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Problem Set 2	: Suggested Solutions	

1. .

(a) In this case, the capital accumulation equation becomes:

$$K_{C,(t+1)} + K_{I,(t+1)} = (1-\delta)K_{C,t} + (1-\delta)K_{I,t} + G(K_{I,t},(L-L_{C,t}))$$

So the dynamic programming problem is :

$$\begin{aligned} V(K_{C,t},K_{I,t}) &= \max_{\{C_t,L_t,K_{C,(t+1)},K_{I,(t+1)}\}_{t=0}^{\infty}} \left\{u(C_t) + \beta V(K_{C,(t+1)},K_{I,(t+1)})\right\} \\ & \quad s.t. \\ C_t &= F(K_{C,t},L_{C,t}) \\ K_{C,(t+1)} + K_{I,(t+1)} &= (1-\delta)K_{C,t} + (1-\delta)K_{I,t} + G(K_{I,t},(L-L_{C,t})) \end{aligned}$$

Substituting in the constraints:

$$V(K_{C,t},K_{I,t}) = \max_{\{L_t,K_{C,(t+1)}\}_{t=0}^{\infty}} \left\{ \begin{array}{l} u(F(K_{C,t},L_{C,t}))+ \\ \beta V(K_{C,(t+1)},(1-\delta)K_{C,t} + (1-\delta)K_{I,t} + G(K_{I,t},(L-L_{C,t})) - K_{C,(t+1)}) \end{array} \right\}$$

Our state variables are: $K_{C,t}$, $K_{I,t}$

Our choice (control) variables are: $L_{C,t}$, $I_{C,t}$

(b) First order conditions

With respect to $L_{C,t}$:

$$u'(C_t)\frac{F_2(K_{C,t},L_{C,t})}{G_2(K_{I,t},(L-L_C))} = \beta V_2(K_{C,(t+1)},K_{I,(t+1)})$$

With respect to $K_{C,(t+1)}$:

$$V_1(K_{C,(t+1)},K_{I,(t+1)}) = V_2(K_{C,(t+1)},K_{I,(t+1)})$$

And hence:

$$V_1(K_{C,(t+1)},K_{I,(t+1)}) = \frac{u'(C_t)}{\beta}\frac{F_2(K_{C,t},L_{C,t})}{G_2(K_{I,t},(L-L_{C,t}))}.$$

Envelope Theorem:

$$\begin{aligned} V_1(K_{C,t},K_{I,t}) &= u'(C_t)F_1(K_{C,t},L_{C,t}) + \beta V_1(K_{C,(t+1)},K_{I,(t+1)})(1-\delta) \\ V_2(K_{C,t},K_{I,t}) &= \beta V_2(K_{C,(t+1)},K_{I,(t+1)})(1-\delta+G_1(K_{I,t},(L-L_{C,t},))) \end{aligned}$$

Update one period:

$$\begin{aligned} V_1(K_{C,(t+1)},K_{I,(t+1)}) &= u'(C_{t+1})F_1(K_{C,(t+1)},L_{C,(t+1)}) + \beta V_2(K_{C,(t+2)},K_{I,(t+2)})(1-\delta) \\ V_2(K_{C,(t+1)},K_{I,(t+1)}) &= \beta V_2(K_{C,(t+2)},K_{I,(t+2)})(1-\delta+G_1(K_{I,(t+1)},(L-L_{C,(t+1)}))) \end{aligned}$$

(c) The set of equations to determine the steady state values of capital and labor in each sector (i.e., K_C, L_C, K_I, L_I):
The Euler equations are:

$$\begin{aligned}
1 &= \beta\frac{u'(C_{t+1})}{u'(C_t)}\left[F_1(K_{C,(t+1)},L_{C,(t+1)})\frac{G_2(K_{I,t},(L-L_{C,t}))}{F_2(K_{C,t},L_{C,t})}+(1-\delta)\frac{\frac{F_2(K_{C,(t+1)},L_{C,(t+1)})}{G_2(K_{I,(t+1)},(L-L_{C,(t+1)}))}}{\frac{F_2(K_{C,t},L_{C,t})}{G_2(K_{I,t},(L-L_{C,t}))}}\right]\\
1 &= \beta\frac{u'(C_{t+1})}{u'(C_t)}\frac{\frac{F_2(K_{C,(t+1)},L_{C,(t+1)})}{G_2(K_{I,(t+1)},(L-L_{C,(t+1)}))}}{\frac{F_2(K_{C,t},L_{C,t})}{G_2(K_{I,t},(L-L_{C,t}))}}\left[(1-\delta)+G_1(K_{I,(t+1)},(L-L_{C,(t+1)}))\right]
\end{aligned}$$

