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The model we will study is called the Solow model (after the Nobel Prize-winning economist Robert Solow at M.I.T.).
The Aggregate Production Function Revisited
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Let’s study the shape of the aggregate production function (again, holding technology and employment constant).
This production function exhibits diminishing returns to capital: the extra output from a little bit more capital decreases as $K$ increases.
Put differently, decreasing returns to capital means that the slope of the production function decreases as capital increases:

The slope of the tangent line is higher at $K_1$ than at $K_2$. 
Diminishing Marginal Product of Capital
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- The marginal product of capital is the amount by which output increases when capital increases by a (very) small amount.
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- The slope of the production function is called the marginal product of capital.
- The marginal product of capital is the amount by which output increases when capital increases by a (very) small amount.
- The declining marginal product of capital suggests that it will be difficult to generate sustained growth simply by increasing capital over time.
Sergey Brin, co-founder of Google, on diminishing returns to capital.

Mr. Brin said that he saw no end to other innovations. “You might imagine the lower-hanging fruit has been picked,” he said, “but at the same time we have built ladders and are reaching for larger, higher-hanging fruit.”

From: New York Times Business Section, 10/20/06

link to full article: http://www.nytimes.com/2006/10/20/technology/20google.html
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- To keep things simple, suppose that the entire current stock of capital is depleted (or used up) through depreciation during the course of production.
- In other words, if the economy does not invest today, there will be no capital with which to produce tomorrow.
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- Because capital depreciates completely during production, investment (\( I_t \)) is the only source of capital goods in the future: \( K_{t+1} = I_t \). (Note: We are assuming that it takes one year to build and install new capital goods.)
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Putting it all together:

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Putting it all together:

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This is the law of motion for the economy’s capital stock.
An Alternative Expression for the Law of Motion
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- $\Delta K_{t+1}$ is zero if $sK_t^\alpha = K_t$. 
A Useful Graph

\[ y = K^\alpha \]
\[ I = sK^\alpha \]

When \( K = \bar{K} \),
\[ s\bar{K}^\alpha = K, \text{ so} \]
\[ \Delta K = 0. \]

\( \bar{K} \) is the steady state value of capital.
Let the initial period be $t = 0$.

If $K_0 = \bar{K}$, then the economy's capital stock remains at $\bar{K}$.

Period 0: $K_1 = sK_0^\alpha = s\bar{K}^\alpha = \bar{K}$

This equation defines $\bar{K}$.

Period 1: $K_2 = sK_1^\alpha = s\bar{K}^\alpha = \bar{K}$

Period 2: $K_3 = sK_2^\alpha = s\bar{K}^\alpha = \bar{K}$

Periods 3, 4, 5, ...: more of the same
Solving for the Steady State
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\[ \bar{K} = s\bar{K}^\alpha \implies \bar{K} = s^{1/(1-\alpha)} . \]
Solving for the Steady State

\[ \bar{K} = s \bar{K}^\alpha \quad \Rightarrow \quad \bar{K} = s^{1/(1-\alpha)}. \]

- The steady-state value of the capital stock depends on the savings rate \( s \) and the exponent \( \alpha \) in the production function.
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The higher is the savings rate, the higher is the steady-state capital stock.
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- The steady-state value of the capital stock depends on the savings rate $s$ and the exponent $\alpha$ in the production function.
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- Steady-state output (GDP) is: $\bar{Y} = \bar{K}^\alpha$. 
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- The higher is the savings rate, the higher is the steady-state capital stock.
- Steady-state output (GDP) is: \( \overline{Y} = \overline{K}^\alpha. \)
- Steady-state consumption is: \( \overline{C} = (1 - s)\overline{Y}. \)
THE GOLDEN RULE
(discovered by Edmund Phelps, last year's winner of the Nobel Prize in Economics)

\[ \bar{c} \]

At \( s = 0 \), \( \bar{K} = 0 \), so \( \bar{c} = 0 \)

At \( s = 1 \), \( \bar{c} = (1-s)\bar{Y} = 0 \)

At \( s = \alpha \), \( \bar{c} \) is maximized

\( S = 0 \quad S = \alpha \quad S = 1 \)
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Question: If the economy doesn’t start at the steady-state capital stock $\bar{K}$, does it ever get there?
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Short answer: The economy always converges to $\bar{K}$ (as long as the initial capital stock is positive).
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Short answer: The economy always converges to $\bar{K}$ (as long as the initial capital stock is positive).

However, it takes an infinite amount of time to get to the steady state.
When $K < \bar{K}$, $sK^\alpha - K > 0$, so $K$ increases.

When $K > \bar{K}$, $sK^\alpha - K < 0$, so $K$ decreases.
Dynamics using algebra

\[ K_1 = s K_0^\alpha \]
\[ K_2 = s K_1^\alpha \]
\[ K_3 = s K_2^\alpha \]
\[ \text{etc.} \]

\[ c_0 = y_0 - I_0 = k_0^\alpha - k_1 \]
\[ c_1 = y_1 - I_1 = k_1^\alpha - k_2 \]
\[ c_2 = y_2 - I_2 = k_2^\alpha - k_3 \]
\[ \text{etc.} \]

“Eventually”, both \( K_t \) and \( C_t \) converge to their steady-state values \( \bar{K} \) and \( \bar{C} \).
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An Important Theoretical Discovery

- Growth in the long run is ZERO!
- The savings rate does NOT affect growth in the long run (that is, after the economy converges to its steady state).
- Increases in the savings rate DO affect growth in the short run but NOT in the long run.
An Increase in the Savings Rate

$I = s' K^\alpha \quad (s' > s)$

$s' K^\alpha - \bar{K} = 0$

$I = s K^\alpha$

When $s$ increases, the economy moves to a new higher steady state.
An Improvement in Technology

When technology improves (from $A$ to $A'$), the economy moves to a new higher steady state.
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- Improvements in technology overcome the problem of diminishing returns to capital.
- This is what Sergey Brin means by “building ladders to larger, higher-hanging fruit.”