Econ 510a (second half)
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Solutions for Homework #1

Question 1

We know that $L_{ct} = \theta L$ and thus $L_{it} = (1-\theta)L$. It is also true that $k_t = k_{ct} + k_{it}$. Moreover we have that:

$$\frac{k_{ct}}{L_{ct}} = \frac{k_{it}}{L_{it}} \Leftrightarrow \frac{k_{ct}}{\theta L} = \frac{k_t - k_{ct}}{(1-\theta)L} \Leftrightarrow$$
$$(1-\theta)k_{ct} = \theta(k_t - k_{ct}) \Leftrightarrow k_{ct} = \theta k_t$$

and:

$$k_{it} = (1-\theta)k_t$$

Thus we can write the capital-labor ratios above as:

$$\frac{k_{ct}}{L_{ct}} = \frac{\theta k_t}{\theta L_t} = \frac{k_t}{L_t} = \frac{k_{it}}{L_{it}}$$

Since both $F(\bullet)$ and $G(\bullet)$ have constant returns to scale we can write:

$$F(k_{ct,}L_{ct}) = L_{ct}f\left(\frac{K_{ct}}{L_{ct}}\right) = \theta Lf\left(\frac{k_t}{L_t}\right)$$
$$G(k_{it,}L_{it}) = L_{it}g\left(\frac{K_{it}}{L_{it}}\right) = (1-\theta)Lg\left(\frac{k_t}{L_t}\right)$$

Since the total amount of labor is fixed, with out loss of generality we can normalize it so that $L = 1$:

$$F(k_{ct,}1) = \theta f(k_t)$$
$$G(k_{it,}1) = (1-\theta)g(k_t)$$

Total capital evolves according to:

$$k_{t+1} = (1-\delta)k_t + i_t \Leftrightarrow$$
$$k_{t+1} = (1-\delta)k_t + (1-\theta)g(k_t)$$

We will assume the following:

- $g(k_t)$ is strictly increasing in k_t
- $g(k_t)$ is strictly concave in k_t
- $g(0) = 0$
- $(1-\theta)g'(0) > \delta$
- $\lim_{k\to\infty} (1-\delta) + (1-\theta)g'(k) < 1$

Therefore based on all the above, as we did in class we see that if we graph

$k_{t+1} = (1-\delta)k_t + (1-\theta)g(k_t)$, it will start at zero and cross the 45° degree line from above only once at k^*, where k^* is such that $k^* = (1-\delta)k^* + (1-\theta)g(k^*)$ ($\Leftrightarrow \delta k^* = (1-\theta)g(k^*)$) .

Thus for $k_t < k^*$ it will be the case that $k_{t+1} - k_t > 0$ (since $k_{t+1} = (1-\delta)k_t + (1-\theta)g(k_t)$ is above the 45° degree line) and for $k_t > k^*$ it will be the case that $k_{t+1} - k_t < 0$ (since $k_{t+1} = (1-\delta)k_t + (1-\theta)g(k_t)$ is below the 45° degree line).

Thus k_t is monotone bounded sequence. Since it is also bounded, it has a limit which is k^*.

Question 2

Let's define two new function $h(\bullet)$ and $l(\bullet)$ such that:

$$h(k_t) \equiv f(g(k_t))$$
$$l(k_t) \equiv g(f(k_t))$$

Obviously it will be the case that:

$$k_{t+2} = h(k_t) \text{ if } t \text{ is even}$$
$$k_{t+2} = l(k_t) \text{ if } t \text{ is odd}$$

We now want to see whether there exists k_h such that $h(k_h) = k_h$. We know that since $f(\bullet)$ and $g(\bullet)$ are strictly increasing and strictly concave, $h(\bullet)$ has to be strictly increasing and strictly concave. Moreover we have that:

$$f(0) = g(0) = 0 \Rightarrow h(0)$$

$$f'(0)g'(0) > 1 \Leftrightarrow h'(0) > 1$$

and:

$$\lim_{k\to\infty} f'(g(k))g'(k) < 1 \Leftrightarrow \lim_{k\to\infty} h'(k) < 1$$

Therefore based on all the above, as we did in class we see that if we graph $h(k)$, it will start at zero and cross the 45° degree line from above only once at k_h.

Thus for $k_t < k_h$ and t even, it will be the case that $k_{t+2} - k_t = h(k_t) - k_t > 0$ and for $k_t > k_h$ and t even, it will be the case that $k_{t+2} - k_t = h(k_t) - k_t < 0$ (since $h(\bullet)$ is now below the 45° degree line).

