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Problem Set 1	: Suggested Solutions	

1. (a) In the Solow growth model we have:

$$k_{t+1} = (1-\delta)k_t - sf(k_t)$$

where

$$f(k_t) = L_t F(\frac{K_t}{L_t}, 1)$$

i. We know that in the steady state $k_{t+1} = k_t = k^*$, and so

$$\begin{aligned} k^* &= (1-\delta)k^* - sf(k^*) \\ \delta k^* &= sf(k^*) \end{aligned}$$

If $\delta = 0$ and if we assume there is a non-zero savings rate, i.e., $s \neq 0$, then

$$k^* = 0.$$

And so there is no non-trivial steady state.

ii. The growth rate of capital stock is defined as:

$$\begin{aligned} \frac{k_{t+1} - k_t}{k_t} &= \frac{sf(k_t)}{k_t} - \delta \\ \lim_{t\to\infty} \frac{k_{t+1} - k_t}{k_t} &= \lim_{t\to\infty} \frac{sf(k_t)}{k_t} - \lim_{t\to\infty} \delta \\ &= \lim_{t\to\infty} \frac{sf(k_t)}{k_t} && \text{(since } \delta = 0\text{)} \\ &= \lim_{t\to\infty} sf'(k_t) && \text{(by l'Hôpital's rule)} \\ &= 0 && \text{(by Inada condition).} \end{aligned}$$

(b) In the neoclassical growth model we have the Euler equation:

$$u'\left(F\left(K\right) - K'\right) = \beta u'\left(F\left(K'\right) - K''\right) F'\left(K'\right)$$

$$F\left(K\right) = f(K) - (1-\delta)K$$

At the steady state $k_{t+1} = k_t = k^*$, and so:

$$\begin{aligned} u'\left(F\left(K^*\right) - K^*\right) &= \beta u'\left(F\left(K^*\right) - K^*\right) F'\left(K^*\right) \\ F'\left(K^*\right) &= \beta^{-1} \\ f'(K^*) - (1-\delta) &= \beta^{-1} \\ f'(K^*) &= \beta^{-1} + (1-\delta) \end{aligned}$$

If $\delta = 0$ we have

$$f'(K^*) = \beta^{-1} + 1$$

Since all of our assumptions on f still hold, a steady state exists.

2. Recall that near the steady state, the speed of convergence is inversely related to the slope of the optimal decision rule (policy function) at the steady state:

$$\begin{aligned} k' &= g(k) \\ &\approx g(k^*) + g'(k^*)(k - k^*) \end{aligned}$$

And so we have:

$$k' - k^* = g'(k^*)(k - k^*)$$

To find an expression for the slope of the decision rule we start with our functional Euler equation:

$$-u'\left(f\left(k\right) - g\left(k\right)\right) + \beta u'\left(f\left(g\left(k\right)\right) - g\left(g\left(k\right)\right)\right) f'\left(g\left(k\right)\right) = 0$$

Differentiate both sides with respect to k, we have

$$\begin{aligned} 0 = & -u''\left(f\left(k\right) - g\left(k\right)\right)\left(f'\left(k\right) - g'\left(k\right)\right) \\ & +\beta u''\left(f\left(g\left(k\right)\right) - g\left(g\left(k\right)\right)\right)\left[f'\left(g\left(k\right)\right) g'\left(k\right) - \left(g'\left(k\right)\right)^2\right] f'\left(g\left(k\right)\right) \\ & +\beta u'\left(f\left(g\left(k\right)\right) - g\left(g\left(k\right)\right)\right) f''\left(g\left(k\right)\right) g'\left(k\right) \end{aligned}$$

Evaluate this equation at $k = k^*$, and notice that $g\left(k^*\right) = k^*$ and $f\left(k^*\right) = \beta^{-1}$, we have

$$\begin{aligned} 0 = & -u''\left(f\left(k^*\right) - k^*\right)\left(f'\left(k^*\right) - g'\left(k^*\right)\right) \\ & +\beta u''\left(f\left(k^*\right) - k^*\right)\left[f'\left(k^*\right) g'\left(k^*\right) - \left(g'\left(k^*\right)\right)^2\right] \\ & +\beta u'\left(f\left(k^*\right) - k^*\right) f''\left(k^*\right) g'\left(k^*\right) \end{aligned}$$

Plug in $c^* = f\left(k^*\right) - k^*$, after some manipulation, we get

$$\left(g'\left(k^*\right)\right)^2 - \left(1 + \frac{1}{\beta} + \frac{u'\left(c^*\right)}{u''\left(c^*\right)} \frac{f''\left(k^*\right)}{f'\left(k^*\right)}\right) g'\left(k^*\right) + \frac{1}{\beta} = 0$$

Or:

$$\lambda^2 - \left(1 + \frac{1}{\beta} + \frac{u'(c^*)}{u''(c^*)}\frac{f''(k^*)}{f'(k^*)}\right)\lambda + \frac{1}{\beta} = 0$$

where λ is equal to $g'(k^*)$.

