Instructor	: Tony Smith	October 27, 2005
T.A.	: Carolina Silva	
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Suggested Solutions	: Problem Set 1	

1. (a) Given the wage w, the interest r and the consumer's endowment K_0 of capital, the consumer have to solve:

$$\max_{c,l} \log(c) + A\log(l)$$
$$s.t. \quad c = w(1-l) + rK_0$$

where, as renting the capital bring only utility to the consumer, we are not including the rented capital as a decision variable (he just rent all of it). Replacing the expression for c, given by the budget constraint, in the utility function, we get the following FOC:

$$\frac{-w}{w(1-l)+rK_0} + \frac{A}{l} = 0$$

which implies that

$$n^* = 1 - l^* = \frac{w - ArK_0}{w(1+A)} \tag{1}$$

Now, the firms solve the following problem

$$\max_{k,n} zk^\alpha n^{1-\alpha} - rk - wn$$

and the FOC's of this problem imply that prices must satisfy:

$$w = z(1-\alpha)\left(\frac{k}{n}\right)^\alpha \quad \text{and} \quad r = z\alpha\left(\frac{k}{n}\right)^{\alpha-1} \tag{2}$$

Replacing equations (2) in eq.(1), and noting that in equilibrium the supply and demand of capital and labor must be equal, that is, capital demand must be K_0 and labor demand n^*, we get

$$\begin{aligned} n^* &= \frac{z(1-\alpha)(K_0/n^*)^\alpha - AK_0 z\alpha(K_0/n^*)^{\alpha-1}}{z(1-\alpha)(K_0/n^*)^\alpha(1+A)} \\ &= \frac{(1-\alpha) - An^*\alpha}{(1-\alpha)(1+A)} \end{aligned}$$

$$\Rightarrow n^*(1-\alpha)(1+A) = (1-\alpha) - An^*\alpha$$
$$\Rightarrow n^* = \frac{1-\alpha}{1-\alpha+A} \tag{3}$$

In equilibrium we have that output and consumption are the same (remember that as we imposed clearing market conditions for labor and capital, the last market, the goods market, will clear too), using eq.(3) we obtain the following expression for it

$$y^* = zK_0^\alpha\left(\frac{1-\alpha}{1-\alpha+A}\right)^{1-\alpha} = c^*$$

Therefore,

- $\uparrow z \Rightarrow$ no change in n^* and $\uparrow y^*, c^*$. There are no effects on the labor supply of an increase in z because, given we are working with a log utility, the substitution effect (which alone would imply an increase in labor supply) and the income effect (which alone would imply an increase in the consumption of leisure) cancel each other out.
- $\uparrow A \Rightarrow \downarrow n^*$ and $\downarrow y^*, c^*$. With an increase in A, the marginal utility of leisure increases, and thus people want to consume more leisure, and thus supply less labor. As the labor supply decreases, total output, and thus consumption, decreases.

(b) Now, given K_0, w, r, τ and T, the consumer must solve

$$\max_{c,l} \log(c) + A\log(l)$$

$$s.t. \quad c = (1-\tau)w(1-l) + rK_0 + T$$

Replacing the expression of consumption in the utility function we obtain the following FOC:

$$\frac{-w(1-\tau)}{rK_0 + (1-\tau)w(1-l) + T} + \frac{A}{l} = 0$$

which implies

$$n^* = 1 - l^* = \frac{w(1-\tau) - ArK_0 - AT}{w(1-\tau)(1+A)} \tag{4}$$

The firm's problem has not changed, thus we have the same FOC as in part (a). Replacing those FOC's in equation (4) we get (and using clearing market conditions)

$$n^* = \frac{z(1-\alpha)(K_0/n^*)^\alpha(1-\tau) - AK_0 z\alpha(K_0/n^*)^{\alpha-1} - A\tau n^* z(1-\alpha)(K_0/n^*)^\alpha}{z(1-\alpha)(K_0/n^*)^\alpha(1-\tau)(1+A)}$$

$$= \frac{(1-\alpha)(1-\tau) - A\alpha n^* - A\tau n^*(1-\alpha)}{(1-\alpha)(1-\tau)(1+A)}$$

$$\Rightarrow n^*(1-\alpha)(1-\tau)(1+A) = (1-\alpha)(1-\tau) - A\alpha n^* - A\tau n^*(1-\alpha)$$

$$\Rightarrow n^* = \frac{(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau) + A}$$