Thus k_t is monotone bounded sequence. Since it is also bounded, it has a limit which is k_h.

Similarly for $l(\bullet)$ we want to whether there exists k_l such that $l(k_l) = k_l$. We know that since $f(\bullet)$ and $g(\bullet)$ are strictly increasing and strictly concave, $l(\bullet)$ has to be strictly increasing and strictly concave. Moreover we have that:

$$f(0) = g(0) = 0 \Rightarrow l(0)$$

$$f'(0)g'(0) > 1 \Leftrightarrow l'(0) > 1$$

and:

$$\lim_{k\to\infty} g'(f(k))f'(k) < 1 \Leftrightarrow \lim_{k\to\infty} l'(k) < 1$$

Therefore based on all the above, as we did in class we see that if we graph $l(k)$, it will start at zero and cross the 45° degree line from above only once at k_l.

Thus for $k_t < k_l$ and t odd, it will be the case that $k_{t+2} - k_t = l(k_t) - k_t > 0$ and for $k_t > k_l$ and t odd, it will be the case that $k_{t+2} - k_t = l(k_t) - k_t < 0$ (since $l(\bullet)$ is now below the 45° degree line).

Thus k_t is monotone bounded sequence. Since it is also bounded, it has a limit which is k_l.

In other words there is global convergence to a ”two cycle” in which k_t oscillates between k_h and k_l.

If we know assume that $f(k_t) = ak_t$ and $g(k_t) = bk_t$, where a and b are positive constants,

we will have that for t even:

$$h(k_t) = f(g(k_t)) = abk_t$$

and for t odd:

$$l(k_t) = g(f(k_t)) = bak_t$$

Clearly if $ab > 1$ capital grows indefinitely and if $ab < 1$, capital will shrink and converge to zero. If $ab = 1$ capital will stay at its initial level.

Question 3

The functional Euler equation is

$$-u'(f(k) - g(k)) + \beta u'(f(g(k)) - g(g(k)))f'(g(k)) = 0$$

Differentiate both sides with respect to k, we have

$$
\begin{aligned}
0 = & -u''(f(k)-g(k))(f'(k)-g'(k)) \\
& +\beta u''(f(g(k))-g(g(k)))\left[f'(g(k))g'(k)-(g'(k))^2\right]f'(g(k)) \\
& +\beta u'(f(g(k))-g(g(k)))f''(g(k))g'(k)
\end{aligned}
$$

Evaluate this equation at $k = k^*$, and notice that $g(k^*) = k^*$ and $f(k^*) = \beta^{-1}$, we have

$$
\begin{aligned}
0 = & -u''(f(k^*)-k^*)(f'(k^*)-g'(k^*)) \\
& +u''(f(k^*)-k^*)\left[f'(k^*)g'(k^*)-(g'(k^*))^2\right] \\
& +\beta u'(f(k^*)-k^*)f''(k^*)g'(k^*)
\end{aligned}
$$

Plug in $c^* = f(k^*)-k^*$, after some manipulation, we get

$$
(g'(k^*))^2-\left(1+\frac{1}{\beta}+\frac{u'(c^*)}{u''(c^*)}\frac{f''(k^*)}{f'(k^*)}\right)g'(k^*)+\frac{1}{\beta}=0
$$

Or:

$$
\lambda^2-\left(1+\frac{1}{\beta}+\frac{u'(c^*)}{u''(c^*)}\frac{f''(k^*)}{f'(k^*)}\right)\lambda+\frac{1}{\beta}=0
$$

where λ is equal to $g'(k^*)$. We also know that the speed of convergence near the steady state is inversely related to the slope of the decision rule at the steady state (i.e. $g'(k^*)$ or λ).

Here we will see that curvature of production function will speed up convergence, while curvature of utility function will retard it. The economic intuition is as follows: (a) the higher the curvature of production in the steady state, the sharper the change in the marginal return of capital when we are perturbed from steady state. Therefore, people want to invest more to make use of this opportunity, which speeds up convergence. (b) the higher the curvature of utility function in the steady state, the sharper the change in the marginal utility when perturbed from the steady state. Since people wants to smooth their marginal utility, they will consume more today to offset the change, which slows down the capital accumulation.

