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Course	: Econ 510a (Macroeconomics)	
Term	: Fall 2005, second half	
Suggested Solutions	: Problem Set 2	

1. (a) In this question we have two types of consumers: there are a fraction θ of type 1 consumers, who are endowed with k_0^1 units of capital in period 0. The rest of the consumers, a fraction of $1-\theta$, are endowed with k_0^2, where $k_0^1 > k_0^2$.

Then, given the log utility function, the problem that a consumer of type $i = 1, 2$ must solve is:

$$\max_{c_0^i, c_1^i} \log(c_0^i) + \beta \log(c_1^i)$$

$$st \;\; c_0^i = r_0 k_0^i + w_0 - k_1^i$$

$$c_1^i = r_1 k_1^i + w_1$$

which implies the FOC:

$$r_1 k_1^i + w_1 = \beta r_1 (r_0 k_0^i + w_0 - k_1^i) \quad i = 1, 2$$

On the other hand, each period $i = 0, 1$, the firm must solve:

$$\max_{\bar{K}_i, \bar{N}_i} \bar{K}_i^\alpha \bar{N}_i^{1-\alpha} - \bar{K}_i r_i - \bar{N}_i w_i$$

which implies the competitive prices:

$$w_1 = (1-\alpha)\bar{K}_1^\alpha \quad \text{and} \quad r_1 = \alpha \bar{K}_1^{\alpha-1}$$

Also, as we saw in class, the CRS production technology implies that:

$$r_0 \bar{K}_0 + w_0 \bar{N}_0 = y_0 = \bar{K}_0^\alpha \bar{N}_0^{1-\alpha} \tag{1}$$

Finally, the equilibrium conditions are: $\bar{K}_i = \theta k_i^1 + (1-\theta) k_i^2$ for $i = 1, 2$ and as labor is inelastically supplied, $\bar{N}_i = 1$ for $i = 0, 1$.

Now we are ready to solve for the equilibrium. First, multiplying each type's FOC equation by its fraction in the population and summing up both equations, we get:

$$r_1(\theta k_1^1 + (1-\theta)k_1^2) + w_1 = \beta r_1 [r_0(\theta k_0^1 + (1-\theta)k_0^2) + w_0 - (\theta k_1^1 + (1-\theta)k_1^2)]$$

Using the equilibrium conditions and eq.(1):

$$r_1 \bar{K}_1 + w_1 = \beta r_1 [\bar{K}_0^\alpha - \bar{K}_1]$$

then, replacing the competitive prices,

$$\alpha \bar{K}_1^{\alpha-1} \bar{K}_1 + (1-\alpha)\bar{K}_1^\alpha = \beta \alpha \bar{K}_1^{\alpha-1} [\bar{K}_0^\alpha - \bar{K}_1]$$

which implies

$$\bar{K}_1 = \frac{\alpha \beta \bar{K}_0^\alpha}{1 + \alpha\beta}$$

Thus, as period 1 aggregate savings only depends on aggregate period 0 savings, and not on how much has each consumer, if this aggregate is not changed, $\bar{K}_1$ won't change either. And as prices depend also only on aggregates, changes in the distribution of capital endowments, that leave the aggregate equal, won't affect them.

(b) So now we just have to change the consumers utility function. Each type's problem is:

$$\max_{c_0^i, c_1^i} \frac{(c_0^i)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(c_1^i)^{1-\sigma} - 1}{1-\sigma}$$

$$st \;\; c_0^i = r_0 k_0^i + w_0 - k_1^i$$

$$c_1^i = r_1 k_1^i + w_1$$

which implies the FOC:

$$r_0 k_0^i + w_0 - k_1^i = (\beta r_1)^{-1/\sigma}(r_1 k_1^i + w_1)$$

The firm's problem has not changed, so we have the same equations for prices. Following the same steps of part (a), we get that:

$$r_0 \bar{K}_0 + w_0 - \bar{K}_1 = (\beta r_1)^{-1/\sigma}(r_1 \bar{K}_1 + w_1) \tag{2}$$

then, remembering that wages and interest rates only depend on aggregates, from eq.(2), which implicitly defines $\bar{K}_1$, we can conclude that if $\bar{K}_0$ does not change then $\bar{K}_1$ won't change also, and thus, the aggregation results holds here too.

