Econ 510a (second half)
Yale University
Fall 2006
Prof. Tony Smith

HOMEWORK #1

This homework assignment is due at 5PM on Friday, November 3 in Marnix Amand's mailbox.

1. Consider a growth model with capital accumulation equation $k_{t+1} = f(k_t)$ if t is even and $k_{t+1} = g(k_t)$ if t is odd. Assume that:

(i) $f(0) = g(0) = 0$.

(ii) $f'(0)g'(0) > 1$.

(iii) $\lim_{k\to\infty} f'(g(k))g'(k) < 1$ and $\lim_{k\to\infty} g'(f(k))f'(k) < 1$.

(iv) f and g are strictly increasing and strictly concave.

Show that, from any initial condition $k_0 > 0$, there is global convergence to a "two-cycle" in which k_t oscillates between two values. How are these values determined?

Solution We have:

- $k_{2t} = g(f(k_{2t-2}))$
- $k_{2t+1} = f(g(k_{2t-1}))$

Let h be defined as $h(k) = g(f(k)) - k$ for all k. Should the sequence k_{2t} be converging, the limit necessarily is a solution of $h(k) = 0$ (cf. your favorite math course on sequences). Using properties (i) and (ii), we have that $h(0) = 0$ and h is increasing at 0. By differentiating twice, h is strictly concave. Using property (iii), $h' < 0$ for k sufficiently large, and hence $\lim_{k\to\infty} h(k) = -\infty$ because, by concavity, h is "under" its tangents. It is thus proved that $h(k) = 0$ has one solution, and only one (using concavity again), for $k > 0$, called $\hat{k}$.

Furthermore, since $g \circ f$ is increasing, we have, for $k < \hat{k}$, $g(f(k)) < g(f(\hat{k})) = \hat{k}$, and, for $k > \hat{k}$, $g(f(\hat{k})) < \hat{k}$. So the sequence k_{2t} is increasing and bounded for $k_0 < \hat{k}$ and decreasing and bounded for $k_0 > \hat{k}$. In both cases, k_{2t} is converging, necessarily toward $\hat{k}$ which is the only possible limit.

Same story for k_{2t+1}, which converges to the unique solution of $f(g(k)) = k$. This the global convergence to the "two-cycle".

2. Consider a neoclassical growth model similar to the one that we have discussed in lecture, but in which the level of technology oscillates deterministically between two values A_H and A_L, where $A_H > A_L$. In particular, period-t output y_t equals $A_H F(k_t)$ if t is even and equals $A_L F(k_t)$ if t is odd. The planner ("Robinson Crusoe") seeks to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$, given $k_0 > 0$, subject to the resource constraint that $c_t + k_{t+1} = y_t + (1-\delta)k_t$ and to the nonnegativity constraint $k_{t+1} \geq 0$ for all t.

(a) Formulate the planner's problem recursively. (Hint: Consider two value functions, one for periods in which the level of technology is high and one for periods in which the level of technology is low. Find a pair of Bellman equations that these functions must satisfy.)

(b) Let the felicity function u be logarithmic, let $y_t = A_t k_t^{\alpha}$, and assume that capital depreciates fully in one period (i.e., set $\delta = 1$). Use a guess-and-verify method to find the two value functions in part (a). Describe fully the dynamic behavior of the capital stock.

Solution **(a)** When $A = A_H$, the planner's problem is given by the following value function:

$$V_H(K) = \max_{K'} \{u(A_H F(K) + (1-\delta) K - K') + \beta V_L(K')\} \tag{1}$$

when $A = A_L$ the planner's problem is going to be given by:

$$V_L(K) = \max_{K'} \{u(A_L F(K) + (1-\delta) K - K') + \beta V_H(K')\} \tag{2}$$

(b) If $u(c) = \log c$, $\delta = 1$ and $F(K_t) = K_t^a$, then the above value functions become:

$$\begin{aligned} V_H(K) &= \max_{K'} \{\log(A_H K^a - K') + \beta V_L(K')\} \\ V_L(K) &= \max_{K'} \{\log(A_L K^a - K') + \beta V_H(K')\} \end{aligned}$$

