face a bankruptcy constraint. Albeit similar from a technical perspective, the driving forces of both models differ substantially. In particular, in the present paper, contestants trade-off higher costs versus a winning probability, whereas in Seel and Strack (2009) the trade-off is between winning probability and risk. Also, the applications of Seel and Strack (2009) are related to finance, while the present paper is in spirit of the contest literature.

Taylor (1995) also analyzes a model in which players only receive feedback about their own research success. In his analysis, however, only the highest draw in a single period determines this success. This allows him to construct an equilibrium stopping rule, which is independent of previous research success and time elapsed in the contest.

The paper entails a direct extension of the literature on *silent timing games*—see, e.g., Karlin (1953), which analyzes the same setting for the case without uncertainty. Intuitively, adding uncertainty allows us to have a model with partial non-deterministic feedback throughout the contest.

We proceed as follows. Section 2 sets up the model. In Section 3, we prove that an equilibrium exists and is unique. Section 4 discusses the relation to all-pay contests and derives the main comparative statics results. Section 5 concludes. Most proofs are relegated to the appendix.

2. THE MODEL

There are $n < \infty$ agents indexed by $i \in \{1, 2, \dots, n\} = N$ who face a stopping problem in continuous time. At each point in time $t \in \mathbb{R}_+$, agent i privately observes the realization of a stochastic process $(X_t^i)_{t \in \mathbb{R}_+}$ with

$$X_t^i = x_0 + \mu t + \sigma B_t^i \,.$$

The constant x_0 denotes the starting value of all processes; without loss of generality, we assume $x_0 = 0$. The drift $\mu \in \mathbb{R}_+$ is the common expected change of each process X_t^i per time, i.e., $\mathbb{E}(X_{t+\Delta}^i - X_t^i) = \mu\Delta$. The noise term is an n-dimensional Brownian motion (B_t) scaled by $\sigma \in \mathbb{R}_+$.

2.1. *Strategies*

A pure strategy of player i is a stopping time τ^i. This stopping time depends only on the realization of his process X_t^i, as the player only observes his own process.[1] Mathematically, the agents' stopping decision until time t has to be $\mathcal{F}_t^i$-measurable, where $\mathcal{F}_t^i = \sigma(\{X_s^i : s < t\})$ is the sigma algebra induced by the possible observations of the process X_s^i before time t. We require stopping times to be bounded by a real number $T < \infty$ such that $\tau^i < T$ almost surely.

To incorporate mixed strategies, we allow for randomized stopping times—progressively $\mathcal{F}_t^i$ measurable functions $\tau^i(\cdot)$ such that for every $r^i \in [0,1]$ the value $\tau^i(r^i)$ is a stopping time. Intuitively, agents draw a random number r^i from the uniform distribution on $[0,1]$ before the game and play a stopping strategy $\tau^i(r^i)$.[2]

2.2. *Payoffs*

The player who stops his process at the highest value wins a prize $p > 0$. Ties are broken randomly. Each player incurs a flow cost $c : \mathbb{R} \to \mathbb{R}_{++}$ until he stops. The payoff π^i is thus given by

$$\pi^i = \frac{p}{k}\,\mathbf{1}_{\{X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}} - \int_0^{\tau^i} c^i(X_t^i)dt \ ,$$

where $k = |\{i \in N : X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}|$ is the number of agents who stopped with the highest value. All agents maximize their expected profit $\mathbb{E}(\pi^i)$. We henceforth normalize p to 1, since agents only care about the trade-off between winning probability and cost-prize ratio.

The cost function $c : \mathbb{R} \to \mathbb{R}_{++}$ is continuous and satisfies the following conditions

Assumption 1 *For every $x \in \mathbb{R}$ the cost function $c(\cdot)$ is bounded away from zero on $[x,\infty)$.*

Note that Assumption 1 is satisfied for every non-decreasing cost function.

2.3. *A Brief Discussion of the Technology*

The Brownian Motion specification entails the possibility that research success might decrease over time. There are several possible interpretations of this feature. In an R&D setting, for instance, the value of the innovation (say prototype

[1] The equilibrium of the model would be the same if the stopping decision was reversible and stopped processes were constant.

[2] Mixed strategies will turn out to be irrelevant as the equilibrium we are going to construct will be in pure strategies and we will prove that the equilibrium outcome is unique.

of a fighter jet) may depend on the market price of the components. Hence, one might interpret the decrease in innovation value for one player as an increase in the prize of a component of that players prototype. Another interpretation is that a part of the innovation gets destroyed (for instance a car component). Similarly, a worker might be hired by another firm or simply forget something.

3. EQUILIBRIUM CONSTRUCTION

In this section, we first establish some necessary conditions on the distribution functions in equilibrium. In a second step, we prove existence and uniqueness of the Nash equilibrium outcome, and calculate the equilibrium distributions depending on the cost function.

