Econ 511b (Part I), Spring 2004
Suggested Solutions to Homework #1

1. **Consider the dynamic programming problem**

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right), \text{ given } x_0,$$

subject to $x_{t+1} \in \Gamma\left(x_t\right)$. **Let** F **be a quadratic function of its two arguments.**

(a) Show that the Euler equation for this problem is a linear second-order difference equation. (You may assume that this problem has an interior solution.)

We proceed in two steps. In the first step, we derive the Euler equation for the general formulation. Here we can see that Euler equation is a second-order (in general nonlinear) difference equation. Then we show that if F is a quadratic function of its two arguments, the difference equation is indeed linear.

The recursive formulation of the sequence problem is

$$v\left(x\right) = \max_{y \in \Gamma(x)} F\left(x, y\right) + \beta v\left(y\right)$$

Assuming interior condition, we can find F.O.C.

$$F_2\left(x, y\right) + \beta v^{'}\left(y\right) = 0$$

From Envelope theorem, we have

$$v^{'}\left(x\right) = F_1\left(x, y\right)$$

Iterate forward and plug back time subscripts, we get the Euler equation

$$F_2\left(x_t, x_{t+1}\right) + \beta F_1\left(x_{t+1}, x_{t+2}\right) = 0$$

Here we can see that it is a second order (in general nonlinear) difference equation.

We know that F is a quadratic function. Without loss of generality, we assume that

$$F(x_t, x_{t+1}) = x_t + a x_{t+1} - b x_t^2 - c x_{t+1}^2 + d x_t x_{t+1}$$

To make the optimization well defined, we need to impose some restrictions on these coefficients. For example, if we let F to be strict concave functions, we should have $b > 0, c > 0$,and $4bc - d^2 < 0$. Here we assume that the coefficients are such that the problem is well defined. Then Euler equation becomes

$$\beta d x_{t+2} - 2\left(\beta b + c\right) x_{t+1} + d x_t + a + \beta = 0$$

which is a linear second-order difference equation.

(b) Show that if one of the two eigenvalues associated with the second-order difference equation in part (a) is less than one in absolute value, then the other eigenvalue is larger than β^{-1} in absolute value.

From the theory of solutions of second order difference equation (we derived that in the class from the solutions of stacked first-order system), the characteristic equation is

$$\beta d\lambda^2 - 2(\beta b + c)\lambda + d = 0$$

From the property of roots of this equation, we know that

$$\lambda_1 \lambda_2 = \frac{d}{\beta d} = \frac{1}{\beta} \Rightarrow |\lambda_1||\lambda_2| = \frac{1}{\beta}$$

Therefore, if $|\lambda_1| < 1$,we have $|\lambda_2| > \frac{1}{\beta}$.

2. **(a) For the neoclassical growth model that we developed in lecture, show that the derivative of the policy function at the steady state can be found by differentiating the recursive (or functional) version of the Euler equation. In particular, show that this derivative (evaluated at the steady state) solves a quadratic equation whose two roots correspond to the two eigenvalues of the linearized second-order difference equation that we discussed in class.**

The functional Euler equation is

$$-u'(f(k) - g(k)) + \beta u'(f(g(k)) - g(g(k)))\, f'(g(k)) = 0$$

Differentiate both sides with respect to k, we have

$$\begin{aligned} 0 = & -u''(f(k) - g(k))\left(f'(k) - g'(k)\right) \\ & + \beta u''(f(g(k)) - g(g(k)))\left[f'(g(k))\, g'(k) - \left(g'(k)\right)^2\right] f'(g(k)) \\ & + \beta u'(f(g(k)) - g(g(k)))\, f''(g(k))\, g'(k) \end{aligned}$$

Evaluate this equation at $k = k^*$, and notice that $g(k^*) = k^*$ and $f(k^*) = \beta^{-1}$, we have

$$\begin{aligned} 0 = & -u''(f(k^*) - k^*)\left(f'(k^*) - g'(k^*)\right) \\ & + u''(f(k^*) - k^*)\left[f'(k^*)\, g'(k^*) - \left(g'(k^*)\right)^2\right] \\ & + \beta u'(f(k^*) - k^*)\, f''(k^*)\, g'(k^*) \end{aligned}$$

Plug in $c^* = f(k^*) - k^*$, after some manipulation, we get

$$\left(g^{'}\left(k^{*}\right)\right)^{2}-\left(1+\frac{1}{\beta}+\frac{u^{'}\left(c^{*}\right)}{u^{"}\left(c^{*}\right)}\frac{f^{"}\left(k^{*}\right)}{f^{'}\left(k^{*}\right)}\right)g^{'}\left(k^{*}\right)+\frac{1}{\beta}=0$$