We know guess that the value functions are of the form:

$$\begin{aligned} V_H &= E + F \log K \\ V_L &= G + J \log K \end{aligned}$$

In that case, the equation above for V_H becomes:

$$E + F \log K = \max_{K'} \{\log(A_H K^a - K') + \beta (G + J \log K')\} \tag{3}$$

The first-order conditions are:

$$\begin{aligned}
-\frac{1}{A_H K^a - K'} + \frac{\beta J}{K'} &= 0 \Leftrightarrow \\
\frac{\beta J}{K'} &= \frac{1}{A_H K^a - K'} \Leftrightarrow \\
K' &= A_H \beta J K^a - \beta J K' \Leftrightarrow \\
g_H(K) &= K' = \frac{\beta J A_H K^a}{1+\beta J}
\end{aligned}$$

Plugging the decision rule derived above back in the original equation:

$$\begin{aligned}
E + F\log K &= \log\left(A_H K^a - \frac{J A_H \beta K^a}{1+\beta J}\right) + \beta\left(G + J\log\frac{J A_H \beta K^a}{1+\beta J}\right) \\
&= \log A_H + a\log K - \log(1+\beta J) + \beta G + \beta J \log J A_H \beta \\
&- \beta J\log(1+\beta J) + \beta J a \log K
\end{aligned}$$

Therefore:

$$E = \log A_H - (1+\beta J)\log(1+\beta J) + \beta G + \beta J \log J A_H \beta \tag{4}$$

and:

$$F = a + \beta a J \tag{5}$$

Similarly the value function when $A = A_L$, is given by:

$$G + J\log K = \max_{K''}\{\log(A_L K^a - K') + \beta(E + F\log K')\} \tag{6}$$

The first-order condition gives the following decision rule:

$$g_L(K) = K' = \frac{\beta F A_L K^a}{1+\beta F} \tag{7}$$

Plugging the decision rule back into the equation above, we get:

$$\begin{aligned}
G + J\log K &= \log(A_L K^a - \frac{\beta F A_L K^a}{1+\beta F}) + \beta\left(E + F\log\frac{\beta F A_L K^a}{1+\beta F}\right) \\
&= \log A_L + a\log K - \log(1+\beta F) + \beta E + \beta F\log\beta F A_L \\
&+ \beta F a\log K - \beta F\log(1+\beta F)
\end{aligned}$$

Therefore:

$$G = \log A_L - (1+\beta F)\log(1+\beta F) + \beta E + \beta F \log \beta F A_L \tag{8}$$

and

$$J = a + \beta a F \tag{9}$$

combining the expression above for J with the one we found for F, gives us:

$$\begin{aligned} J &= a + \beta a (a + \beta a F) \Leftrightarrow \\ J &= \frac{a+\beta a^2}{1-\beta^2 a^2} = \frac{a(1+\beta a)}{(1+\beta a)(1-\beta a)} = \frac{a}{1-\beta a} \end{aligned}$$

and thus:

$$F = a + \beta a \frac{a}{1-\beta a} = \frac{a}{1-\beta a} = J \tag{10}$$

Therefore the equation for E now becomes:

$$\begin{aligned} E &= \log A_H - \left(1 + \frac{\beta a}{1-\beta a}\right)\log\left(1+\frac{\beta a}{1-\beta a}\right) + \beta G + \frac{\beta a}{1-\beta a}\log A_H \frac{\beta a}{1-\beta a} = \\ &= \frac{1}{1-\beta a}\log A_H + \beta G + \frac{\beta a}{1-\beta a}\log\frac{\beta a}{1-\beta a} - \frac{1}{1-\beta a}\log\frac{1}{1-\beta a} \end{aligned}$$