Every strategy of agent i induces a (potentially non-smooth) cumulative distribution function (cdf) $F^i : \mathbb{R} \to [0,1]$ of his stopped process $F^i(x) = \mathbb{P}(X^i_{\tau^i} \le x)$. We denote the winning probability of player i if he stops at $X^i_{\tau^i} = x$, given the other players distributions, by

$$u^i(x) = \mathbb{P}(\max_{j \neq i} X^j_{\tau^j} \le x) = \prod_{j \neq i} F^j(x)\,.$$

Denote the endpoints of the support of the cdf of player i by $\overline{x}^i = \sup\{x : F^i(x) < 1\}$ and $\underline{x}^i = \inf\{x : F^i(x) > 0\}$. Let $\underline{x} = \max_{i \in N} \underline{x}_i$ and $\overline{x} = \max_{i \in N} \overline{x}_i$. Moreover, we write $\tau^i_{(a,b)}(x)$ shorthand for $\inf\{t : X^i_t \notin (a,b) | X^i_s = x\}$. In the next step, we establish a series of auxiliary results that are crucial to prove uniqueness of the equilibrium distribution.

LEMMA 1 *At least two players stop with positive probability on every interval* $I = (a,b) \subset [\underline{x}, \overline{x}]$.

LEMMA 2 *No player places a mass point in the interior of the state space, i.e., for all i, for all $x > \underline{x}$: $\mathbb{P}(X^i_{\tau^i} = x) = 0$. At least one player has no mass at the left endpoint, i.e., $F^i(\underline{x}) = 0$ for at least player i.*

We omit the proof of Lemma 2, since it is just a specialization of the standard logic in static game theory with a continuous state space; see, e.g., Burdett and Judd (1983). Intuitively, in equilibrium, no player can place a mass point in the interior of the state space, since no other player would then stop slightly below the mass point. This contradicts Lemma 1.

LEMMA 3 *All players have the same right endpoint, $\overline{x}^i = \overline{x}$, for all i.*

LEMMA 4 *All players have the same expected profit in equilibrium. Moreover, each player loses for sure at $\underline{x}$, i.e., $u^i(\underline{x}) = 0$ for all i.*

LEMMA 5 *All players have the same equilibrium distribution function* $F^i = F$.

As players have symmetric distributions, we henceforth drop the superscript i. The previous lemmata imply that each player is indifferent between any stopping strategy on his support. By Itô's lemma, it follows from the indifference inside the support that for every point for $x \in (\underline{x}, \overline{x})$ the function $u(\cdot)$ must satisfy the second order ordinary differential equation (ODE)

$$c(x) = \mu u'(x) + \frac{\sigma^2}{2} u''(x)\,. \tag{1}$$

As (1) is a second order ODE, we need two boundary conditions to determine $u(\cdot)$ uniquely. One boundary condition is given by $u(\underline{x}) = 0$. We determine the other one in the following lemma:

LEMMA 6 *In equilibrium,* $u'(\underline{x}) = 0$.

The idea of the proof in the appendix is simple. If there derivative was negative, $u'(\underline{x}) < 0$, there would a profitable deviation at $\underline{x}$, which stops in the neighborhood of $\underline{x}$ rather than at the point itself.

Thus, imposing the two boundary conditions, the solution to equation (1) is unique. To calculate it, we define $\phi(x) = \exp(\frac{-2\mu x}{\sigma^2})$ as a solution of the homogeneous equation $0 = \mu u'(x) + \frac{\sigma^2}{2} u''(x)$. To solve the inhomogeneous equation (1), we apply the variation of the constants formula. We then use the two boundary conditions to calculate the unique solution candidate. Finally, we rearrange with Fubini's Theorem to get

$$u(x) \;=\; \begin{cases} 0 & \text{for } x < \underline{x} \\ \frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z & \text{for } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for } \overline{x} < x\,. \end{cases}$$

By symmetry of the equilibrium strategy, the function $F : \mathbb{R} \to [0,1]$ satisfies $F(x) = \sqrt[n-1]{u(x)}$. Consequently, the unique candidate for an equilibrium distribution F is given by

$$F(x) = \begin{cases} 0 & \text{for all } x < \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z} & \text{for all } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for all } \overline{x} < x\,. \end{cases}$$

In the next step, we verify that F is a cumulative distribution function, i.e., that F is nondecreasing and that $\lim_{x\to\infty} F(x) = 1$.

LEMMA 7 *F is a cumulative distribution function.*

PROOF: By construction of F, $F(\underline{x}) = 0$. Clearly, F is increasing on $(\underline{x}, \overline{x})$, as the derivative with respect to x,

$$F'(x) = \frac{F(x)^{2-n}}{(n-1)} \left(\frac{2}{\sigma^2} \int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z\right),$$

is greater than zero for all $x > \underline{x}$. It remains to show that there exists an $x > \underline{x}$ such that $F(x) = 1$.

$$\begin{aligned} F(x)^{n-1} &= \frac{1}{\mu} \int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z \\ &\geq \frac{1}{\mu} \inf_{y \in [\underline{x}, \infty)} c(y) \left(x - \underline{x} - \frac{\sigma^2}{2\mu}(1-\phi(x-\underline{x}))\right) \\ &\geq \frac{1}{\mu} \inf_{y \in [\underline{x}, \infty)} c(y)(x - \underline{x} - \frac{\sigma^2}{2\mu}) \end{aligned}$$

Assumption 1 implies that the cost function $c(\cdot)$ is bounded away from zero an consequently $\inf_{y \in [\underline{x}, \infty)} c(y)$ is strictly greater than zero. Continuity of F implies that there exists a point $\overline{x} > \underline{x}$ such that $F(\overline{x}) = 1$. *Q.E.D.*

The next lemma derives a necessary condition for a distribution F to be the outcome of a strategy τ.