At the steady state:

$$\begin{aligned}
1 &= \beta\frac{u'(C^*)}{u'(C^*)}\left[F_1(K_C^*,L_C^*)\frac{G_2(K_I^*,(L-L_C^*))}{F_2(K_C^*,L_C^*)}+(1-\delta)\frac{\frac{F_2(K_C^*,L_C^*)}{G_2(K_I^*,(L-L_C^*))}}{\frac{F_2(K_C^*,L_C^*)}{G_2(K_I^*,(L-L_C^*))}}\right]\\
&= \beta\left[F_1(K_C^*,L_C^*)\frac{G_2(K_I^*,(L-L_C^*))}{F_2(K_C^*,L_C^*)}+(1-\delta)\right]\\
\frac{1}{\beta} &= F_1(K_C^*,L_C^*)\frac{G_2(K_I^*,(L-L_C^*))}{F_2(K_C^*,L_C^*)}+(1-\delta)
\end{aligned}$$

And

$$\begin{aligned}
1 &= \beta\frac{u'(C^*)}{u'(C^*)}\frac{\frac{F_2(K_C^*,L_C^*)}{G_2(K_I^*,(L-L_C^*))}}{\frac{F_2(K_C^*,L_C^*)}{G_2(K_I^*,(L-L_C^*))}}\left[(1-\delta)+G_1(K_I^*,(L-L_C^*))\right]\\
&= \beta\left[(1-\delta)+G_1(K_I^*,(L-L_C^*))\right]\\
\frac{1}{\beta} &= (1-\delta)+G_1(K_I^*,(L-L_C^*))
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
F_1(K_C^*,L_C^*)\frac{G_2(K_I^*,(L-L_C^*))}{F_2(K_C^*,L_C^*)}+(1-\delta) &= (1-\delta)+G_1(K_I^*,(L-L_C^*))\\
\frac{F_1(K_C^*,L_C^*)}{G_1(K_I^*,(L-L_C^*))} &= \frac{F_2(K_C^*,L_C^*)}{G_2(K_I^*,(L-L_C^*))}
\end{aligned}$$

Additionally we have:

$$\begin{aligned}
V_2(K_C^*,K_I^*) &= \beta V_2(K_C^*,K_I^*)(1-\delta+G_1(K_I^*,(L-L_C^*,)))\\
G_1(K_I^*,(L-L_C^*,)) &= \beta^{-1}-(1-\delta).
\end{aligned}$$

Finall we have the law of motion for capital:

$$\begin{aligned}
K_C^*+K_I^* &= (1-\delta)K_C^*+(1-\delta)K_I^*+G(K_I^*,(L-L_C^*))\\
\delta(K_C^*+K_I^*) &= G(K_I^*,(L-L_C^*)).
\end{aligned}$$

(d) We have $F(K_{C,t}, L_{C,t}) = K_{C,t}^{\alpha} L_{C,t}^{(1-\alpha)}$and $G(K_{I,t}, L_{I,t}) = K_{I,t}^{\gamma} L_{I,t}^{(1-\gamma)}$.
And so:

$$\begin{aligned}
F_1(K_C^*, L_C^*) &= \alpha \left(\frac{K_C^*}{L_C^*} \right)^{(\alpha-1)} \\
F_2(K_C^*, L_C^*) &= (1-\alpha) \left(\frac{K_C^*}{L_C^*} \right)^{\alpha} \\
G_1(K_I^*, (L - L_C^*)) &= \gamma \left(\frac{K_I^*}{(L - L_C^*)} \right)^{(\gamma-1)} \\
G_2(K_I^*, (L - L_C^*)) &= (1-\gamma) \left(\frac{K_I^*}{(L - L_C^*)} \right)^{\gamma}
\end{aligned}$$