Plug into the curvature of utility and production functions, we have

$$
\lambda^2-\left(1+\frac{1}{\beta}+\frac{\beta c^*}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2}\right)\lambda+\frac{1}{\beta}=0
$$

in which

$$
\begin{aligned}
k^* &= (f')^{-1}\left(\frac{1}{\beta}\right)=\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \\
c^* &= f(k^*)-k^* = k^*(k^{*\alpha-1}-\delta)
\end{aligned}
$$

Define

$$\begin{aligned} B &= 1 + \frac{1}{\beta} + \frac{\beta c^*}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2} \\ &= 1 + \frac{1}{\beta} + \frac{\beta k^*(k^{*\alpha-1} - \delta)}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2} \\ &= 1 + \frac{1}{\beta} + \frac{\beta\alpha(1-\alpha)}{\sigma}(k^*)^{\alpha-1}(k^{*\alpha-1} - \delta) \\ &= 1 + \frac{1}{\beta} + \frac{(1-\alpha)(1-\beta+\beta\delta)(1-\beta+\beta\delta-\alpha\beta\delta)}{\alpha\beta\sigma} \end{aligned}$$

Now we know that the solutions of characteristic equation are

$$\lambda = \frac{B \pm \sqrt{B^2 - 4\beta^{-1}}}{2}$$

We are interested in the smaller root, i.e.

$$\lambda_1 = \frac{B - \sqrt{B^2 - 4\beta^{-1}}}{2}$$

Now we study the effect of different parameters in turn. For each case, first we derive the sign mathematically; then we give the economic intuition for this.

First, the effect of α. Differentiate w.r.t. α, we get

$$\frac{\partial\lambda_1}{\partial\alpha} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial\alpha}$$

Obviously the first term is negative. The second term is

$$\begin{aligned} \frac{\partial B}{\partial\alpha} &= \frac{(1-\beta+\beta\delta)}{\beta\sigma}\left[-\frac{1}{\alpha^2}(1-\beta+\beta\delta-\alpha\beta\delta) + \left(\frac{1}{\alpha} - 1\right)(-\beta\delta)\right] \\ &= -\frac{(1-\beta+\beta\delta)}{\beta\sigma}\,\frac{1-\beta+(1-\alpha^2)\beta\delta}{\alpha^2} \\ &< 0 \end{aligned}$$

Therefore, we have $\frac{\partial\lambda_1}{\partial\alpha} > 0$. This confirms our economic intuition before. Since the total effect of increasing α reduces the curvature ($\frac{\partial B}{\partial\alpha} < 0$), it makes the speed of convergence slower (*higher* λ).

Second, the effect of σ. Differentiate w.r.t. σ, we get

$$\frac{\partial\lambda_1}{\partial\sigma} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial\sigma}$$

Obviously the first term is negative. The second term is

$$\frac{\partial B}{\partial \sigma} = -\frac{(1-\alpha)(1-\beta+\beta\delta)(1-\beta+\beta\delta-\alpha\beta\delta)}{\alpha\beta\sigma^2}$$
$$< 0$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \sigma} > 0$. The economic intuition goes as follows. Since in the steady state there is no intertemporal issues, σ won't influence steady-state capital level. Therefore, the only effect of σ on convergence is on the direct effect of intertemporal substitution. If σ increases, the curvature of utility function increases, which makes convergence slows down. Consequently, an increase in α causes slower convergence (*higher* λ).

Third, the effect of δ. Differentiate w.r.t. δ, we get

$$\frac{\partial \lambda_1}{\partial \delta} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \delta}$$

Obviously the first term is negative. The second term is

$$\frac{\partial B}{\partial \delta} = \frac{1-\alpha}{\alpha\beta\sigma}[\beta(1-\beta+\beta\delta-\alpha\beta\delta)+\beta(1-\beta+\beta\delta)(1-\alpha)]$$
$$> 0$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \delta} < 0$. The economic intuition is similar to the case of α. First, given k^*, increasing δ makes the curvature of production function $\left(\frac{\alpha(1-\alpha)k^{*(\alpha-2)}}{\alpha k^{*(\alpha-1)}+1-\delta}\right)$ increase, which speeds up convergence rate. Second, increasing δ will reduce steady-state capital stock, which again increases the curvature. Two effects operate together to speed up convergence further.

