Econ 510a (second half)
Yale University
Fall 2006
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HOMEWORK #2

This homework assignment is due at 5PM on Friday, November 10 in Marnix Amand's mailbox.

1. Consider a consumer with the following sequence of budget constraints:

$$c_t + a_{t+1} = R_t a_t + w_t, \quad t = 0, 1, 2, \ldots,$$

where $w_t \geq 0$ is the consumer's labor income in period t. The consumer's consumption cannot be negative. The period-t gross interest rate R_t is greater than 1 for all t.

(a) Find the no-Ponzi-game restriction and use it to derive a consolidated, or intertemporal, budget constraint for the consumer. Interpret your answer.

(b) Suppose that $w_t = w > 0$ and $R_t = R > 1$ for all t. Show that the nPg restriction is equivalent to imposing a constraint that the consumer's asset holdings never fall below a fixed amount B, where B is allowed to be negative. In other words, show that there is a borrowing limit B such that the set of feasible consumption levels defined by the sequential budget constraints and the nPg restriction is identical to the set of feasible consumption levels defined by the sequential budget constraints and the borrowing constraint $a_{t+1} \geq B$ for all t. Express B in terms of R and w. (Hint: Imagine a consumer who has zero consumption and whose asset holdings do not change over time.)

(c) Suppose again that w_t and R_t are time-varying. What would the borrowing constraint have to look like in order to obtain a result like the one in part (b)?

Solution (a) Let's use the budget constraints iteratively:

$$a_1 = Ra_0 + w - c_0$$

$$a_2 = Ra_1 + w - c_1 \quad \Rightarrow \quad a_2 = R(Ra_0 + w - c_0) + w - c_1 = R^2 a_0 + Rw - Rc_0 + w - c_1$$

$$\begin{aligned} a_3 = Ra_2 + w - c_2 \quad \Rightarrow \quad a_3 &= R(R^2 a_0 + Rw - Rc_0 + w - c_1) + w - c_2 \\ &= R^3 a_0 + R^2 w - R^2 c_0 + Rw - Rc_1 + w - c_2 \end{aligned}$$

thus, following this steps we will get to:

$$a_t = R^t a_0 + w(R^{t-1} + R^{t-2} + ... + 1) - (R^{t-1}c_0 + R^{t-2}c_1 + ... + c_{t-1})$$

then,

$$\frac{a_t}{R^{t-1}} = Ra_0 + \sum_{i=0}^{t-1} \frac{w}{R^i} - \sum_{i=0}^{t-1} \frac{c_i}{R^i}$$

Taking limits we get

$$\lim_{t\to\infty} \sum_{i=0}^{t-1} \frac{c_i}{R^i} = Ra_0 + \frac{Rw}{R-1} - \lim_{t\to\infty} \frac{a_t}{R^{t-1}}$$

but the no Ponzi game conditions state that

$$\lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

and then, we have that

$$\sum_{i=0}^{\infty} \frac{c_i}{R^i} \leq Ra_0 + \frac{Rw}{R-1}$$

(b) Consider first an agent with zero consumption and such that $a_t = \bar{a} \quad \forall t$, then the budget constraints imply that:

$$\bar{a} = R\bar{a} + w \quad \Rightarrow \quad \bar{a} = \frac{w}{1-R} \equiv B$$

Let's prove now that

$$a_t \geq B \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

- Note that

$$a_t \geq B \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq \lim_{t\to\infty} \frac{B}{R^{t-1}} = 0 \quad (R > 1)$$

- Suppose $\exists n$ such that $a_n < B$, then $\exists \delta > 0$ such that $a_n = B - \delta$. Let's see what the BC imply:

$$\begin{aligned}
a_{n+1} &= Ra_n + w - c_n \\
&\leq Ra_n + w \qquad (c_n \geq 0) \\
&= R(B - \delta) + w \\
&= B - R\delta \qquad (R = \frac{w}{1-R})
\end{aligned}$$

$$\begin{aligned} a_{n+2} &= Ra_{n+1} + w - c_{n+1} \\ &\leq R(B - R\delta) + w \\ &= B - R^2\delta \end{aligned}$$

thus, we can see that $a_{n+t} = B - R^t\delta$, and then

$$\lim_{t\to\infty} \frac{a_{n+t}}{R^{n+t-1}} \leq \lim_{t\to\infty} \left[\frac{B}{R^{n+t-1}} - \frac{\delta}{R^{n-1}}\right] = 0 - \frac{\delta}{R^{n-1}} < 0$$

Summarizing, we have

$$a_t \geq B \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

and

$$a_n < B \text{ for some } n \quad \Rightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} < 0$$

and then,

$$a_t \geq B \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{a_t}{R^{t-1}} \geq 0$$

2. Consider a consumer with the following optimization problem:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{given } a_0 > 0,$$

subject to the sequence of budget constraints in the first problem and a no-Ponzi-game restriction. The felicity function u is strictly increasing, strictly concave, twice continuously differentiable, and satisfies the Inada condition $\lim_{c\to 0} u'(c) = \infty$.