The eigenvalues of the linearized second-order difference equation solves the equation

$$\lambda^{2}-\left(1+\frac{1}{\beta}+\frac{u^{'}\left(c^{*}\right)}{u^{"}\left(c^{*}\right)}\frac{f^{"}\left(k^{*}\right)}{f^{'}\left(k^{*}\right)}\right)\lambda+\frac{1}{\beta}=0$$

Therefore, $g^{'}\left(k^{*}\right)$ solves a quadratic equation whose two roots correspond to the two eigenvalues of the linearized second-order difference equation that we discussed in class.

(b) In the neoclassical growth model developed in lecture, assume that the felicity function u has constant elasticity of intertemporal substitution σ^{-1} and that $f\left(k\right)=k^{\alpha}+\left(1-\delta\right)k$. Discuss how β, α, δ, and σ affect the speed of convergence to the steady state.

Plug $u\left(c\right)=\frac{c^{1-\sigma}-1}{1-\sigma}$, $f\left(k\right)=k^{\alpha}+\left(1-\delta\right)k$, and $f^{'}\left(k^{*}\right)=\frac{1}{\beta}$ into the characteristic equation of the linearized second-order difference equation

$$\lambda^{2}-\left(1+\frac{1}{\beta}+\frac{u^{'}\left(c^{*}\right)}{u^{"}\left(c^{*}\right)}\frac{f^{"}\left(k^{*}\right)}{f^{'}\left(k^{*}\right)}\right)\lambda+\frac{1}{\beta}=0$$

Here we can see that curvature of production function will speed up convergence, while curvature of utility function will retard it. The economic intuition is as follows: (a) the higher the curvature of production in the steady state, the sharper the change in the marginal return of capital when we are perturbed from steady state. Therefore, people want to invest more to make use of this opportunity, which speeds up convergence. (b) the higher the curvature of utility function in the steady state, the sharper the change in the marginal utility when perturbed from the steady state. Since people wants to smooth their marginal utility, they will consume more today to offset the change, which slows down the capital accumulation.

Plug into the curvature of utility and production functions, we have

$$\lambda^{2}-\left(1+\frac{1}{\beta}+\frac{\beta c^{*}}{\sigma}\alpha\left(1-\alpha\right)\left(k^{*}\right)^{\alpha-2}\right)\lambda+\frac{1}{\beta}=0$$

in which

$$\begin{aligned}k^{*} &= \left(f^{'}\right)^{-1}\left(\frac{1}{\beta}\right)=\left(\frac{1-\beta\left(1-\delta\right)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}}\\ c^{*} &= f\left(k^{*}\right)-k^{*}=k^{*}\left(k^{*\alpha-1}-\delta\right)\end{aligned}$$

Define

$$\begin{aligned} B &= 1+\frac{1}{\beta}+\frac{\beta c^*}{\sigma}\alpha\left(1-\alpha\right)\left(k^*\right)^{\alpha-2} \\ &= 1+\frac{1}{\beta}+\frac{\beta k^*\left(k^{*\alpha-1}-\delta\right)}{\sigma}\alpha\left(1-\alpha\right)\left(k^*\right)^{\alpha-2} \\ &= 1+\frac{1}{\beta}+\frac{\beta\alpha\left(1-\alpha\right)}{\sigma}\left(k^*\right)^{\alpha-1}\left(k^{*\alpha-1}-\delta\right) \\ &= 1+\frac{1}{\beta}+\frac{\left(1-\alpha\right)\left(1-\beta+\beta\delta\right)\left(1-\beta+\beta\delta-\alpha\beta\delta\right)}{\alpha\beta\sigma} \end{aligned}$$

Now we know that the solutions of characteristic equation are

$$\lambda=\frac{B\pm\sqrt{B^2-4\beta^{-1}}}{2}$$

We are interested in the smaller root, i.e.

$$\lambda_1=\frac{B-\sqrt{B^2-4\beta^{-1}}}{2}$$

Now we study the effect of different parameters in turn. For each case, first we derive the sign mathematically; then we give the economic intuition for this.