LEMMA 8 *If $\tau \leq T < \infty$ is a bounded stopping time that induces the continuous distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$, then $1 = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x$.*

PROOF: Observe that $(\phi(X_t))_{t \in \mathbb{R}_+}$ is a martingale. Hence, by Doob's optional stopping theorem, for any bounded stopping time τ,

$$1 = \phi(X_0) = \mathbb{E}[\phi(X_\tau)] = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x\,.$$

Q.E.D.

We use the necessary condition derived in Lemma 8 to prove that the equilibrium distribution is unique.

PROPOSITION 1 *There exists a unique pair $\underline{x}, \overline{x} \in \mathbb{R}$ such that the distribution*

$$F(x) = \begin{cases} 0 & \text{for all } x \leq \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))dz} & \text{for all } x \in (\underline{x}, \overline{x}) \\ 1 & \text{for all } x \geq \overline{x} \end{cases}$$

is the unique candidate for an equilibrium distribution.

PROOF: As F is continuous, the right endpoint $\overline{x}$ satisfies $1 = \int_{\underline{x}}^{\overline{x}} F'(x;\underline{x},\overline{x})\mathrm{d}x$. Since $F'(x;\underline{x},\overline{x})$ is independent of $\overline{x}$, we drop the dependency in our notation. By the implicit function theorem,

$$\frac{\partial \overline{x}}{\partial \underline{x}} = -\frac{-\overbrace{F'(\underline{x};\underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})} = -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})}\,. \tag{2}$$

Lemma 8 states that any feasible distribution satisfies $1 = \int_{\underline{x}}^{\overline{x}} F'(x;\underline{x})\phi(x)\mathrm{d}x$. Applying the implicit function theorem to this equation gives us

$$\begin{aligned}\frac{\partial \overline{x}}{\partial \underline{x}} &= -\frac{-\overbrace{F'(\underline{x};\underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\phi(x)\mathrm{d}x}{F'(\overline{x};\underline{x})\phi(\overline{x})} \\ &= -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\overbrace{\phi(x-\overline{x})}^{<1}\mathrm{d}x}{F'(\overline{x};\underline{x})} \\ &< -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})}\,.\end{aligned} \tag{3}$$

The last step follows, because $\frac{\partial}{\partial \underline{x}} F'(x;\underline{x}) \geq 0$. Hence, conditions 2 and 3 cross exactly once. Thus, the left and the right endpoint in equilibrium are unique. *Q.E.D.*

Henceforth, we write $F(\cdot)$ to refer to the unique equilibrium distribution.

LEMMA 9 *Every strategy that induces the unique distribution F from Proposition 1 is an equilibrium strategy.*

PROOF: Define $\Psi(\cdot)$ as the unique solution to (1) with the boundary conditions $\Psi(\underline{x}) = 0$ and $\Psi'(\underline{x}) = 0$. By construction, the process $\Psi(X_t^i) - \int_0^t c(X_s^i)\mathrm{d}s$ is a martingale and $\Psi(x) = u(x)$ for all $x \in [\underline{x},\overline{x}]$. As $\Psi'(x) < 0$ for $x < \underline{x}$ and $\Psi'(x) > 0$ for $x > \overline{x}$, $\Psi(x) > u(x)$ for all $x \notin [\underline{x},\overline{x}]$. For every stopping time S, we use Itô's Lemma to calculate the expected value

$$\begin{aligned}\mathbb{E}[u(X_S) - \int_0^S c(X_t)\mathrm{d}t] &\leq \mathbb{E}[\Psi(X_S) - \int_0^S c(X_t)\mathrm{d}t] \\ &= \Psi(X_0) = u(X_0) = \mathbb{E}(u(X_\tau))\,.\end{aligned}$$

The last equality results from the indifference of every agent to stop immediately with the expected payoff $u(X_0)$ or to play the equilibrium strategy with the expected payoff $\mathbb{E}(u(X_\tau))$. *Q.E.D.*

So far, we have verified that a bounded stopping time $\tau \leq T < \infty$ is an equilibrium strategy if and only if it induces the distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$. To show that the game has a Nash equilibrium, it remains to establish the existence a bounded stopping time inducing $F(\cdot)$. The problem of finding a stopping time τ such that a Brownian motion stopped at τ has a given centered probability distribution F, i.e., $F \sim W_\tau$, is known in the probability literature as the Skorokhod embedding problem (SEP). Since its initial formulation in Skorokhod (1961, 1965), many solutions have been derived; for a survey article, see Obłój (2004). Ankirchner and Strack (2011) find conditions guaranteeing the existence of stopping times τ that are *bounded* by some real number $T < \infty$, and embed a given distribution in Brownian motion, possibly with drift.[3] They define $g(x) = F^{-1}(\Phi(x))$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(\frac{z^2}{2}) \mathrm{d}z$ is the density function of the normal distribution.