(c) Now the labor supply each period is going to be $\bar{\varepsilon} = \theta\varepsilon_1 + (1-\theta)\varepsilon_2$, and the type $i = 1, 2$ consumer problem is:

$$\max_{c_0^i, c_1^i} \log(c_0^i) + \beta \log(c_1^i)$$

$$st \;\; c_0^i = r_0 k_0 + \varepsilon_i w_0 - k_1$$

$$c_1^i = r_1 k_1 + \varepsilon_i w_1$$

which implies the FOC:

$$r_1 k_1^i + \varepsilon_i w_1 = \beta r_1 (r_0 k_0 + \varepsilon_i w_0 - k_1^i) \quad i = 1, 2$$

and we have the following equilibrium conditions:

$$\bar{K}_0 = k_0 \quad \text{and} \quad \bar{K}_1 = \theta k_1^1 + (1-\theta) k_1^2$$

$$w_1 = f_1(\bar{K}_1, \bar{\varepsilon}) = (1-\alpha)(\bar{K}_1/\bar{\varepsilon})^{\alpha} \quad \text{and} \quad r_1 = f_2(\bar{K}_1, \bar{\varepsilon}) = \alpha(\bar{K}_1/\bar{\varepsilon})^{\alpha-1}$$

and

$$\text{CRS} \Rightarrow r_0 \bar{K}_0 + w_0 \bar{\varepsilon} = y_0 = \bar{K}_0^{\alpha} \bar{\varepsilon}_0^{1-\alpha}$$

Following exactly the same steps as in part (a), ie, multiply FOCs by corresponding fractions, adding them up and using equilibrium conditions, we get that

$$\bar{K}_1 = \frac{\alpha\beta \bar{K}_0^{\alpha} \bar{\varepsilon}^{1-\alpha}}{1 + \alpha\beta}$$

(d) If we follow the same steps as before, we will arrive to an equation that no longer depends only on aggregates, so we will work on each consumer's FOC separately. Consumers of type 1 has the following FOC:

$$r_1 k_1^1 + w_1 = \beta_1 r_1 (r_0 k_0 + w_0 - k_1^1)$$

replacing the competitive prices (firm's problem is the same as in part (a)) and using the implication of the CRS production function, we get

$$\alpha \bar{K}_1^{\alpha-1} k_1^1 + (1-\alpha)\bar{K}_1^{\alpha} = \beta_1 \alpha \bar{K}_1^{\alpha-1}(r_0 k_0 + w_0 - k_1^1)$$

which implies

$$k_1^1 = \frac{\alpha\beta_1 \bar{K}_0^{\alpha} - (1-\alpha)\bar{K}_1}{\alpha(1+\beta_1)} \tag{3}$$

Analogously,

$$k_2^1 = \frac{\alpha\beta_2 \bar{K}_0^{\alpha} - (1-\alpha)\bar{K}_1}{\alpha(1+\beta_2)} \tag{4}$$

Finally, replacing eqs.(3) and (4) in the consistency condition $\bar{K}_1 = \theta k_1^1 + (1-\theta)k_1^2$, we get that:

$$\bar{K}_1 = \frac{\alpha \bar{K}_0^{\alpha}(\bar{\beta} + \beta_1\beta_2)}{\alpha(1 = \beta_1)(1+\beta_2) + (1-\alpha)(1+\bar{\beta})}$$

where $\bar{\beta} = \theta\beta_1 + (1-\theta)\beta_2$.

