General-equilibrium models of wealth inequality based on uninsurable idiosyncratic risk: Bewley-Huggett-Aiyagari Models

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Why market incompleteness?

- Consider a model without shocks but with two agents. What predictions are there for wealth inequality?
  - Assume two infinitely-lived households with identical preferences but different initial capital and different income streams.
  - What is the (set of) steady state(s)?
  - What kind of preference heterogeneity breaks this result?
  - Does finite lives break this result?
  - What else breaks this result?

- Add income shocks but complete markets: how is the result influenced?


- Do we “need” to depart from representative-agent macroeconomics?
The incomplete-markets setting: the consumer

- **Shock process:** labor income is $\epsilon \in \{\epsilon_h, \epsilon_l\}$, with first-order Markov transitions $\Pi = \begin{pmatrix} \pi_h & 1 - \pi_h \\ \pi_l & 1 - \pi_l \end{pmatrix}$.

- **Preferences:** $E \sum_{t=0}^{\infty} \beta^t u(c_t)$.

- **Asset markets:** no insurance market but a riskless bond, priced at $q_t$. A **borrowing constraint:** $a_{t+1} \geq a$ for all $t$.

- **Consumption possibility set:** budget $c_t + q_t a_{t+1} = \epsilon_t + a_t$, borrowing constraint $a_{t+1} \geq a$, and $c_t \geq 0$. 
Aggregate environment in which the consumer lives

- Other people: initially, at $t = 0$, a continuum of consumers with different asset holdings and different labor income statuses. Joint distribution: the measure $\Gamma_0(a, \epsilon)$.

- Aggregate shocks: none. Labor income shocks *uncorrelated* across agents. Law of large numbers holds. If $\mu_t$ is the fraction with $\epsilon = \epsilon_h$ at time $t$, we have $\mu_{t+1} = \mu_t \pi_h + (1 - \mu_t) \pi_l$.

- Market clearing for the asset: $\int_a a_{i,t+1} di = 0$ for all $t$, with initial condition $\int_a a_{i,0} di = 0$; $i$ denotes agent $i$.

- Aggregate resource constraint: an endowment economy, and thus $\int c_{i,t} di = \int \epsilon_{i,t} di$ for all $t$.

- An equilibrium: stochastic processes for all individual variables and a deterministic sequence of prices $q_t$ such that the individual variables (i) solve the maximization problems given the $q_t$ sequence and (ii) satisfy the market-clearing condition for assets and the resource constraints at all points in time.
Stationary equilibrium

- Conceptually: an equilibrium, as defined above, such that $q_t$ and the distribution $\Gamma_t(a, \epsilon)$ do not depend on $t$. That is, the distribution of people over asset holdings and endowments looks the same every period, but individuals move around within it. E.g., the number of people with $\epsilon = \epsilon_h$ and with $a < 2$ is the same every period, even though each consumer only belongs to this group now and then.

- Formally: we use recursive methods. The consumer problem is

$$V_s(a) = \max_{a' \in [a, (\epsilon_s + a)/q]} u(\epsilon_s + a - qa') + \beta \left( \pi_s V_h(a') + (1 - \pi_s) V_l(a') \right)$$

for all $s$ and $a$.

- Decision rule: the problem delivers optimal asset holdings $a' = g_s(a)$ satisfying, for all $s$ and $a$,

$$g_s(a) = \arg\max_{a'} u(\epsilon_s + a - qa') + \beta \left( \pi_s V_h(a') + (1 - \pi_s) V_l(a') \right)$$

s.t. $a' \in [a, (\epsilon_s + a)/q]$. 
Stationary equilibrium, continued

- Formal recursive definition: a $q$, functions $V_s(a)$ and $g_s(a)$, and a measure $\Gamma(a, \epsilon)$, such that (i) given $q$, $V_s(a)$ solves the dynamic-programming problem and $g_s(a)$ attains the maximum in this problem; (ii) the asset market clears; and (iii) $g_s(a)$ and the Markov process generates $\Gamma(a, \epsilon)$ as a stationary distribution.