LEMMA 10 (Ankirchner and Strack (2011), Theorem 2) *Suppose that g is Lipschitz-continuous with Lipschitz constant $\sqrt{T}$. Then F can be embedded in $X_t = \mu t + B_t$.*

This auxiliary result enables us to prove the main result of this section:

THEOREM 1 *The game has a Nash equilibrium, i.e., there exists a bounded stopping time τ that induces F.*

The proof in the appendix verifies Lipschitz continuity of the function g. Thus, a Nash equilibrium in bounded time stopping strategies exists, and, by Proposition 1, it has a unique distribution.

4. EQUILIBRIUM ANALYSIS

4.1. *Convergence to the All-pay Contest*

This subsection considers the relationship between our model and the literature on all-pay contests. In a first step, we show that for vanishing noise, the left endpoint of the equilibrium distribution converges to the starting point.

LEMMA 11 *If the noise vanishes $\sigma \to 0$, the left endpoint $\underline{x}$ of the equilibrium distribution converges to zero, i.e., $\lim_{\sigma \to 0} \underline{x} = 0$.*

PROOF: For any bounded stopping time, for any $\sigma > 0$, feasibility implies that $\underline{x} \leq 0$. By contradiction, assume there exists a constant ϵ such that $\underline{x} \leq \epsilon < 0$

[3] Ankirchner and Strack (2011) use a construction of the stopping time introduced for the Brownian motion without drift in Bass (1983) and for the case with drift in Ankirchner, Heyne, and Imkeller (2008).

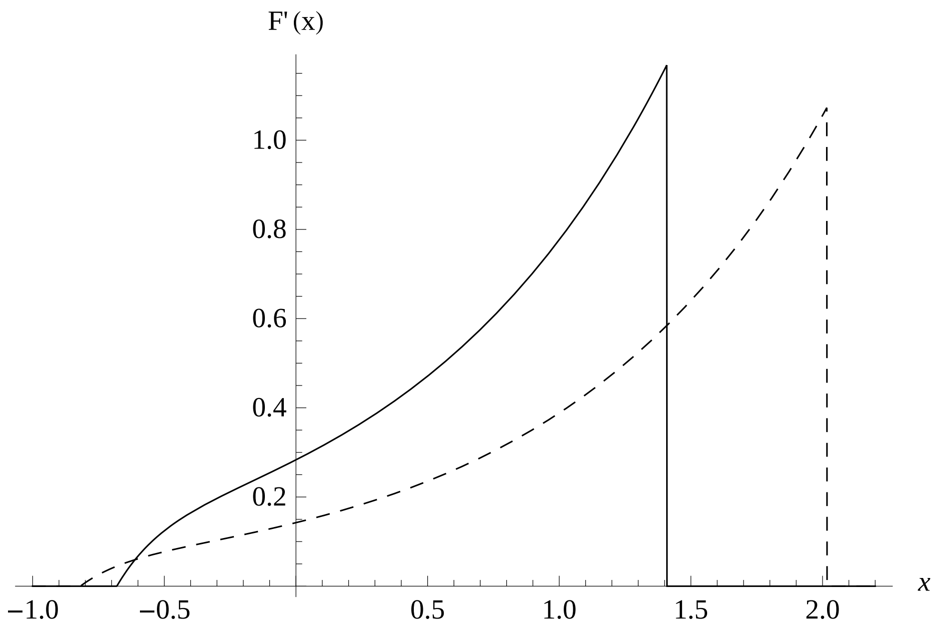

FIGURE 1.— The density function $F'(\cdot)$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \exp(x)$ solid line and $c(x) = \frac{1}{2}\exp(x)$ dashed line. Note that the socially efficient cut-off point is $\log(3) \approx 0.477$ respectively $\log(6) \approx 0.778$

for all $\sigma > 0$. Then F' is bounded away from zero by

$$\begin{aligned} F'(x) &= \frac{F(x)^{2-n}}{n-1}\frac{2}{\sigma^2}\Big(\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z\Big) \\ &\geq \frac{1}{n-1}\frac{2}{\sigma^2}\int_{\epsilon}^{x} c(z)\phi(x-z)\mathrm{d}z \\ &= \frac{1}{\mu(n-1)}\left(\inf_{y\in[\epsilon,\infty)} c(y)\right)(1-\phi(x-\epsilon)). \end{aligned}$$

For every point $x < 0$, $\lim_{\sigma\to 0}\phi(x) = \infty$. Thus, $\lim_{\sigma\to 0}\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x > 1$, which contradicts feasibility, because $\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x \leq \int_{\underline{x}}^{\overline{x}} F'(x)\phi(x)\mathrm{d}x = 1$. *Q.E.D.*

Taking the limit $\sigma \to 0$, the equilibrium distribution converges to

$$\lim_{\sigma\to 0} F(x) = \sqrt[n-1]{\frac{1}{\mu}\int_0^x c(z)\mathrm{d}z}.$$

This condition is well-known in the literature on static all-pay contests (Siegel, 2009; Siegel, 2010), which yields us the following proposition.