First, the effect of α.Differentiate w.r.t. α, we get

$$\frac{\partial\lambda_1}{\partial\alpha}=\frac{1}{2}\left(1-\frac{B}{\sqrt{B^2-4\beta^{-1}}}\right)\frac{\partial B}{\partial\alpha}$$

Obviously the first term is negative. The second term is

$$\begin{aligned} \frac{\partial B}{\partial\alpha} &= \frac{\left(1-\beta+\beta\delta\right)}{\beta\sigma}\left[-\frac{1}{\alpha^2}\left(1-\beta+\beta\delta-\alpha\beta\delta\right)+\left(\frac{1}{\alpha}-1\right)\left(-\beta\delta\right)\right] \\ &= -\frac{\left(1-\beta+\beta\delta\right)}{\beta\sigma}\frac{1-\beta+\left(1-\alpha^2\right)\beta\delta}{\alpha^2} \\ &< 0 \end{aligned}$$

Therefore, we have $\frac{\partial\lambda_1}{\partial\alpha}>0$. This confirms our economic intuition before. Since the total effect of increasing α reduces the curvature ($\frac{\partial B}{\partial\alpha}<0$), it makes the speed of convergence slower (*higher* λ).

Second, the effect of σ.Differentiate w.r.t. σ, we get

$$\frac{\partial\lambda_1}{\partial\sigma}=\frac{1}{2}\left(1-\frac{B}{\sqrt{B^2-4\beta^{-1}}}\right)\frac{\partial B}{\partial\sigma}$$

Obviously the first term is negative. The second term is

$$\begin{aligned}\frac{\partial B}{\partial \sigma} &= -\frac{(1-\alpha)(1-\beta+\beta\delta)(1-\beta+\beta\delta-\alpha\beta\delta)}{\alpha\beta\sigma^2} \\ &< 0\end{aligned}$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \sigma} > 0$. The economic intuition goes as follows. Since in the steady state there is no intertemporal issues, σ won't influence steady-state capital level. Therefore, the only effect of σ on convergence is on the direct effect of intertemporal substitution. If σ increases, the curvature of utility function increases, which makes convergence slows down. Consequently, increase in α causes slower convergence (*higher* λ).

Third, the effect of δ.Differentiate w.r.t. δ, we get

$$\frac{\partial \lambda_1}{\partial \delta} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \delta}$$

Obviously the first term is negative. The second term is

$$\begin{aligned}\frac{\partial B}{\partial \delta} &= \frac{1-\alpha}{\alpha\beta\sigma}\left[\beta(1-\beta+\beta\delta-\alpha\beta\delta)+\beta(1-\beta+\beta\delta)(1-\alpha)\right] \\ &> 0\end{aligned}$$

Therefore, we have $\frac{\partial \lambda_1}{\partial \delta} < 0$. The economic intuition is similar to the case of α. First, given k^*, increasing δ makes the curvature of production function $\left(\frac{\alpha(1-\alpha)k^{*(\alpha-2)}}{\alpha k^{*(\alpha-1)}+1-\delta}\right)$ increase, which speeds up convergence rate. Second, increasing δ will reduce steady-state capital stock, whch again increase the curvature. Two effects operate together to speed up convergence further.

Finally, the effect of β.Differentiate w.r.t. β, we get

$$\frac{\partial \lambda_1}{\partial \beta} = \frac{1}{2}\left(1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}}\right)\frac{\partial B}{\partial \beta} - \frac{1}{\beta^2\sqrt{B^2 - 4\beta^{-1}}}$$

It is easy to see that

$$\begin{aligned}1 - \frac{B}{\sqrt{B^2 - 4\beta^{-1}}} &< 0 \\ -\frac{1}{\beta^2\sqrt{B^2 - 4\beta^{-1}}} &< 0\end{aligned}$$

Now we determine the term

$$\begin{aligned}\frac{\partial B}{\partial \beta} &= \underbrace{-\frac{1}{\beta^2}}_{<0} + \underbrace{\frac{1-\alpha}{\alpha\sigma}}_{>0}\left[\underbrace{\left(-\frac{1}{\beta^2}\right)}_{<0}\underbrace{(1-\beta+\beta\delta-\alpha\beta\delta)}_{>0} + \underbrace{\left(\beta^{-1}-1+\delta\right)}_{>0}\underbrace{(-1+\delta-\alpha\delta)}_{<0}\right] \\ &< 0\end{aligned}$$

Now we have

$$\begin{aligned}\frac{\partial \lambda_1}{\partial \beta} &= \frac{1}{2}\underbrace{\left(1-\frac{B}{\sqrt{B^2-4\beta^{-1}}}\right)}_{<0}\underbrace{\frac{\partial B}{\partial \beta}}_{<0} - \underbrace{\frac{1}{\beta^2\sqrt{B^2-4\beta^{-1}}}}_{>0} \\ &\gtreqless 0\end{aligned}$$

Therefore, the effect of β on convergence is ambiguous.