From

$$\begin{aligned}
G_1(K_I^*, (L - L_C^*,)) &= \beta^{-1} - (1-\delta) \\
\gamma \left(\frac{K_I^*}{(L - L_C^*)} \right)^{(\gamma-1)} &= \beta^{-1} - (1-\delta) \\
\frac{K_I^*}{(L - L_C^*)} &= \left(\frac{\left(\beta^{-1} - (1-\delta)\right)}{\gamma} \right)^{\frac{1}{(\gamma-1)}}.
\end{aligned}$$

Plugging these back into

$$\begin{aligned}
\frac{G_2(K_I^*, (L - L_C^*))}{G_1(K_I^*, (L - L_C^*))} &= \frac{F_2(K_C^*, L_C^*)}{F_1(K_C^*, L_C^*)} \\
\frac{(1-\gamma) \left(\frac{K_I^*}{(L - L_C^*)} \right)^{\gamma}}{\gamma \left(\frac{K_I^*}{(L - L_C^*)} \right)^{(\gamma-1)}} &= \frac{(1-\alpha) \left(\frac{K_C^*}{L_C^*} \right)^{\alpha}}{\alpha \left(\frac{K_C^*}{L_C^*} \right)^{(\alpha-1)}} \\
\frac{K_I^*}{(L - L_C^*)} &= \frac{\gamma}{(1-\gamma)} \frac{(1-\alpha)}{\alpha} \left(\frac{K_C^*}{L_C^*} \right) \\
\frac{K_C^*}{L_C^*} &= \frac{(1-\gamma)}{\gamma} \frac{\alpha}{(1-\alpha)} \left(\frac{\left(\beta^{-1} - (1-\delta)\right)}{\gamma} \right)^{\frac{1}{(\gamma-1)}}.
\end{aligned}$$

Finally, plugging these results into the budget constraint

$$\begin{aligned}
\delta(K_C^* + K_I^*) &= G(K_I^*, (L - L_C^*)) \\
\delta(K_C^* + K_I^*) &= (K_I^*)^{\gamma} (L - L_C^*)^{(1-\gamma)} \\
\delta\left(\frac{(K_C^* + K_I^*)}{(L - L_C^*)}\right) &= \left(\frac{K_I^*}{(L - L_C^*)}\right)^{\gamma} \\
\frac{K_C^*}{L_C^*}\frac{L_C^*}{(L - L_C^*)} &= \frac{1}{\delta}\left(\frac{K_I^*}{(L - L_C^*)}\right)^{\gamma} - \frac{K_I^*}{(L - L_C^*)} \\
\frac{L_C^*}{(L - L_C^*)} &= \frac{\frac{1}{\delta}\left(\frac{K_I^*}{(L-L_C^*)}\right)^{\gamma} - \frac{K_I^*}{(L-L_C^*)}}{\frac{K_C^*}{L_C^*}} \\
&= \frac{\frac{1}{\delta}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{\gamma}{(\gamma-1)}} - \left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}}}{\frac{(1-\gamma)}{\gamma}\frac{\alpha}{(1-\alpha)}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}}} \\
&= \frac{\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}}\left[\frac{1}{\delta}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right) - 1\right]}{\frac{(1-\gamma)}{\gamma}\frac{\alpha}{(1-\alpha)}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}}} \\
&= \frac{\gamma}{(1-\gamma)}\frac{(1-\alpha)}{\alpha}\left[\frac{1}{\delta}\left(\frac{\left(\beta^{-1} - (1-\delta)\right)}{\gamma}\right) - 1\right]
\end{aligned}$$

So

$$L_C^* = \frac{\frac{\gamma}{(1-\gamma)}\frac{(1-\alpha)}{\alpha}\left[\frac{1}{\delta}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right) - 1\right]}{\left(\frac{\gamma}{(1-\gamma)}\frac{(1-\alpha)}{\alpha}\left[\frac{1}{\delta}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right) - 1\right]\right) + 1} L.$$

If we let

$$A = \frac{\gamma}{(1-\gamma)}\frac{(1-\alpha)}{\alpha}\left[\frac{1}{\delta}\left(\frac{\left(\beta^{-1} - (1-\delta)\right)}{\gamma}\right) - 1\right]$$