Looking at this quadratic equation we can see that curvature of the production function will speed up convergence, while curvature of the utility function will slow it down.

The economic intuition is as follows:
(a) the greater the curvature of the production function in the steady state, the greater the change in the marginal return of capital when we are perturbed from steady state. And so, people want to invest more to make use of this opportunity, which speeds up convergence
(b) the greater the curvature of the utility function in the steady state, the greater the change in the marginal utility when perturbed from the steady state. Since people want to smooth consumption, they will consume more today to offset the change, which slows down the capital accumulation.

To analyze the parameters we plug in the utility and production functions:

$$\lambda^2 - \left(1 + \frac{1}{\beta} + \frac{\beta c^*}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2}\right)\lambda + \frac{1}{\beta} = 0$$

in which

$$\begin{aligned} k^* &= (f')^{-1}\left(\frac{1}{\beta}\right) = \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \\ c^* &= f(k^*) - k^* = k^*\left(k^{*\alpha-1} - \delta\right) \end{aligned}$$

Define

$$\begin{aligned} B &= 1 + \frac{1}{\beta} + \frac{\beta c^*}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2} \\ &= 1 + \frac{1}{\beta} + \frac{\beta k^*\left(k^{*\alpha-1}-\delta\right)}{\sigma}\alpha(1-\alpha)(k^*)^{\alpha-2} \\ &= 1 + \frac{1}{\beta} + \frac{\beta\alpha(1-\alpha)}{\sigma}(k^*)^{\alpha-1}\left(k^{*\alpha-1}-\delta\right) \\ &= 1 + \frac{1}{\beta} + \frac{(1-\alpha)(1-\beta+\beta\delta)(1-\beta+\beta\delta-\alpha\beta\delta)}{\alpha\beta\sigma} \end{aligned}$$

We know that the solutions of characteristic equation are

$$\lambda = \frac{B \pm \sqrt{B^2 - 4\beta^{-1}}}{2}$$

We are interested in the smaller root, i.e.

$$\lambda_1 = \frac{B - \sqrt{B^2 - 4\beta^{-1}}}{2}$$

First, the effect of α. Differentiating w.r.t. α, we get

$$\frac{\partial \lambda_1}{\partial \alpha} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \alpha}$$

Obviously the first term is negative. The second term is

$$\begin{aligned}
\frac{\partial B}{\partial \alpha} &= \frac{(1-\beta+\beta\delta)}{\beta\sigma}\left[-\frac{1}{\alpha^2}(1-\beta+\beta\delta-\alpha\beta\delta) + \left(\frac{1}{\alpha}-1\right)(-\beta\delta)\right] \\
&= -\frac{(1-\beta+\beta\delta)}{\beta\sigma}\frac{1-\beta+\left(1-\alpha^2\right)\beta\delta}{\alpha^2} \\
&< 0
\end{aligned}$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \alpha} > 0$. Since the total effect of increasing α reduces the curvature ($\frac{\partial B}{\partial \alpha} < 0$), it makes the speed of convergence slower (*higher* λ).

Second, the effect of σ.Differentiating w.r.t. σ, we get

$$\frac{\partial \lambda_1}{\partial \sigma} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \sigma}$$

Obviously the first term is negative. The second term is

$$\begin{aligned}
\frac{\partial B}{\partial \sigma} &= -\frac{(1-\alpha)(1-\beta+\beta\delta)(1-\beta+\beta\delta-\alpha\beta\delta)}{\alpha\beta\sigma^2} \\
&< 0
\end{aligned}$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \sigma} > 0$. The economic intuition goes as follows. Since in the steady state there are no intertemporal issues, σ won't influence steady-state capital level. Therefore, the only effect of σ on convergence wil be on the direct effect of intertemporal substitution. If σ increases, the curvature of the utility function increases and convergence slows down. Consequently, an increase in α causes slower convergence (*higher* λ).

3. Our first step is to derive the Euler equation for this problem, and then we can solve the question numerically.

   The consumer's problem is:

$$v_t(k_t) = \max_{k_{t+1}} \ln(f(k_t) - k_{t+1}) + \beta v_{t+1}(k_{t+1})$$

   Note that here value function depends on the time subscript t. F.O.C. for $t \leq (T-1)$ is

$$-\frac{1}{c_t} + \beta v'_{t+1}(k_{t+1}) = 0$$

From the Envelope Theorem, we have:

$$v_t'(k_t) = \frac{1}{c_t} f'(k_t)$$

If we update and then substitute for $v_{t+1}'(k_{t+1})$ we have our Euler equation:

$$\beta \frac{1}{c_{t+1}} f'(k_{t+1}) = \frac{1}{c_t}$$

We have

$$\begin{aligned} c_t &= f(k_t) - k_{t+1} \\ f(k_t) &= Ak_t^\alpha + (1-\delta)k_t \end{aligned}$$

and so

$$\begin{aligned} k_{t+2} &= f(k_{t+1}) - \beta(f(k_t) - k_{t+1})f'(k_{t+1}) \\ k_{t+2} &= \left(Ak_{t+1}^\alpha + (1-\delta)k_{t+1}\right) - \beta(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})\left(A\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right) \\ k_{t+2} &= A(1+\alpha\beta)k_{t+1}^\alpha + (1-\delta)(1+\beta)k_{t+1} - \beta(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})\left(A\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right) \end{aligned}$$

We can see that it is a second-order nonlinear difference equation, with the boundary condition $k_0 = 10$ and $k_{T+1} = 0$.

Before we go on to the numerical calculation, we solve for A. The infinite horizon steady state k^* solves

$$\begin{aligned} f'(k^*) &= \beta^{-1} \\ A &= \frac{\beta^{-1} - (1-\delta)}{\alpha(k^*)^{\alpha-1}} \\ &= \frac{(0.95)^{-1} - (1-0.1)}{0.4(100)^{0.4-1}} \\ A &= 6.0476 \end{aligned}$$

(a) Savings: $0.48, 0.45, 0.42$, and so on

(b) For an error bound $|k_{T+1}| < 0.01$, when time horizon T exceeds 37, the highest value of capital between periods 0 and T exceeds 90. The approximate period-one capital stock for $T = 37$ is $k_1 = 16.3833123$.

4. We have the functional equation

$$v(k) = u\left(F(k) - g(k)\right) + \beta v(g(k))$$

which we differentiate twice

$$
\begin{aligned}
v'(k) &= u'\left(F(k)-g(k)\right)\left(F'(k)-g'(k)\right)+\beta v'(g(k))g'(k) \\
v''(k) &= \left[u''\left(F(k)-g(k)\right)\left(F'(k)-g'(k)\right)^2+u'\left(F(k)-g(k)\right)\left(F''(k)-g''(k)\right)\right] \\
&\quad +\beta\left[v''(g(k))\left(g'(k)\right)^2+v'(g(k))g''(k)\right]
\end{aligned}
$$

Evaluating at the steady state k^* we have $k=k^*$, $g\left(k^*\right)=k^*$ and $F\left(k^*\right)=\beta^{-1}$:

$$
\begin{aligned}
v''(k^*) &= \left[u''\left(F(k^*)-g(k^*)\right)\left(F'(k^*)-g'(k^*)\right)^2+u'\left(F(k^*)-g(k^*)\right)\left(F''(k^*)-g''(k^*)\right)\right] \\
&\quad +\beta\left[v''(g(k^*))\left(g'(k^*)\right)^2+v'(g(k^*))g''(k^*)\right] \\
v''(k^*) &= u''\left(F(k^*)-k^*\right)\left(\beta^{-1}-g'(k^*)\right)^2+u'\left(F(k^*)-k^*\right)F''(k^*) \\
&\quad -u'\left(F(k^*)-k^*\right)g''(k^*)+\beta v''(k^*)\left(g'(k^*)\right)^2+\beta v'(k^*)g''(k^*) \\
v''(k^*)-\beta v''(k^*)\left(g'(k^*)\right)^2 &= u''\left(F(k^*)-k^*\right)\left(\beta^{-1}-g'(k^*)\right)^2+u'\left(F(k^*)-k^*\right)F''(k^*) \\
&\quad +g''(k^*)\left[\beta v'(k^*)-u'\left(F(k^*)-k^*\right)\right]
\end{aligned}
$$

And so we have:

$$
v''(k^*)=\frac{u''(F(k^*)-k^*)(\beta^{-1}-g'(k^*))^2+u'(F(k^*)-k^*)F''(k^*)+g''(k^*)\left[\beta v'(k^*)-u'(F(k^*)-k^*)\right]}{1-\beta(g'(k^*))^2}
$$

We know that:

$v''(k^*)<0$ (strict concavity)

$u''\left(F(k^*)-k^*\right)<0$ (strict concavity)

$F''(k^*)<0$ (strict concavity)

$u'\left(F(k^*)-k^*\right)>0$ (strictly increasing)

$\beta v'(k^*)-u'\left(F(k^*)-k^*\right)=0$

So the LHS is less than zero while the denominator on the RHS is greater than zero. In this case we need:

$$
\begin{aligned}
1-\beta\left(g'(k^*)\right)^2 &> 0 \\
\left(g'(k^*)\right)^2 &< \beta^{-1}
\end{aligned}
$$

And therefore:

$$
g'(k^*)<\beta^{-1}
$$