PROPOSITION 2 *For vanishing noise, the equilibrium distribution converges to the symmetric equilibrium distribution of an all-pay contest. In the case of constant costs, it converges to the symmetric equilibrium distribution of an all-pay auction.*

Thus, our model supports the use of all-pay auctions to analyze contests in which the variance is negligible. In figure 4.1 we illustrate the similarity with the all-pay auction equilibrium if the variance σ and the costs $c(\cdot)$ are small in comparison to the drift μ.

Moreover, the symmetric all-pay auction has multiple equilibria—for a full characterization see Baye, Kovenock, and de Vries (1996). This paper offers a selection criterion in favor of the symmetric equilibrium, in which no participant places a mass point at zero. Intuitively, all other equilibria of the symmetric all-pay auction include mass points at zero for some players, which is not possible in our model for any positive σ by Lemma 2.

4.2. *Rent Dispersion and Comparative Statics*

Proposition 2 created a link between all-pay contests with complete information and our model for the case of vanishing noise. In the following, we scrutinize how the predictions are altered if there is a positive noise. One prediction of a symmetric all-pay contests with complete information is that agents make zero

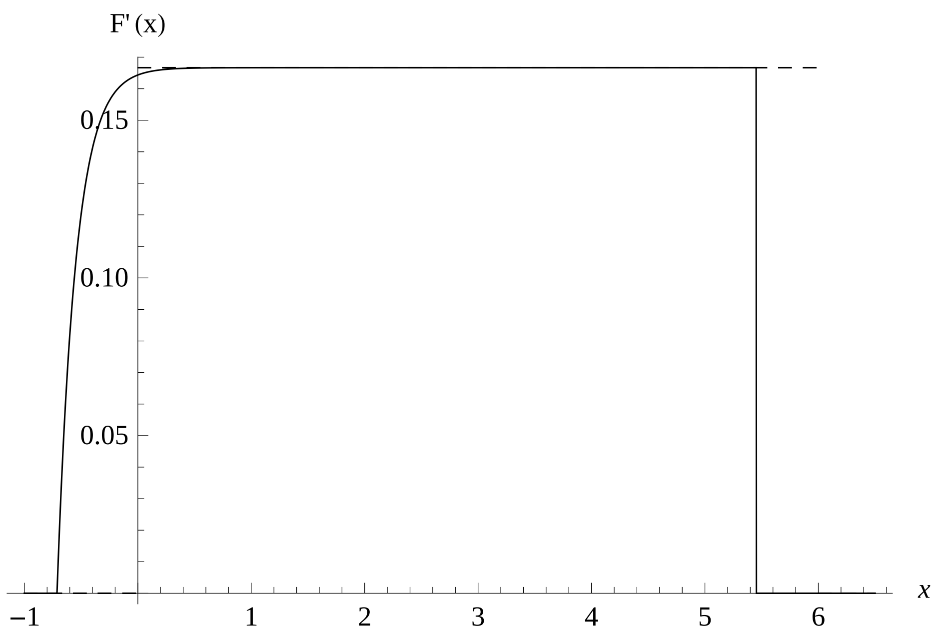

FIGURE 2.— This picture shows the density function $F'(\cdot)$ with support $[-0.71, 5.45]$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \frac{1}{2}$ (solid line) and for the same parameters the equilibrium density of the all-pay auction with support $[0, 6]$ (dashed line).

profits in equilibrium. This does no longer hold true for any positive level of variance σ:

PROPOSITION 3 *In equilibrium, all agents make strictly positive expected profits.*

PROOF: In equilibrium, agents are indifferent between stopping immediately and the equilibrium strategy. Their expected profit is thus given by $u(0)$, which is strictly positive. *Q.E.D.*

Intuitively, private information about their research progress enables the agents to generate informational rents. A similar result is known in the literature on all-pay contests with incomplete information, see, e.g., Hillman and Riley (1989), Amann and Leininger (1996), and Moldovanu and Sela (2001). In their models, participants take a draw from a distribution prior to the contest, which determines their valuation. The valuation is private information. In contrast to this, private information about one's progress arrives continuously over time in our model.

In the next paragraphs, we derive additional comparative statics results.

PROPOSITION 4 *If the number of agents n increases and $c(x) = c$, the expected profit of each agent decreases.*

PROOF: As n increases, the function $u(x)$ remains constant up to its endpoints. As both endpoints move to the left, $u(0) > \tilde{u}(0)$. Thus, the expected value of stopping immediately, which is an optimal strategy in both cases, decreases as n increases. *Q.E.D.*

LEMMA 12 *If the cost function $c(\cdot)$ increases point-wise the support gets smaller, i.e. the left endpoint $\underline{x}$ increases and the right endpoint $\overline{x}$ decreases.*

PROOF: Let $0 < c(x) \leq d(x)$ for all $x \in \mathbb{R}$. Let us denote the equilibrium distribution corresponding to the cost function $c(\cdot)$ by F with the endpoints $\underline{x}, \overline{x}$ and the equilibrium distribution corresponding to the cost function $d(\cdot)$ by G with the endpoints $\underline{y}, \overline{y}$. To be feasible, $F(x)$ and $G(x)$ need to intersect at least once on their support—otherwise one would stochastically dominate the other one which is not possible as agents have to use bounded time stopping strategies. Assume $\underline{y} < \underline{x}$ then, for all $\underline{y} \leq x$,

$$G(x) = \sqrt[n-1]{\int_{\underline{y}}^{x} \frac{d(z)}{\mu}(1 - \phi(x-z))\mathrm{d}z} \geq \sqrt[n-1]{\int_{\underline{x}}^{x} \frac{c(z)}{\mu}(1 - \phi(x-z))\mathrm{d}z} = F(x)\,.$$