We have

$$\begin{aligned}
L_C^* &= \frac{A}{(A+1)}L \\
K_C^* &= \frac{(1-\gamma)}{\gamma}\frac{\alpha}{(1-\alpha)}\left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}} L_C^* \\
L_I^* &= L - L_C^* \\
&= \frac{1}{(A+1)}L \\
K_I^* &= \left(\frac{\left(\beta^{-1}-(1-\delta)\right)}{\gamma}\right)^{\frac{1}{(\gamma-1)}} L_I^*.
\end{aligned}$$

2. (a) We start with the first period's budget constraint:

$$\begin{aligned}
c_0 + qb_1 &= b_0 + \omega \\
b_1 &= q^{-1}\left(b_0 + \omega - c_0\right).
\end{aligned}$$

For $t = 1$ we have:

$$\begin{aligned}
c_1 + qb_2 &= b_1 + \omega \\
b_2 &= q^{-1}\left(b_1 + \omega - c_1\right) \\
&= q^{-1}\left(q^{-1}\left(b_0 + \omega - c_0\right) + \omega - c_1\right) \\
&= q^{-2}b_0 + q^{-2}\omega + q^{-1}\omega - q^{-2}c_0 - q^{-1}c_1.
\end{aligned}$$

For $t = 2$ we have:

$$\begin{aligned}
c_2 + qb_3 &= b_2 + \omega \\
b_3 &= q^{-1}\left(b_2 + \omega - c_2\right) \\
&= q^{-1}\left(q^{-2}b_0 + q^{-2}\omega + q^{-1}\omega - q^{-2}c_0 - q^{-1}c_1 + \omega - c_2\right) \\
&= q^{-3}b_0 + q^{-3}\omega + q^{-2}\omega + q^{-1}\omega - q^{-3}c_0 - q^{-2}c_1 - q^{-1}c_2.
\end{aligned}$$

And so on, until we have:

$$\begin{aligned}
b_{(t+1)} &= q^{-(t+1)}b_0 + \omega(q^{-(t+1)} + ... + q^{-1}) - \left(q^{-(t+1)}c_0 + q^{-t}c_1 + ... + q^{-1}c_t\right) \\
q^{(t+1)}b_{(t+1)} &= b_0 + \omega(1 + ... + q^t) - \left(c_0 + qc_1 + ... + q^t c_t\right) \\
&= b_0 + \omega\sum_{k=0}^{t} q^k - \sum_{k=0}^{t} q^k c_k.
\end{aligned}$$

Taking the limit gives us:

$$\begin{aligned}
\lim_{t\to\infty} q^{(t+1)}b_{(t+1)} &= b_0 + \omega\lim_{t\to\infty}\sum_{k=0}^{t} q^k - \lim_{t\to\infty}\sum_{k=0}^{t} q^k c_k \\
0 &= b_0 + \frac{\omega}{1-q} - \sum_{t=0}^{\infty} q^t c_t
\end{aligned}$$

where we used the no-Ponzi game condition on the LHS.
So the consumer's consolidated (or lifetime) budget constraint is

$$\sum_{t=0}^{\infty} q^t c_t = b_0 + \frac{\omega}{1-q}.$$

(b) The transversality condition is:

$$\lim_{t\to\infty} \beta^t u'(c_t^*) b_t^* = 0$$

or,

$$\lim_{t\to\infty} \beta^t u'(b_t - qb_{(t+1)} + \omega) b_t = 0.$$

The Euler equation is

$$qu'(b_t - qb_{(t+1)} + \omega) = \beta u'(b_{(t+1)} - qb_{(t+2)} + \omega).$$

We want to show

$$\lim_{t\to\infty} q^t b_t = 0.$$

Note that the Euler equation holds $\forall t$:

$$\begin{aligned} qu'(b_0 - qb_1 + \omega) &= \beta u'(b_1 - qb_2 + \omega) \\ qu'(b_1 - qb_2 + \omega) &= \beta u'(b_2 - qb_3 + \omega) \end{aligned}$$

and

$$q^2 u'(b_0 - qb_1 + \omega) = \beta^2 u'(b_2 - qb_3 + \omega).$$

So we have

$$q^t u'(b_0 - qb_1 + \omega) = \beta^t u'(b_t - qb_{(t+1)} + \omega).$$

Multiplying both sides by b_t and taking limits:

$$\begin{aligned} \lim_{t\to\infty} q^t u'(b_0 - qb_1 + \omega) b_t &= \lim_{t\to\infty} \beta^t u'(b_t - qb_{(t+1)} + \omega) b_t \\ u'(b_0 - qb_1 + \omega) \lim_{t\to\infty} q^t b_t &= 0 \qquad \text{(using the transversality condition).} \end{aligned}$$