Let $M = \frac{\beta a}{1-\beta a}\log\frac{\beta a}{1-\beta a} - \frac{1}{1-\beta a}\log\frac{1}{1-\beta a}$. Then E becomes:

$$E = \frac{1}{1-\beta a}\log A_H + \beta G + M \tag{11}$$

And similarly plugging in for E, in the equation above we had for G we have:

$$G = \frac{1}{1-\beta a}\log A_L + \beta E + M \tag{12}$$

Thus solving the above system of 2 equations and 2 unknowns, we get:

$$\begin{aligned} E &= \left(1-\beta^2\right)^{-1}\left((1-\beta a)^{-1}(\log A_H + \beta \log A_L) + M + \beta M\right) \\ G &= \left(1-\beta^2\right)^{-1}\left((1-\beta a)^{-1}(\log A_L + \beta \log A_H) + M + \beta M\right) \end{aligned}$$

Therefore the decision rule when $A = A_H$ is given by:

$$g_H(K) = \beta a A_H K^a \tag{13}$$

and when $A = A_{L,}$ is given by:

$$g_L(K) = \beta a A_L K^a \tag{14}$$

We will now show that there is a "global convergence" to a "two-cycle" in which K_t oscillates between two values. These 2 values are:

$$\begin{aligned} K_H &= \beta a A_H K_H^a \Leftrightarrow \\ K_H &= (\beta a A_H)^{\frac{1}{1-a}} \end{aligned}$$

when $A = A_H$ and similarly:

$$K_L = (\beta a A_L)^{\frac{1}{1-a}} \tag{15}$$

when $A = A_{L.}$ We will show that the assumptions of Question 2 of Homework #1 hold indeed in this case and thus there is a global convergence to a "two-cycle". We have:

$\cdot g_H(0) = g_L(0) = 0$

$\cdot$ Moreover we have that: $g'_H(K) = \beta a^2 A_H K^{a-1}$ and $g'_L(K) = \beta a^2 A_L K^{a-1}$ and thus $g'_H(0) = \infty$ and $g'_L(0) = \infty$ since $a - 1 < 0$. Therefore:

$$g'_H(0)\, g'_L(0) = \infty > 1 \tag{16}$$

$\cdot$ Furthermore:

$$\begin{aligned} g_H(g_L(K)) &= \beta^{a+1} a^{a+1} A_H A_L^a K^{a^2} \\ g'_H(g_L(K)) &= \beta^{a+1} a^{a+3} A_H A_L^a K^{a^2-1} \end{aligned}$$

and:

$$\begin{aligned} g'_H(g_L(K))\, g'_L(K) &= \beta^{a+1} a^{a+3} A_H A_L^a K^{a^2-1} \beta a^2 A_L K^{a-1} = \\ &= \beta^{a+2} a^{a+5} A_H A_L^{a+1} K^{a^2+a-2} \end{aligned}$$

and since $a^2 + a - 2 < 0$ (remember $0 < a < 1$) we have that:

$$\lim_{K\to\infty} g'_H(g_L(K))\, g'_L(K) = 0 \tag{17}$$

Similarly:

$$g'_L\left(g_H\left(K\right)\right)g'_H\left(K\right) = \beta^{a+2}a^{a+5}A_L A_H^{a+1}K^{a^2+a-2} \tag{18}$$

and:

$$\lim_{K\to\infty} = g'_L\left(g_H\left(K\right)\right)g'_H\left(K\right) = 0 \tag{19}$$

· Finally g_H and g_L are strictly increasing and strictly concave

3. Consider a neoclassical growth model in which the felicity function u has constant elasticity of intertemporal substitution σ^{-1}:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

where $\sigma > 0$ and $u(c) = \log(c)$ if $\sigma = 1$. In addition, assume that $f(k) = Ak^\alpha + (1-\delta)k$, where $A > 0$, $\alpha \in (0, 1)$, and $\delta \in [0, 1]$. What happens to the speed of convergence to the steady state as u becomes linear (i.e., as σ approaches 0)? As f becomes linear (i.e., as α approaches 1)? Try to provide economic intuition for your findings. (Note: As discussed in Section 4.2 in Chapter 4 of the lecture notes, the speed of convergence near the steady state is inversely related to the slope of the decision rule at the steady state.)