- Asset-market clearing, more precisely:

$$\sum_s \int_a g_s(a) \Gamma(da, \epsilon_s) = 0.$$

- The stationary distribution: for all $(B, \epsilon)$, where $B$ is an interval $[0, b]$,

$$\Gamma(B, \epsilon) = \sum_s \pi_{\epsilon|\epsilon_s} \int_{a:g_s(a) \in B} \Gamma(da, \epsilon_s).$$
Special case I: no shocks \((\pi_h = 1, \pi_l = 0)\)

- Outcome, intuitively: no market frictions, so people compute, using undistorted prices, their present-value total wealth and consume the return on it in every period.

- Asset price: \(q = \beta\).

- Decision rule: for all \(s\) and \(a\), \(g_s(a) = a\).

- Present-value wealth: for individual \(i\), \(a_{i0} + \frac{\epsilon_h}{1-q}\).

- Consumption: for individual \(i\), in each period, \((1 - q)a_{i0} + \epsilon_h\).

- Stationary asset distribution: \(\Gamma_0(a, \epsilon)\), i.e., whatever it was in period 0. There is a large number of stationary asset distributions: any distribution works (special case of discussion on first page of notes!).
Special case II: the tightest borrowing constraint

- Constraint: \( a = 0 \).
- Outcome, intuitively: if no one can borrow, no one can lend, implying autarky. The bond price is set by the “lender” who most wants it; this \((\epsilon_h-)\)individual is unconstrained but has \( a = 0 \).
- Decision rule: for all \( s \), \( g_s(0) = 0 \). (Note: we do not need to define \( g \) for \( a > 0 \) here.) Consumption is \( \epsilon_s \).
- Asset price: given by Euler equation of the \( h \) individual:

\[
qu'(\epsilon_h) = \beta \left( \pi_h u'(\epsilon_h) + (1 - \pi_h) u'(\epsilon_l) \right) \quad \Rightarrow \quad q = \beta \left( \pi_h + (1 - \pi_h) \frac{u'(\epsilon_l)}{u'(\epsilon_h)} \right) > \beta,
\]

so that the gross riskless rate is less than \( 1/\beta \).
- Stationary asset distribution: \( \Gamma_0(a, \epsilon) = 0 \) for \( a \neq 0 \).
The general case: numerical solution

- Now, thus: $a < 0$.
- Outcome, intuitively: a nontrivial outcome; people move around in the distribution, and their consumption responds to endowment realizations. The interest rate is below $1/\beta$—reflecting that assets have an insurance value in addition to the market return—but above the value when $a = 0$.
- Loosest possible borrowing constraint: $a^* = \frac{\epsilon}{1-q}$, which is the value below which a consumer’s debt explodes, even with zero consumption in all periods.
- Decision rule: $g_h(a) \geq g_l(a), g_s(a)$ strictly increasing (and convex, using typical preferences) for $a$ large enough, and $g_l(a) = a$ for $a$ low enough.
- Model solution: exactly as in the Aiyagari model, but iterating on $q$ instead of on aggregate capital.
The Aiyagari model with valued leisure

The neoclassical growth model without aggregate shocks and with idiosyncratic shocks. For simplicity, $\epsilon \in \{0, 1\}$, so 0 mean unemployment.

- The consumer’s problem: for all $(\omega, \epsilon)$,

$$V(\omega, \epsilon) = \max_{k', n} u(\omega + s + n\epsilon w - k', 1 - n) + \beta E[V(k'(1 - \delta + r), \epsilon')|\epsilon]$$

s.t. $k' \geq k$, $n \in [0, 1]$. ($s$ is a transfer/home production.) This leads to decision rules $h^k(\omega, \epsilon)$ and $h^n(\omega, \epsilon)$.

As in the Huggett model, labor income is $\epsilon \in \{\epsilon_h, \epsilon_l\}$, with first-order Markov transitions $\Pi = \begin{pmatrix} \pi_h & 1 - \pi_h \\ \pi_l & 1 - \pi_l \end{pmatrix}$.

Note: Aiyagari (1994) has wage shocks (the individual wage is AR(1) with lognormal shocks).