Which is a contradiction to feasibility. Thus, $\underline{x} \leq \underline{y}$. *Q.E.D.*

PROPOSITION 5 *If $c(x) = c$ increases to $\tilde{c}(x) = c\alpha$, where $\alpha > 1$, then each participant makes more profit.*

PROOF:

$$u(0) = \frac{c}{\mu}[-\underline{x} - \frac{\sigma^2}{2\mu}(1 - \phi(-\underline{x}))]$$

$$u(\overline{x}) = \frac{c}{\mu}[\Delta - \frac{\sigma^2}{2\mu}(1 - \phi(\Delta))] = 1$$

$$\frac{\mathrm{d}\frac{1}{c}}{\mathrm{d}\Delta} = \frac{1}{\mu}[1 - \phi(\Delta)] \leq \frac{1}{\mu}$$

Q.E.D.

LEMMA 13 *If $n \nearrow$ and $c(x) = c$, then $\underline{x} \nearrow$, $\overline{x} \nearrow$, and $\overline{x} - \underline{x}$ remains constant.*

PROOF: If $c(x) = c$, $F(\overline{x}) - F(\underline{x})$ clearly depends only on $\overline{x} - \underline{x}$. Hence, for $F(\overline{x}) - F(\underline{x}) = 1$, $\overline{x} - \underline{x}$ has to be constant. As F gets more concave if n increases, by feasibility $\underline{x} \nearrow$ and $\overline{x} \nearrow$. (Both functions intersect twice on $(\underline{x}, \overline{x})$.) *Q.E.D.*

LEMMA 14 *If $\sigma \nearrow$, then $\underline{x} \searrow$ and $\underline{x} \nearrow$.*

PROOF: As in Lemma 12, $F(x) - \tilde{F}(x)$ is increasing in x and has exactly one intersection (on the support) by feasibility. Hence, $\underline{x} \searrow$ and $\underline{x} \nearrow$. *Q.E.D.*

4.3. *Head Starts and Activity*

In this section, we consider a situation in which players start with different values x_0. This models a situation in which some participants already have a better technology than others when the competition starts. For motivating examples, as well as many results about head starts in an all-pay auction setting, see Siegel (2011). If players compete for one prize in the auction, only the bidders with the two highest valuations bid with positive probability, i.e., are *active*.[4] Analogously, we say that a player is *active* in our setting, if he does not stop the process immediately with probability 1.

PROPOSITION 6 *Assume $n = 2$ and $x_0^1 > x_0^2$. Then both players are active in equilibrium if and only if there exists a value $\underline{x} < x_0^2$ such that $\mathbb{P}(X_\tau = x_0^1 | \tau_{(\underline{x}, x_0^1)}(x_0^2)) - Ec(\tau_{(\underline{x}, x_0^1)}(x_0^2)) > 0$.*

PROPOSITION 7 *Assume $n \geq 3$ and $x_0^1 \geq x_0^2 \geq x_0^3 \geq \ldots \geq x_0^n$. Denote the support of the equilibrium distributions if only player 1 and 2 are active $[\underline{x}, \overline{x}]$. In any equilibrium, player 3 is also active if and only if $x_0^3 \in (\underline{x}, \overline{x})$. In this case, at least 3 players make positive profits.*

[4]Siegel (2011) shows under a mild genericity assumption that even for an n-player auction with $m < n$ identical prizes, in any equilibrium only $m + 1$ bidders are active.

Hence, two of the well-known results in the literature on all-pay auctions—with or without head starts—, namely activity of only two players and positive profits for at most one player, do no longer hold in general if the research success is not entirely deterministic. However, if the head start of one or two participants is too high, the players with an inferior starting value prefer to stay out of the contest.

5. APPENDIX

Proof of Lemma 1:: We distinguish two cases and show that both lead to a contradiction.
(i) Assume there exists an interval $I = (a, b)$ such that exactly one player i stops with positive probability on every subinterval $I' = [c, d] \subset I$. Hence, in particular, player i has to be indifferent to stop if $X_t^i = \frac{c+d}{2}$ or to continue with $\tau^i_{(c,d)}(\frac{c+d}{2})$. However, as no other player stops in $[c, d]$, the continuation strategy is strictly worse, because it is costly and leaves the winning probability unchanged.
(ii) Now assume there exists an interval $I = (a, b) \subset [\underline{x}, \overline{x}]$ on which no player stops with positive probability. Denote the infimum of points $x \geq b$ at which a player stops by $\tilde{x}$ and the player who stops at $\tilde{x}$ by j. Hence, an optimal strategy of player j is $\tau^j_{(\underline{x}^j,\tilde{x})}(\frac{a+b}{2})$. If no other player i places a mass point at $\tilde{x}$, the continuation strategy is worse than to stop at $X_t^j = \frac{a+b}{2}$, because it is costly and leaves the winning probability unchanged. If another player places a mass point at $\tilde{x}$, by continuity of c, there exists an $\epsilon > 0$ such that, $\tau^j_{(\underline{x}^j,\tilde{x}+\epsilon)}(\frac{a+b}{2})$ is strictly better than $\tau^j_{(\underline{x}^j,\tilde{x})}(\frac{a+b}{2})$. This contradicts optimality of $\tau^j_{(\underline{x}^j,\tilde{x})}(\frac{a+b}{2})$. *Q.E.D.*