And then

$$y^* = c^* = zK_0^\alpha \left(\frac{(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau) + A}\right)^{1-\alpha}$$

Therefore

- $\uparrow z \Rightarrow \Delta n^* = 0$ and $\uparrow y^*, c^*$. Same as in part (a).
- $\uparrow A \Rightarrow \downarrow n^*$ and $\downarrow y^*, c^*$. Same as in part (a).
- $\uparrow \tau \Rightarrow \downarrow n^*$ and $\downarrow y^*, c^*$. The income tax implies that now the relative price of leisure as smaller, and thus people want to consume more leisure and thus supply less labor (as total wealth do not change, there is no income effect). A decrease in labor supply implies a decrease in total output and consumption.

(c) The Pareto optimal allocation is the one computed in part (a), and thus we can easily see that the equilibrium labor supply in part (b) is smaller that the Pareto optimal one. Hence, the labor income tax is creating distortions in the leisure decision of the consumer, the larger the tax, the smaller the labor supply.

(d) If instead we had a proportional tax on capital, then, as capital is inelastically supplied in this example, this tax will not distort the consumer decision. Hence, in this case, the Pareto optimal allocations and the competitive equilibrium will be the same.

2. The consumer's problem is

$$\max_{c,l} \frac{c^{1-\sigma}-1}{1-\sigma} + A\frac{l^{1-\gamma}-1}{1-\gamma}$$

$$s.t. \quad c = w(1-l) + rK_0$$

and thus the FOC is

$$-(w(1-l)+rk_0)^{-\sigma}w + Al^{-\gamma} = 0 \tag{5}$$

Replacing in (5) the formulas for the competitive prices $w = z(1-\alpha)\left(\frac{k}{n}\right)^{\alpha}$ and $r = z\alpha\left(\frac{k}{n}\right)^{\alpha-1}$, and noting that $n = (1-l)$, we get that

$$A(1-n)^{-\gamma} = z(1-\alpha)\left(\frac{k}{n}\right)^{\alpha}\left(z(1-\alpha)\left(\frac{k}{n}\right)^{\alpha} n + z\alpha\left(\frac{k}{n}\right)^{\alpha-1} K_0\right)^{-\sigma}$$

after some algebra, this equation leads to

$$A(1-n)^{-\gamma} = (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha))} \tag{6}$$

which implicitly solves for the optimal labor supply n^*. To see how increases in z affect the equilibrium outcomes, let's compute total derivative of eq. (6) (I will omit the "*" for a clearer notation):

$$\begin{aligned} 0 =& [-(1-\alpha)(1-\sigma)z^{-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha))}]dz + \\ &+ [A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}]dn \end{aligned}$$

and then

$$\frac{dn}{dz} = \frac{(1-\alpha)(1-\sigma)z^{-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha))}}{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}}$$

Thus, the denumerator is positive and the sign of the numerator depends on whether σ is larger or smaller that 1:

$$\frac{dn^*}{dz} \text{ is } \begin{cases} >0 & \text{if } \sigma < 1 \\ <0 & \text{if } \sigma > 1 \\ =0 & \text{if } \sigma = 1 \end{cases}$$

Thus, if $\sigma > 1$ then the income effect dominates and the agent supply less labor, if $\sigma < 1$ then the substitution effect dominates and the agent supply more labor and if $\sigma = 1$, ie, log utility, these effects cancel out.