Solution Using the notes (section 4.2), especially equation 4.2 on p. 47 (simply use the algebraic expressions for u and f):

- for u becoming linear, we have $\lambda \approx 0$, so convergence is immediate. The intuition is that with linear utility, there is hardly any intertemporal smoothing, so the agent immediately "jumps" to the equilibrium
- for $\alpha \approx 1$ becoming linear, we have $\lambda \approx 1$, so convergence is very slow (in fact, there is no convergence). The intuition is that there are no decreasing returns to scale for saving, so the agent doesn't keep investing until he reaches equilibrium.

4. Consider the planning problem for a basic finite-horizon neoclassical growth model:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t \log(c_t),$$

given $k_0 = 10$ and subject to the resource constraint that $c_t + k_{t+1} = Ak_t^\alpha + (1-\delta)k_t$ and to the nonnegativity constraint $k_{t+1} \geq 0$ for all $t \leq T$. Set $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.4$. Choose A so that the steady-state value of capital in the corresponding infinite-horizon model is 100.

Solve the model numerically (say, in Matlab) using the "shooting" method described in lecture on October 23: start by guessing a value for k_1, solve for k_2 from the Euler equation at time 0, then solve for k_3 from the Euler equation at time 1, and so on, until k_{T+1} is found (T being the time horizon). Then vary k_1 and repeat until the appropriate value of k_{T+1} (what is it?) is found. Find the lowest value for T such that the highest value of capital between periods 0 and T exceeds 90.

Solution We first derive the Euler equation for this problem, then we solve the question numerically. We can solve for Euler equation either from nonlinear programming or dynamic programming method. Here we use dynamic programming. Define $f(k_t) = Ak_t^\alpha + (1-\delta) k_t$. The recursive formulation for this problem is

$$v_t(k_t) = \max_{k_{t+1}} \ln (f(k_t) - k_{t+1}) + \beta v_{t+1}(k_{t+1})$$

Note that here value function depends on the time subscript t. F.O.C. for $t \leq T-1$ is

$$-\frac{1}{c_t} + \beta v'_{t+1}(k_{t+1}) = 0$$

¿From Envelope theorem, we have

$$v'_t(k_t) = \frac{1}{c_t} f'(k_t)$$

Iterate forward and plug back time subscripts, we get the Euler equation

$$\beta \frac{1}{c_{t+1}} f'(k_{t+1}) = \frac{1}{c_t}$$

Plug in $c_t = f(k_t) - k_{t+1}$ and $f(k_t) = Ak_t^\alpha + (1-\delta) k_t$, we have

$$\begin{aligned}
& k_{t+2} = f(k_{t+1}) - \beta (f(k_t) - k_{t+1}) f'(k_{t+1}) \\
\Rightarrow \quad & k_{t+2} = \left(Ak_{t+1}^\alpha + (1-\delta) k_{t+1}\right) - \beta \left(Ak_t^\alpha + (1-\delta) k_t - k_{t+1}\right) \left(A\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right) \\
\Rightarrow \quad & k_{t+2} = A(1+\alpha\beta) k_{t+1}^\alpha + (1-\delta)(1+\beta) k_{t+1} - \beta \left(Ak_t^\alpha + (1-\delta) k_t\right) \left(A\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right)
\end{aligned}$$

We can see that it is a second-order nonlinear difference equation, which has boundary condition $k_0 = 10, k_{T+1} = 0$.

Before we go on to the numerical step, we solve for A. The infinite horizon steady state k^* solves

$$f'(k^*) = \beta^{-1} \Rightarrow A = \frac{\beta^{-1} - (1-\delta)}{\alpha (k^*)^{\alpha-1}} = \frac{29}{76} 10^{\frac{6}{5}} \approx 6.0476$$

Now we start to solve for it numerically by using "shooting" method. For an error bound $|k_{T+1}| < 0.01$, when time horizon T exceeds 37, the highest value of capital between periods 0 and T exceeds 90. The approximate period-1 capital stock for $T = 37$ is $k_1 = 16.3833123$. If you use grid point search, to get such a precision you have to define the step size of grid as fine as 10^{-7}.