2. (a) We have an exchange economy where consumers differ in their endowments (both groups have an equal measure):

$$\text{type 1} \longrightarrow w_0^1 = 2 \text{ and } w_1^1 = 0$$

$$\text{type 2} \longrightarrow w_0^2 = 0 \text{ and } w_1^2 = 2$$

Taking period 0 good as the numeraire, the date-0 trading problem consumers have to solve is

$$\max_{c_0^i, c_1^i} \log(c_0^i) + \beta \log(c_1^i)$$

$$st \;\; c_0^i + pc_1^i = w_0^i + pw_1^i$$

which implies the FOC:

$$\frac{-p}{w_0^i + pw_1^i - pc_1^i} + \frac{\beta}{c_1^i} = 0 \quad i = 1, 2$$

and then

$$c_1^i = \frac{\beta(w_0^i + pw_1^i)}{p(1+\beta)} \quad \text{and} \quad c_0^i = \frac{w_0^i + pw_1^i}{1+\beta}$$

Replacing the values of endowments and using the equilibrium condition $c_0^1 + c_0^2 = 2$, we get

$$p = \beta \quad \text{and then} \quad c_0^1 = c_1^1 = \frac{2}{1+\beta} \quad \text{and} \quad c_0^2 = c_1^2 = \frac{2\beta}{1+\beta}$$

Therefore, individuals of type 1 are going to consume more, which is very intuitive, given that they are the ones that have a positive endowment in the period when it is more valuable (period 0).

(b) The planner's problem is

$$\max_{c_0^1,c_1^1,c_0^2,c_1^2} \alpha[\log(c_0^1)+\beta\log(c_1^1)]+(1-\alpha)[\log(c_0^2)+\beta\log(c_1^2)]$$

$$st \;\; c_0^1+c_0^2=2$$
$$c_1^1+c_1^2=2$$

Replacing the expressions for c_0^2 and c_1^2, given by the budget constraints, in the utility function we get the following FOC:

$$0=\frac{\alpha}{c_0^1}-\frac{1-\alpha}{2-c_0^1} \quad \text{and} \quad 0=\frac{\alpha\beta}{c_1^1}-\frac{(1-\alpha)\beta}{2-c_1^1}$$

which implies $c_0^1=c_1^1=2\alpha$ and so $c_0^2=c_1^2=2(1-\alpha)$.

(c) In this problem, the Pareto efficient (PE) allocations are those feasible allocations where the intertemporal marginal rates of substitution of both kind of consumers are equated,

$$\text{equal IMRS} \Rightarrow \frac{u'(c_1^1)}{u'(c_0^1)}=\frac{u'(c_1^2)}{u'(c_0^2)} \quad \text{and} \quad \text{feasibility} \Rightarrow c_0^1+c_0^2=c_1^1+c_1^2=2$$

Thus,

$$(c_0^1+c_1^1) \;\; \text{is PE} \quad \Leftrightarrow \quad \frac{c_0^1}{c_1^1}=\frac{2-c_0^1}{2-c_1^1} \quad \Rightarrow \quad c_0^1=c_1^1$$

Therefore, the PE allocations are those such that $c_0^1=c_1^1=\tau$ with $\tau\in(0,2)$ and $c_0^2=c_1^2=2-\tau$, which are exactly the allocations we obtain from the planner's problem as α varies.

(d) For the allocations from the planner and the competitive equilibrium problems to be the same, we need:

$$2\alpha=\frac{2}{1+\beta} \quad \Rightarrow \quad \bar{\alpha}=\frac{1}{1+\beta}$$

For a type 1 consumer to have more consumption in the planner's problem, he needs to be given a higher weight (note that the only thing that matters when distributing consumption in the planner's problem is the weight of each consumer), and as we saw, the competitive equilibrium implies a larger consumption to type 1 consumers, so α needs to be larger that $(1-\alpha)$, which is what we have here $(1/(1+\beta)$ vs. $\beta/(1+\beta))$.