Now,

$$y^* = zK_0^{\alpha}(n^*)^{1-\alpha} \quad \Rightarrow \quad \frac{dy^*}{dz} = K_0^{\alpha}(n^*)^{1-\alpha} + \frac{dn^*}{dz}z(1-\alpha)K_0^{\alpha}(n^*)^{-\alpha}$$

the first term is positive and the sign of the second term depends on the sign of dn^*/dz. If $(dn^*/dz) > 0$ then $(dn^*/dz) > 0$ too, but if $(dy^*/dz) < 0$ then we have to analyze the relation between both terms. Let's see:

$$\frac{dy^*}{dz} = K_0^\alpha n^{1-\alpha}\left[1 + \frac{dn^*}{dz} z(1-\alpha)n^{-1}\right]$$

$$= K_0^\alpha n^{1-\alpha}\left[1 + \frac{(1-\alpha)^2(1-\sigma)z^{1-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha)+1)}}{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}}\right]$$

$$= \frac{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha)+1)}[(1-\alpha)(1-\sigma)+\alpha+\sigma(1-\alpha)]}{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}}$$

$$= \frac{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}n^{-(\alpha+\sigma(1-\alpha)+1)}}{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}}$$
$$> 0$$

as $c^* = y^*$, the same result holds for equilibrium consumption.

Finally, for the effects of changes in A, from (6):

$$\frac{dn^*}{dA} = \frac{-(1-n)^{-\gamma}}{A\gamma(1-n)^{-(1+\gamma)} + (1-\alpha)z^{1-\sigma}K_0^{\alpha(1-\sigma)}(\alpha+\sigma(1-\alpha))n^{-(\alpha+\sigma(1-\alpha)+1)}} < 0$$

and

$$\frac{dc^*}{dA} = \frac{dy^*}{dA} = \frac{n^*}{dA} z(1-\alpha)K_0^\alpha(n^*)^{-\alpha} < 0$$

3. (a) The consumer's problem is:

$$max_{c_0,c_1,k_1} \log(c_0) + \beta\log(c_1)$$

$$s.t. \quad c_0 + k_1 = r_0k_0 + w_0 \quad \text{and} \quad c_1 = r_1k_1 + w_1$$

$$\Leftrightarrow \max_{k_1} \log(-k_1 + r_0k_0 + w_0) + \beta\log(r_1k_1 + w_1)$$

and the implied FOC is:

$$\frac{1}{-k_1 + r_0k_0 + w_0} = \frac{\beta r_1}{r_1k_1 + w_1}$$

The firm's optimal solution plus market clearing conditions ($n = 1$ and capital in period 0 is k_0 and k_1^* in period 1) imply that:

$$w_1 = z(1-\alpha)(k_1^*)^\alpha \quad \text{and} \quad r_1 = z\alpha(k_1^*)^{\alpha-1} \tag{7}$$

Next note that, as we have CRS, at equilibrium $r_0k_0 + w_0 = y_0 = zK_0^\alpha$. Replacing this equality in the FOC, and using the formulas in (7), we get that

$$k_1^* = \frac{\alpha\beta}{1+\alpha\beta} zk_0^\alpha$$

and then

$$r_1 = z\alpha(k_1^*)^{\alpha-1} = z^\alpha \alpha \left(\frac{\alpha\beta}{1+\alpha\beta}\right)^{\alpha-1} k_0^{\alpha(\alpha-1)}$$

and the consumption allocation is

$$c_0 = y_0 - k_1^* = zk_0^\alpha \left(1 - \frac{\alpha\beta}{1+\alpha\beta}\right) = \frac{zk_0^\alpha}{1+\alpha\beta}$$

and

$$c_1 = y_1 = z(k_1^*)^\alpha = z^{1+\alpha} \left(\frac{\alpha\beta}{1+\alpha\beta}\right)^\alpha k_0^{\alpha^2}$$

and thus an increase in z will increase the equilibrium consumption in both periods.

(b) The planner problem does not involve prices (as the planner decides the allocations, prices are not needed anymore), he only have to consider the clearing market conditions. Thus the planner's problem is:

$$\max_{c_0,c_1} \log(c_0) + \beta \log(c_1)$$

$$s.t. \quad c_0 = zk_0^\alpha - k_1 \quad \text{and} \quad c_1 = zk_1^\alpha$$

Replacing the expressions for c_0 and c_1, given in the budget constraint, in the utility function, we get the following FOC:

$$\frac{\alpha\beta}{k_1} = \frac{1}{zk_0^\alpha - k_1} \quad \Rightarrow \quad k_1^* = \frac{\alpha\beta}{1+\alpha\beta} zk_0^\alpha$$

and thus we have the same allocation of part (a). We will see the graph in class.