And so

$$\lim_{t\to\infty} q^t b_t = 0.$$

(c) We want to prove that a sequence $\{b_t^*\}_{t=0}^{\infty} = 0$ that satisfies the transversality condition and the Euler equation maximizes the consumer's objective function, subject to the sequence of budget constraints and the nPg condition.
Modified Proof:
Consider any alternative feasible sequence $b \equiv \{b_t\}_{t=0}^{\infty}$. We want to show that for any such sequence,

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[u(b_t^* - qb_{(t+1)}^* + \omega) - u(b_t - qb_{(t+1)} + \omega) \right] \geq 0.$$

Define

$$A_T(b) = \sum_{t=0}^{T} \beta^t \left[u(b_t^* - qb_{(t+1)}^* + \omega) - u(b_t - qb_{(t+1)} + \omega) \right] \geq 0.$$

We will show that, as T goes to infinity $A_T(b)$ is bounded below by zero.
By concavity of u ,

$$
\begin{aligned}
A_T(b) &\geq \sum_{t=0}^{T}\beta^t\left[u'(b_t^*-qb_{(t+1)}^*+\omega)(b_t^*-b_t)-qu'(b_t^*-qb_{(t+1)}^*+\omega)(b_{(t+1)}^*-b_{(t+1)})\right]\\
&= \left[\sum_{t=0}^{T}\beta^t u'(b_t^*-qb_{(t+1)}^*+\omega)(b_t^*-b_t)\right]-\left[\sum_{t=0}^{T}\beta^t qu'(b_t^*-qb_{(t+1)}^*+\omega)(b_{(t+1)}^*-b_{(t+1)})\right]\\
&= \left[u'(b_0^*-qb_1^*+\omega)(b_0^*-b_0)+\sum_{t=1}^{T}\beta^t u'(b_t^*-qb_{(t+1)}^*+\omega)(b_t^*-b_t)\right]-\\
&\quad \left[\sum_{t=0}^{T-1}\beta^t qu'(b_t^*-qb_{(t+1)}^*+\omega)(b_{(t+1)}^*-b_{(t+1)})-\beta^T qu'(b_T^*-qb_{(T+1)}^*+\omega)(b_{(T+1)}^*-b_{(T+1)})\right]\\
&= \left[u'(b_0^*-qb_1^*+\omega)(b_0^*-b_0)+\sum_{t=0}^{T-1}\beta^{t+1} u'(b_{(t+1}^*-qb_{(t+2)}^*+\omega)(b_{(t+1)}^*-b_{(t+1)})\right]-\\
&\quad \left[\sum_{t=0}^{T-1}\beta^t qu'(b_t^*-qb_{(t+1)}^*+\omega)(b_{(t+1)}^*-b_{(t+1)})-\beta^T qu'(b_T^*-qb_{(T+1)}^*+\omega)(b_{(T+1)}^*-b_{(T+1)})\right]\\
&= u'(b_0^*-qb_1^*+\omega)(b_0^*-b_0)-\beta^T qu'(b_T^*-qb_{(T+1)}^*+\omega)(b_{(T+1)}^*-b_{(T+1)})+\\
&\quad \sum_{t=0}^{T-1}\beta^t(b_{(t+1)}^*-b_{(t+1)})\left[\beta u'(b_{(t+1}^*-qb_{(t+2)}^*+\omega)-qu'(b_t^*-qb_{(t+1)}^*+\omega)\right].
\end{aligned}
$$

Note that:

$$
\begin{aligned}
(b_0^*-b_0) &= 0 \Rightarrow u'(b_0^*-qb_1^*+\omega)(b_0^*-b_0)=0\\
\beta u'(b_{(t+1}^*-qb_{(t+2)}^*+\omega)-qu'(b_t^*-qb_{(t+1)}^*+\omega) &= 0 \qquad \text{(Euler Equation)}.
\end{aligned}
$$