(e) Given our previous results, the marginal utilities of consumption in period 0 in the competitive equilibrium are:

$$\lambda_i=\frac{1}{c_0^i} \quad \Rightarrow \quad \lambda_1=\frac{1+\beta}{2} \text{ and } \lambda_2=\frac{1+\beta}{2\beta}$$

then

$$\frac{1/\lambda_1}{1/\lambda_1+1/\lambda_2}=\frac{1}{1+\beta}=\bar{\alpha} \quad \surd\surd$$

So, if the marginal utility of type 1 consumers is relatively high, he will receive a lower weight, which makes sense because as the marginal utility increases, the amount of consumption needed to get to certain level of utility, decreases.

(a) Let's use the budget constraints iteratively:

$$a_1 = Ra_0 + w - c_0$$

$$a_2 = Ra_1 + w - c_1 \quad \Rightarrow \quad a_2 = R(Ra_0 + w - c_0) + w - c_1 = R^2 a_0 + Rw - Rc_0 + w - c_1$$

$$\begin{aligned} a_3 = Ra_2 + w - c_2 \quad \Rightarrow \quad a_3 &= R(R^2 a_0 + Rw - Rc_0 + w - c_1) + w - c_2 \\ &= R^3 a_0 + R^2 w - R^2 c_0 + Rw - Rc_1 + w - c_2 \end{aligned}$$

thus, following this steps we will get to:

$$a_t = R^t a_0 + w(R^{t-1} + R^{t-2} + ... + 1) - (R^{t-1} c_0 + R^{t-2} c_1 + ... + c_{t-1})$$

then,

$$\frac{a_t}{R^{t-1}} = Ra_0 + \sum_{i=0}^{t-1} \frac{w}{R^i} - \sum_{i=0}^{t-1} \frac{c_i}{R^i}$$

Taking limits we get

$$\lim_{t\to\infty} \sum_{i=0}^{t-1} \frac{c_i}{R^i} = Ra_0 + \frac{Rw}{R-1} - \lim_{t\to\infty} \frac{a_t}{R^{t-1}}$$

but the no Ponzi game conditions state that

$$\lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

and then, we have that

$$\sum_{i=0}^{\infty} \frac{c_i}{R^i} \leq Ra_0 + \frac{Rw}{R-1}$$

(b) Consider first an agent with zero consumption and such that $a_t = \bar{a} \quad \forall t$, then the budget constraints imply that:

$$\bar{a} = R\bar{a} + w \quad \Rightarrow \quad \bar{a} = \frac{w}{1-R} \equiv B$$

Let's prove now that

$$a_t \geq B \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

- Note that

$$a_t \geq B \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq \lim_{t\to\infty} \frac{B}{R^{t-1}} = 0 \quad (R > 1)$$

- Suppose $\exists n$ such that $a_n < B$, then $\exists \delta > 0$ such that $a_n = B - \delta$. Let's see what the BC imply:

$$\begin{aligned} a_{n+1} &= Ra_n + w - c_n \\ &\leq Ra_n + w \qquad (c_n \geq 0) \\ &= R(B - \delta) + w \\ &= B - R\delta \qquad (R = \frac{w}{1-R}) \end{aligned}$$

$$
\begin{aligned}
a_{n+2} &= Ra_{n+1} + w - c_{n+1} \\
&\leq R(B - R\delta) + w \\
&= B - R^2\delta
\end{aligned}
$$

thus, we can see that $a_{n+t} = B - R^t\delta$, and then

$$\lim_{t\to\infty} \frac{a_{n+t}}{R^{n+t-1}} \leq \lim_{t\to\infty} \left[\frac{B}{R^{n+t-1}} - \frac{\delta}{R^{n-1}} \right] = 0 - \frac{\delta}{R^{n-1}} < 0$$

Summarizing, we have

$$a_t \geq B \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

and

$$a_n < B \text{ for some } n \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} < 0$$

and then,

$$a_t \geq B \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$