(a) Find the transversality condition for this problem. Show that the nPg restriction is met if the transversality condition and the Euler equation are both satisfied.

(b) Modify the proof of Proposition 3.4 on p. 23 of the lecture notes by Per Krusell to prove that a sequence $\{a_t^*\}_{t=0}^{\infty}$ that satisfies the transversality condition and the Euler equation maximizes the consumer's objective, subject to the sequence of budget constraints and the nPg restriction. (Note that the proposition in the lecture notes imposes the requirement that the consumer's asset holdings be nonnegative in each period; in this problem, we are imposing instead the nPg restriction.) Before considering the general case in which labor income and the interest vary over time, you might want to study the special case in which they are constant.

Solution (a) We have the following problem:

$$\max_{\{c_t, a_{t+1}\}} \sum_{t\geq 0} \beta^t u(c_t)$$

$$s.t \;\; a_0 \;\text{ given}$$

$$c_t + a_{t+1} = Ra_t + w \quad \forall t \geq 0$$

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^t} \geq 0$$

Then, the transversality condition (TC) for this problem is

$$\lim_{t\to\infty} \beta^t u'(Ra_t^* + w - a_{t+1}^*)Ra_t^* = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \beta^t u'(c_t^*)Ra_t^* = 0$$

Now we have to prove that the TC plus the Euler eq. imply the nPg restriction. The Euler eq. is:

$$\begin{aligned}
& -\beta^{t-1}u'(c_{t-1}) + \beta^t u'(c_t)R = 0 && \forall t \geq 0 \\
& \Leftrightarrow u'(c_{t-1}) = \beta R u'(c_t) && \forall t \geq 0 \\
\text{(1.a)} \qquad & \Rightarrow u'(c_0) = \beta^{t+1}R^{t+1}u'(c_{t+1}) && \forall t \geq 0 \\
& \Rightarrow \frac{a_{t+1}}{R^t} = R\beta^{t+1}a_{t+1}\frac{u'(c_{t+1})}{u'(c_0)} && \forall t \geq 0 \\
& \Rightarrow \lim_{t\to\infty} \frac{a_{t+1}}{R^t} = \frac{1}{u'(c_0)} \lim_{t\to\infty} R\beta^{t+1}a_{t+1}u'(c_{t+1}) = 0 && \text{(because of TC)}
\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^t} = 0, \quad \text{and thus the nPg restriction is met.}$$

(b) We want to prove that a sequence $\{a_t^*\}$ that satisfies the TC and the Euler eq. maximizes the problem stated in part (a). Thus, letting $c_t^* = Ra_t^* + w - a_{t+1}^*$, this sequence is such that:

(1)
$$\lim_{t\to\infty} \beta^t u'(c_t^*)Ra_t^* = 0$$

(2)
$$-\beta^t u'(c_t^*) + \beta^{t+1}Ru'(c_{t+1}^*) = 0$$

Let $\{a_t\}$ be a feasible sequence (ie a_0 is the given one, it satisfies the budget constraints every period, and it met the nPg restriction), defining c_t as $Ra_t + w - a_{t+1}$, we want to prove that

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [u(c_t^*) - u(c_t)] \geq 0$$

Defining $A_T = \sum_{t=0}^{T} \beta^t [u(c_t^*) - u(c_t)]$, we have that:

(by concavity of u)

$$A_T \geq \sum_{t=0}^{T} \beta^t [u'(c_t^*) R(a_t^* - a_t) - u'(c_t^*)(a_{t+1}^* - a_{t+1})]$$

$$= \sum_{t=0}^{T-1} \beta^t (a_{t+1}^* - a_{t+1}) \underbrace{[-u'(c_t^*) + \beta R u'(c_{t+1}^*)]}_{=0 \text{ by (2)}} + R u'(c_0^*) \underbrace{(a_0^* - a_0)}_{=0} - \beta^T u'(c_T^*)(a_{T+1}^* - a_{T+1})$$

$$= -\beta^T u'(c_T^*)(a_{T+1}^* - a_{T+1})$$

(by (2))

$$= \beta^{T+1} R u'(c_{T+1}^*)(-a_{T+1}^* + a_{T+1})$$

Then,

$$\lim_{T\to\infty} A_T \geq \lim_{T\to\infty} \beta^{T+1} R u'(c_{T+1}^*)(-a_{T+1}^* + a_{T+1})$$

(by (1)) $$= \lim_{T\to\infty} \beta^{T+1} R u'(c_{T+1}^*) a_{T+1}$$

$$= \lim_{T\to\infty} \beta^{T+1} R^{T+1} u'(c_{T+1}^*) \frac{a_{T+1}}{R^T}$$

(by (1.a) of part (a)) $$= \lim_{T\to\infty} u'(c_0^*) \frac{a_{T+1}}{R^T}$$

(because $\{a_t\}$ satisfies nPg) $$= 0$$

which is what we wanted to show.

3. Consider an exchange economy with two consumers named A and B. The two consumers have identical preferences: they each value consumption streams according to $\sum_{t=0}^{\infty} \beta^t \log(c_t)$. Consumer i's endowment of consumption goods is $\{\omega_{it}\}_{t=0}^{\infty}$, $i = A, B$. Consumption goods are perishable (i.e., they cannot be stored and used for consumption in future periods).

(a) Carefully define a competitive equilibrium with date-0 trading for this economy.

(b) Suppose that $\omega_{At} = 4$ for all t and $\omega_{Bt} = 1$ for all t. Find the competitive equilibrium allocations and prices.

(c) Suppose now that the endowments fluctuate deterministically: consumer A's endowment stream is $\{4, 1, 4, 1, 4, 1, \ldots\}$ and consumer B's endowment stream is $\{1, 4, 1, 4, 1, 4, \ldots\}$. Find the competitive equilibrium allocations and prices.

(d) In parts (b) and (c) there is no variation in the *aggregate* endowment across time. Suppose that, as in part (b), consumer A's endowment is 4 in every period but that consumer B's endowment fluctuates: his endowment stream is $\{1/2, 2, 1/2, 2, 1/2, 2, \ldots\}$. Find the competitive equilibrium allocations and prices.

(e) The social planning problem for this economy is:

$$\max_{\{c_{At}\}_{t=0}^{\infty}, \{c_{Bt}\}_{t=0}^{\infty}} \left\{ \alpha^A \sum_{t=0}^{\infty} \beta^t \log(c_{At}) + \alpha^B \sum_{t=0}^{\infty} \beta^t \log(c_{Bt}) \right\},$$

subject to the resource constraint $c_{At} + c_{Bt} = \omega_{At} + \omega_{Bt}$ for all t. The numbers α^A and α^B are called *Negishi weights*. For each of the pairs of endowment streams in parts (b), (c), and (d), show that the consumption allocation chosen by the planner coincides with the allocation that arises in competitive equilibrium, provided that the weight α^i is set equal to the inverse of consumer i's marginal utility of income in competitive equilibrium.

(f) Carefully define a competitive equilibrium with sequential trading for this economy. Use your results from parts (b), (c), and (d) to determine the equilibrium interest rates for each pair of endowment streams. In addition, for each case determine how each consumer's asset holdings vary over time (assume that each consumer starts with zero assets in period 0).

Solution

1. A Competitive Equilibrium with date-0 trading for this economy is a vector of prices $\{p_t\}_{t=0}^{\infty}$ and a vector of quantities $\{c_{it}^*\}_{t=0}^{\infty}$ for $i = 1, 2$ such that

   (1) For $i = A, B$

$$\{c_{it}^*\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

$$s.t. \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it}$$

   (2) $c_{At} + c_{Bt} = \omega_{At} + \omega_{Bt}$ for $t = 0, 1, 2...$

2. The FOC for consumer i is

$$\frac{\beta^j u'(c_{i,t+j})}{u'(c_{i,t})} = \frac{p_{t+j}}{p_t} \text{ for } \forall t, j$$