Proof of Lemma 3:: Assume $\overline{x}^j > \overline{x}^i$. For at least two players j, j', the strategy $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ is weakly better than to stop at $X_t^j = \overline{x}^i$ by Lemma 1. By Lemma 2, at least one of these players—denote it j—wins with probability zero at $\underline{x}^j$. Note that $u^i(\overline{x}^i) = \prod_{h\neq i} F^h(\overline{x}^i) < \prod_{h\neq j} F^h(\overline{x}^i) = u^j(\overline{x}^i)$, because $F^i(\overline{x}^i) = 1 > F^j(\overline{x}^i)$.

Optimality of $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ implies

$$u^j(\overline{x}^i) \leq \mathbb{P}(X^j_{\tau^j} = \overline{x}^j | \tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^j(\overline{x}^j) - \mathbb{E}(c(\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

On the other hand,

$$u^i(\overline{x}^i) < u^j(\overline{x}^i) \leq \mathbb{P}(X^i_{\tau^i} = \overline{x}^j | \tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^i(\overline{x}^j) - \mathbb{E}(c(\tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

Hence, at $X_t^i = \overline{x}^i$, player i can profitably deviate to $\tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$; this contradicts the equilibrium assumption. *Q.E.D.*

Proof of Lemma 4:: To prove the first statement, we distinguish two cases.
(i) If at least two players have $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) = 0\ \forall i$. Now assume, there exists a player j who makes less profit than a player i, where $\pi^i = \mathbb{P}(X^i_{\tau^i} =$

$\overline{x}|\tau^i_{(\underline{x}^i,\overline{x})}(0)) - \mathbb{E}(c(\tau^i_{(\underline{x}^i,\overline{x})}(0)))$. If player j deviates to the strategy $\tau^j_{(\underline{x}^j,\overline{x})}(0)$, player j gets a profit equal to π^i; this contradicts optimality of player j's strategy.
(ii) If only one player has $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) > 0$. We now consider the case in which this player i makes a weakly higher payoff than the remaining players, who make the same payoff each—otherwise the argument in the first part of the proof leads to a contradiction.
For any interval $I \in [\underline{x}, \overline{x}]$ in which player i stops with positive probability, by Lemma 1, there exists another player j who also stops in the interval. In particular, for $x \in I$, we get

$$\mathbb{P}(X^i_{\tau^i} = \overline{x}|\tau^i_{(\underline{x},\overline{x})}(x)) + \mathbb{P}(X^i_{\tau^i} = \underline{x}|\tau^i_{(\underline{x},\overline{x})}(x))u^i(\underline{x}) - \mathbb{E}(c(\tau^i_{(\underline{x},\overline{x})}(x))) = u^i(x)$$

and $\mathbb{P}(X^j_{\tau^j} = \overline{x}|\tau^j_{(\underline{x},\overline{x})}(x)) - \mathbb{E}(c(\tau^j_{(\underline{x},\overline{x})}(x))) \geq u^j(x)\ \forall j \neq i$.

The two equations imply that $u^i(x) > u^j(x)$ for all $j \neq i$ and for all x in the support of player i. Hence, $F^i(x) \leq F^j(x)\ \forall j$ and for all x on the support of player i, and, by monotonicity of F^j, on $[\underline{x}, \overline{x}]$. Thus, the distribution of player i stochastically dominates that of all other players. This contradicts feasibility, since all players start at the same value and stopping times have to be bounded. The second statement of the lemma follows immediately from the proof of (ii). *Q.E.D.*

PROOF OF LEMMA 5:: Recall that all players have the same profit, and $u^i(\underline{x}) = 0\ \forall i$. Hence, if players i and j stop in each interval $I' \subset I = (a, b)$, then $u^i(x) = u^j(x)$ for all $x \in (a, b)$, since at 0, it is optimal for both to play until X_t reaches x or one endpoint. By the same argument, for any player h who does not stop on I and any player k who stops on I, we get $u^h(x) = \prod_{l \neq h} F^l(x) \leq \prod_{l \neq k} F^l(x) = u^k(x)$. This implies $F^h(x) \geq F^k(x)$.
Now take the supremum of all points $x \leq \overline{x}$ at which there exists an ϵ such that a player k does not stop in $(x - \epsilon, x)$. Clearly, at x, $F^i(x) = F^j(x)$ for all i, j. Take the highest point $\tilde{x}$ in $(\underline{x}, x - \epsilon)$ at which player k stops. Thus, $F^k(\tilde{x}) > F^i(\tilde{x})$. This contradicts $F^k(\tilde{x}) \leq F^i(\tilde{x})$ from the first part of the proof. *Q.E.D.*