Finally, the effect of β. Differentiate w.r.t. β, we get

$$\frac{\partial \lambda_1}{\partial \beta} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \beta} - \frac{1}{\beta^2\sqrt{B^2 - 4\beta^{-1}}}$$

It is easy to see that

$$1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}} < 0$$
$$-\frac{1}{\beta^2\sqrt{B^2 - 4\beta^{-1}}} < 0$$

Now we determine the term

$$\frac{\partial B}{\partial \beta} = \underbrace{-\frac{1}{\beta^2}}_{<0} + \underbrace{\frac{1-\alpha}{\alpha\sigma}}_{>0}\left[\underbrace{\left(-\frac{1}{\beta^2}\right)}_{<0}\underbrace{(1-\beta+\beta\delta-\alpha\beta\delta)}_{>0} + \underbrace{(\beta^{-1}-1+\delta)}_{>0}\underbrace{(-1+\delta-\alpha\delta)}_{<0}\right]$$
$$< 0$$

Now we have

$$\frac{\partial\lambda_1}{\partial\beta} = \frac{1}{2}\underbrace{\left(1-\frac{B}{\sqrt{B^2-4\beta^{-1}}}\right)}_{<0}\underbrace{\frac{\partial B}{\partial\beta}}_{<0} - \underbrace{\frac{1}{\beta^2\sqrt{B^2-4\beta^{-1}}}}_{>0}$$

$$\lesseqgtr 0$$

Therefore, the effect of β on convergence is ambiguous.

Question 4

If some of you were confused when you first read the problem with s_{1t} and $s_{2t,}$ one way to think about it is that in s_{it}, i denotes the number of periods left to complete the project, so for example s_{2t} which is a choice variable today, will be completed in 2 periods.

(a) As stated in the problem the choice variable every period is $s_{2.}$ The state variables will be the stock of capital as of this period, k (this will be crucial in determining our production this period) and the stock of partially completed projects s_1 (which we will have to complete this period). Making use of the capital accumulation constraint, the resource constraint and the investment equation we can write the Bellman equation as:

$$V(s_1,k) = \max_{s_2} \{u(F(k) - [(1-\phi)s_1 + \phi s_2]) + \beta V(s_1', k')\}$$

where:

$$k' = (1-\delta)k + s_1$$
$$s_1' = s_2$$

(b) The first-order condition for this problem is given by:

$$-\phi u'(c) + \beta V_s(s_2, k') = 0$$

where:

$$c = F(k) - (1-\phi)s_1 - \phi s_2$$

The envelope conditions are:

$$V_s(s_1,k) = -(1-\phi)u'(c) + \beta V_k(s_2,k')$$
$$V_k(s_1,k) = u'(c)F'(k) + (1-\delta)\beta V_k(s_{2,}k')$$

(c) In steady state it is the case that:

$$k = k' = \overline{k}$$
$$s_1 = s_2 = \overline{s}$$
$$c = \overline{c}$$

and from the capital accumulation equation we have:

$$\overline{k} = (1-\delta)\overline{k} + \overline{s} \Leftrightarrow$$
$$\overline{s} = \delta\overline{k}$$

and therefore:

$$\overline{c} = F\left(\overline{k}\right) - \delta\overline{k}$$

From the second envelope condition we derived, we can solve for $V_k\left(\overline{s}, \overline{k}\right)$:

$$V_k\left(\overline{s}, \overline{k}\right) = u'(\overline{c})F'\left(\overline{k}\right) + (1-\delta)\beta V_k\left(\overline{s}, \overline{k}\right) \Leftrightarrow$$
$$V_k\left(\overline{s}, \overline{k}\right) = \frac{u'(\overline{c})F'\left(\overline{k}\right)}{1-\beta(1-\delta)}$$

If we plug this into the l.h.s. of the first envelope condition, we get that:

$$V_s\left(\overline{s}, \overline{k}\right) = -(1-\phi)u'(\overline{c}) + \beta\frac{u'(\overline{c})F'\left(\overline{k}\right)}{1-\beta(1-\delta)}$$

and plugging the above into the first-order conditions and get:

$$\phi u'(\overline{c}) = -\beta(1-\phi)u'(\overline{c}) + \beta^2\frac{u'(\overline{c})F'\left(\overline{k}\right)}{1-\beta(1-\delta)} \Leftrightarrow$$
$$\phi = -\beta(1-\phi) + \beta^2\frac{F'\left(\overline{k}\right)}{1-\beta(1-\delta)}$$

Under the assumptions that $F(k) = k^a$ and $\phi = 1$, the above equation becomes:

$$1 = \frac{\beta^2 a\overline{k}^{a-1}}{1-\beta(1-\delta)} \Leftrightarrow$$
$$\overline{k} = \left[\frac{\beta^2 a}{1-\beta(1-\delta)}\right]^{\frac{1}{1-a}}$$