   This together with budget constraint and market clearing condition determines the competitive equilibrium. Here there are two ways to solve for the equilibrium. One way is to solve for the system of simultaneous equations; another way is to make a guess of solution and check the feasibility for each equations. Due to the special structure of the model, here it is easier to proceed with the second way. Now guess that $c_{it} = c_i$ for $\forall t$. Replacing into the FOC and normalizing $p_0 = 1$, we get

$$p_t = \beta^t$$

Now, plugging into the budget constraint for each individual, we have

$$\begin{aligned} \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it} \quad &\Rightarrow \quad \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^t \omega_i \\ &\Rightarrow \quad c_i = \omega_i \end{aligned}$$

It is easy to check that it satisfies the market clearing condition. Therefore, the competitive equilibrium for this economy is

$$c_{At} = \omega_A = 4, \quad c_{Bt} = \omega_B = 1 \quad \text{and} \quad p_t = \beta^t$$

3. Again guess that $c_{it} = c_i$ for $\forall t$. Replacing these into the FOC and normalizing $p_0 = 1$ we have

$$p_t = \beta^t$$

Plugging into the budget constraint for each individual, we have

$$\begin{aligned} \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it} \quad &\Rightarrow \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^t \omega_i \\ &\Rightarrow \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^{2t} \omega_i^e + \beta \sum_{t=0}^{\infty} \beta^{2t} \omega_i^o \\ &\Rightarrow c_i = \frac{1}{1+\beta}\omega_i^e + \frac{\beta}{1+\beta}\omega_i^o \end{aligned}$$

where ω_i^e and ω_i^o means ω_i in even and odd period, respectively.

It is easy to check that it satisfies the market clearing condition. Therefore, the competitive equilibrium for this economy is

$$c_{At} = \frac{4+\beta}{1+\beta}, \quad c_{Bt} = \frac{1+4\beta}{1+\beta} \quad \text{and} \quad p_t = \beta^t$$

4. Now guess the equilibrium as

$$\begin{aligned}
c_{At} &= c_A^o \text{ for } t \text{ odd} \\
c_{At} &= c_A^e \text{ for } t \text{ even} \\
c_{Bt} &= c_{B2}^o \text{ for } t \text{ odd} \\
c_{Bt} &= c_B^e \text{ for } t \text{ even} \\
p_t &= \beta^{t-1} p^o \text{ for } t \text{ odd} \\
p_t &= \beta^t \text{ for } t \text{ even}
\end{aligned}$$

Replacing into the FOC we get

$$\frac{\beta^j u'\left(c_{i,t+j}\right)}{u'\left(c_{i,t}\right)} = \frac{p_{t+j}}{p_t} \Rightarrow \frac{\beta u'\left(c_i^o\right)}{u'\left(c_i^e\right)} = p^o$$

$$\Rightarrow \beta\left(\frac{c_A^e}{c_A^o}\right) = \beta\left(\frac{c_B^e}{c_B^o}\right) = p^o \tag{1}$$

Our guess must satisfy the budget constraint, that is:

$$\sum_{t\geq 0} \beta^{2t} c_i^e + \sum_{t\geq 0} \beta^{2t} p^o c_i^o = \sum_{t\geq 0} \beta^{2t} \omega_i^e + \sum_{t\geq 0} \beta^{2t} p^o \omega_i^o \qquad i = A, B$$

which implies

$$c_A^e + p^o c_A^o = 4(1+p^o) \quad \text{ and } \quad c_B^e + p^o c_B^o = \frac{1}{2} + 2p^o \tag{2}$$

and it also must satisfy the the market clearing conditions:

$$c_A^e + c_B^e = 4 + \frac{1}{2} \quad \text{ and } \quad c_A^o + c_B^o = 4 + 2 \tag{3}$$

The system of the eqs. in (2) and (3) does not have a unique solution, but we also need

$$\frac{c_A^e}{c_A^o} = \frac{c_B^e}{c_B^o} \qquad \text{(because of (1))} \tag{4}$$

and this last equation pins down the unique solution, which is:

$$\begin{aligned}
p_t &= \frac{3}{4}\beta^t & &\text{for } t \text{ odd} \\
p_t &= \beta^t & &\text{for } t \text{ even}
\end{aligned}$$

and

$$c_A^e = \frac{4+3\beta}{1+\beta} \quad \text{ and } \quad c_A^o = \frac{4}{3}\frac{4+3\beta}{1+\beta}$$

and

$$c_B^e = \frac{1}{2}\frac{1+3\beta}{1+\beta} \quad \text{ and } \quad c_B^o = \frac{2}{3}\frac{1+3\beta}{1+\beta}$$