PROOF OF $u'(\underline{x}) = 0$: By definition, $u(x) = 0$ for all $x \leq \underline{x}$. Hence, the left derivative $\partial_- u(\underline{x})$ is zero. It remains to prove that the right derivative $\partial_+ u(\underline{x})$ is also zero. For a given $u : \mathbb{R} \to \mathbb{R}_+$, let $\Psi : \mathbb{R} \to \mathbb{R}$ be the unique function that satisfies the second order ordinary differential equation $c(x) = \mu\Psi'(x) + \frac{\sigma^2}{2}\Psi''(x)$ with the boundary conditions $\Psi(\underline{x}) = \partial_+ u(\underline{x})$ and $\Psi'(\underline{x}) = \partial_+ u(\underline{x})$. It follows from $\Psi'(\underline{x}) > 0$ that there exists a point $\hat{x} < \underline{x}$ such that $\Psi(\hat{x}) < 0 = u(\hat{x})$. Consider the strategy S that stops when either the point $\hat{x}$ or $\overline{x}$ is reached or at 1,

$$S = \min\{1, \inf\{t \in \mathbb{R}_+ : X^i_t \notin [\hat{x}, \overline{x}]\}\}.$$

As $u(\hat{x}) > \Psi(\hat{x})$ it follows that $\mathbb{E}(u(X_S)) > \mathbb{E}(\Psi(X_S))$ and thus

$$\mathbb{E}(u(X_S) - \int_0^S c(X_t^i)\mathrm{d}t) > \mathbb{E}(\Psi(X_S) - \int_0^S c(X_t^i)\mathrm{d}t)\,.$$

Note that, by Itô's lemma, the process $\Psi(X_t^i) - \int_0^t c(X_s^i)\mathrm{d}s$ is a martingale. By Doob's optional sampling theorem agent i is indifferent between the equilibrium strategy τ and the bounded time strategy S, i.e.,

$$\begin{aligned}\mathbb{E}(\Psi(X_S) - \int_0^S c(X_t^i)\mathrm{d}t) &= \mathbb{E}(\Psi(X_\tau) - \int_0^\tau c(X_t^i)\mathrm{d}t) \\ &= \mathbb{E}(u(X_\tau) - \int_0^\tau c(X_t^i)\mathrm{d}t)\,.\end{aligned}$$

The last step follows because $u(x)$ and $\Psi(x)$ coincide for all $x \in (\underline{x}, \overline{x})$. Consequently, the strategy S is a profitable deviation which is a contradiction. *Q.E.D.*

Proof of Theorem 1:: The density $f(\cdot)$ is given by

$$\begin{aligned} f(x) &= \frac{F(x)^{-n+2}}{n-1}\frac{2}{\sigma^2}\int_{\underline{x}}^x c(z)\phi(x-z)\mathrm{d}z \\ &= \frac{F(x)^{-n+2}}{n-1}\frac{2}{\sigma^2}\left(\int_{\underline{x}}^x c(z)\mathrm{d}z - \mu F(x)^{n-1}\right)\end{aligned}$$

As $f(x) > 0$ for all $x > \underline{x}$ it suffices to show Lipschitz continuity of F^{-1} at 0. We substitute $x = F^{-1}(y)$ and get

$$(f \circ F^{-1})(y) \geq \frac{1}{n-1}\frac{2}{\sigma^2}\left(y^{2-n}(\min_{z\in[\underline{x},\infty)} c(z))((F^{-1}(y) - F^{-1}(0)) - \mu y\right)\,.$$

Rearranging with respect to $F^{-1}(y) - F^{-1}(0)$ gives

$$\begin{aligned} F^{-1}(y) - F^{-1}(0) &\leq \left(\frac{(n-1)\sigma^2}{2}(f\circ F^{-1})(y) + \mu y\right)\frac{y^{n-2}}{\min_{z\in[\underline{x},\infty)} c(z)} \\ &\leq \left(\frac{(n-1)\sigma^2}{2} f(\overline{x}) + \mu\right)\frac{y^{n-2}}{\min_{z\in[\underline{x},\infty)} c(z)}\,.\end{aligned}$$

This proves the Lipschitz continuity of $F^{-1}(\cdot)$ for $n > 2$. Note that for two agents $n = 2$ the function F^{-1} is not Lipschitz continuous. We proof that g is Lipschitz continuous by showing that $|g'|$ is bounded.

$$g'(x) = \frac{\Phi'(x)}{(f\circ F^{-1}\circ\Phi)(x)}$$

Because $\lim_{x\to\underline{x}} \frac{\partial}{\partial x}|(f'\circ F^{-1})(x)| \to \infty$ it follows that $|g'|$ is bounded. *Q.E.D.*

Proof of Proposition 6: If player 1 is inactive, player 2 is only active if the equation is satisfied. But in this case, player 1 also has to active, since player 2 would place a mass point slightly above his starting value. By best response, player 2 has to active then, too. *Q.E.D.*

Proof of Proposition 7: Consider a two-player equilibrium. For any point in the interior $u^3(x) \leq u^2(x) \leq u^1(x)$. Hence, player 3 remains inactive if he starts outside the support of the other players. Could there be eq where player 3 is inactive? *Q.E.D.*

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