- Asset structure: as in the Huggett model, no insurance against idiosyncratic shocks; here, the aggregate asset is capital, so the sum of individual savings is equal to the capital stock.
Stationary equilibrium

Object: prices $r$ and $w$, decision rules $h^k$ and $h^n$, and a stationary distribution $\Gamma$ such that

1. $h^k(\omega, \epsilon)$ and $h^n(\omega, \epsilon)$ attain the maximum in the consumer’s problem for all $(\omega, \epsilon)$.

2. $r = F_k(\bar{k}, \bar{n})$ and $w = F_2(\bar{k}, \bar{n})$, where
   \[
   \bar{k} \equiv (\sum_{\epsilon} \int_{\omega} \omega \Gamma (d\omega, \epsilon)) / (1 - \delta + r) \text{ and }
   \bar{n} \equiv \int_{\omega} h^n(\omega, 1)\Gamma (d\omega, 1).
   \]

3. $\Gamma(B, \epsilon) = \sum_{\tilde{\epsilon}} \pi_{\epsilon|\tilde{\epsilon}} \int_{\omega: h^k(\omega, \epsilon) \in B} \Gamma (d\omega, \tilde{\epsilon})$.

Theoretical properties: shapes of decision rules, existence, etc; see Huggett (1993) and Aiyagari (1994).

Computation: guess on $r$ (or $w$), though now with the labor decision rules as part of the optimization.
Calibration: calibrate $\epsilon$s to observed earnings distribution. Set $a$ arbitrarily. Use model with capital ("Bewley/Aiyagari model"), implying a relation between wages and returns to saving.

Recall: asset inequality is huge (Gini around 0.8; for earnings it is perhaps 0.4), with most of the wealth is held by a few rich people, and with many people at zero to negative assets.
Implications for the wealth distribution: a problem

The distribution of wealth

<table>
<thead>
<tr>
<th></th>
<th>% of wealth held by top</th>
<th>Fraction with wealth &lt; 0</th>
<th>Gini coefficient</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>1% 5% 10% 20% 30%</td>
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</tr>
<tr>
<td>$b = 0$ model</td>
<td>3% 11% 20% 35% 47%</td>
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<td>0.26</td>
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<tr>
<td>$b = -2.4$ model</td>
<td>3% 13% 23% 39% 52%</td>
<td>0.5%</td>
<td>0.33</td>
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<tr>
<td>Data</td>
<td>30% 51% 64% 79% 88%</td>
<td>11%</td>
<td>0.79</td>
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We need the rich to save more. Different possibilities:

- Bigger shocks, somehow (Castañeda, Díaz-Giménez, and Ríos-Rull, 1993).
- Difference in discount factors (on average, the rich families are the patient ones).
- Higher returns for the rich.
- Higher income when poor, say, through unemployment insurance.

Next: $3-\beta$ model with UI 9% of wage.
Preference heterogeneity: people have different $\beta$s

Different degrees of patience should make the wealth distribution fan out:

- with complete markets, it would become degenerate;
- with limited short-selling of future labor income, it will not.

Closely related to Piketty and Zucman (2014). They show that with infinitely-lived agents with random saving rates (and linear decision rules), the wealth distribution becomes Pareto.

Fact in our model: for agents with very large asset holdings, decision rules are (almost) linear! (See our Econometric Society paper for proof in a two-period model.)

Intuition: agents with a lot of wealth are well insured, and if insurance is not an issue, linearity of the decision rules follows from the preference assumptions.
Calibration

Standard, except for $\beta$s: no direct guidance (yet), but the general idea is imperfect passing on of genes across generations.

- Three values of $\tilde{\beta}$: 0.9858, 0.9894, and 0.9930.
- Invariant distribution: 10% at each of the extreme values of $\tilde{\beta}$, 80% at middle value.
- No immediate transitions between extreme values.
- Average duration of highest and lowest $\tilde{\beta}$’s is 50 years (roughly matching the length of a generation).
Results, 3-β model

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<td>64%</td>
<td>79%</td>
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Why are the poor poor in this model? Because they want to be poor.

Contrast this with Castañeda, Díaz-Giménez, and Ríos-Rull (1993): here the poor are just unlucky.

Policy implications differ!