5. This is just a matter of checking. The algebra is straightforward.

   The social planning problem for this economy is the following:

$$\max_{\{c_{1t}\}_{t\geq 0},\{c_{2t}\}_{t\geq 0}} \alpha_A \sum_{t=0}^{\infty} \beta^t u(c_{1t}) + \alpha_B \sum_{t=0}^{\infty} \beta^t u(c_{2t})$$

$$s.t.\ \ c_{1t} + c_{2t} = \omega_{1t} + \omega_{2t} \quad \forall t \geq 0$$

   Now, from the FOC of the consumers' problem (date-0 trading), we have that their marginal utilities of wealth λ_i (ie, the lagrange multipliers) for $i = 1, 2$ are

$$\lambda_i = \frac{\beta^t u'(c_{it})}{p_t} \quad \Rightarrow \quad \lambda_i = u'(c_{i0})/p_0 = c_{i0}^{-1}/p_0$$

   and we know that the Pareto weights that deliver the competitive equilibrium allocation are:

$$\alpha_i = \frac{1/\lambda_i}{1/\lambda_A + 1/\lambda_B}$$

   Then, for each pair of endowments these weights are:
   for part (b),

$$\lambda_A = 4 \quad \text{and} \quad \lambda_B = 1 \quad \Rightarrow \quad \alpha_A = \frac{1}{5} \quad \text{and} \quad \alpha_B = \frac{4}{5}$$

   for part (c),

$$\lambda_A = \frac{4+\beta}{1+\beta} \quad \text{and} \quad \lambda_B = \frac{1+4\beta}{1+\beta} \quad \Rightarrow \quad \alpha_A = \frac{1+4\beta}{5+5\beta} \quad \text{and} \quad \alpha_B = \frac{4+\beta}{5+5\beta}$$

   and for part (d),

$$\lambda_A = \frac{4+3\beta}{1+\beta} \quad \text{and} \quad \lambda_B = \frac{1}{2}\frac{1+3\beta}{1+\beta} \quad \Rightarrow \quad \alpha_A = \frac{1+3\beta}{9+9\beta} \quad \text{and} \quad \alpha_B = 2\frac{4+3\beta}{9+9\beta}$$

6. A Competitive Equilibrium with sequential trading for the economy is a sequence $\{c_{it}^*\}_{t=0}^{\infty}, \{a_{i,t+1}^*\}_{t=0}^{\infty}, \{R_t^*\}_{t=0}^{\infty}$ (where R_t^* means interest rate from t to $t+1$) for $i = 1, 2$ such that

   (1) For $i = A, B$

$$\left\{c_{it}^*, a_{i,t+1}^*\right\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right)$$

$$s.t.\ \ c_{it} + a_{i,t+1} = R_t^* a_{i,t} + \omega_{it}$$

$$\lim_{t\to\infty} a_{i,t+1} \left(\prod_{j=0}^{t+1} R_j^{-1} \right) \geq 0$$

$$a_{i,0} = 0, c_{it} \geq 0$$

(2) $c^*_{At} + c^*_{Bt} = \omega_{At} + \omega_{Bt}$ for $t = 0, 1, 2...$

(3) $a^*_{At} + a^*_{Bt} = 0$ for $t = 0, 1, 2...$

Now we start to solve for equilibrium interest rate and asset holdings for different examples. In each example, it is easy to see that

$$\begin{aligned} R_t &= \frac{p_{t-1}}{p_t} \\ a_{i,t+1} &= R_t a_{i,t} + \omega_{it} - c_{it} \end{aligned}$$

We start with part (b). Plugging in the solution, we have

$$R_t = \frac{1}{\beta} \quad \text{and} \quad a_{i,t} = 0 \quad \forall t \geq 0$$

In part (c), we have

$$\begin{aligned} R_t &= \frac{1}{\beta} \\ a_{A1} &= -a_{B1} = \frac{3\beta}{1+\beta} \end{aligned}$$

In part (d), we have

$$\begin{aligned} R^o &= \frac{4}{3\beta} \\ R^e &= \frac{3}{4\beta} \\ a_{i,0} &= 0 \\ a_{A,1} &= \omega_{A0} - c_{A0} = 4 - \frac{4+3\beta}{1+\beta} \\ a_{B,1} &= -a_{A,1} \\ &\dots \end{aligned}$$