And so we have

$$
\begin{aligned}
A_T(b) &\geq -\beta^T qu'(b_T^*-qb_{(T+1)}^*+\omega)(b_{(T+1)}^*-b_{(T+1)})\\
&= \beta^T qu'(b_T^*-qb_{(T+1)}^*+\omega)(-b_{(T+1)}^*+b_{(T+1)})\\
&= \beta^T \beta u'(b_{(T+1)}-qb_{(T+2)}+\omega)(-b_{(T+1)}^*+b_{(T+1)}).
\end{aligned}
$$

Taking limits we have

$$\lim_{T\to\infty} A_T(b) \geq \lim_{T\to\infty}\beta^{T+1}u'(b_{(T+1)}^*-qb_{(T+2)}^*+\omega)(-b_{(T+1)}^*+b_{(T+1)})$$

Recalling that

$$
\begin{aligned}
q^t u'(b_0^*-qb_1^*+\omega) &= \beta^t u'(b_t^*-qb_{(t+1)}^*+\omega)\\
u'(b_0^*-qb_1^*+\omega) &= \frac{\beta^t}{q^t}u'(b_t^*-qb_{(t+1)}^*+\omega),
\end{aligned}
$$

$$\lim_{t\to\infty} q^t b_t = 0.$$

and

$$\lim_{t\to\infty}\beta^t u'(c_t^*)b_t^* = 0$$

We have

$$
\begin{aligned}
\lim_{T\to\infty} A_T(b) &\geq \lim_{T\to\infty} \beta^{T+1} u'(b^*_{(T+1)} - qb^*_{(T+2)} + \omega)(b_{(T+1)} - b^*_{(T+1)}) \\
&= \lim_{T\to\infty} \frac{\beta^{T+1}}{q^{T+1}} q^{T+1} u'(b^*_{(T+1)} - qb^*_{(T+2)} + \omega) b_{(T+1)} - \lim_{T\to\infty} \beta^{T+1} u'(b^*_{(T+1)} - qb^*_{(T+2)} + \omega) b^*_{(T+1)} \\
&= \lim_{T\to\infty} u'(b^*_0 - qb^*_1 + \omega) q^{T+1} b_{(T+1)} \\
&= 0
\end{aligned}
$$

3. (a) Setting up the Lagrangian for consumer i:

$$
\max_{\{c^*_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i\left(c^i_t\right) + \lambda^i \left(\sum_{t=0}^{\infty} p_t \omega^i_t - \sum_{t=0}^{\infty} p_t c^i_t \right)
$$

The F.O.C. for consumer i with respect to c^i_t :

$$
\beta^t u'\left(c^i_t\right) = \lambda^i p_t
$$

The F.O.C. for consumer i with respect to c^i_{t+1} :

$$
\beta^{t+1} u'\left(c^i_{t+1}\right) = \lambda^i p_{t+1}
$$

Which gives us:

$$
\begin{aligned}
\frac{\beta u'\left(c^i_{t+1}\right)}{u'\left(c^i_t\right)} &= \frac{p_{t+1}}{p_t} \\
\frac{\beta c_{t+1}^{-\sigma_i}}{c_t^{-\sigma_i}} &= \frac{p_{t+1}}{p_t}
\end{aligned}
$$

Or

$$
\frac{\beta^k u'\left(c^i_{t+k}\right)}{u'\left(c^i_t\right)} = \frac{p_{t+k}}{p_t}
$$

Together with the budget constraints

$$
\sum_{t=0}^{\infty} p_t \omega^i_t = \sum_{t=0}^{\infty} p_t c^i_t \qquad i = A, B,
$$

and the market clearing condition

$$
c^A_t + c^B_t = \omega^A_t + \omega^B_t \qquad t = 0, 1, 2, ...,
$$

we can use two alternative approaches to solve for the competitive equilibrium. One option is to solve for the system of simultaneous equations, which can be difficult. A second option is to guess the equilibrium allocation and then check whether our conditions hold.
So let's guess that the individuals consume their endowment in each period. That is, guess that $c^i_t = \omega^i_t$, $\forall t$.

[Note that we could have guessed $c_t^i = c_{t+1}^i$. However, since the consumers are each receiving an unchanging constant stream of endowments, there will be no incentive to trade and so we expect each consumer will consume their own endowment.]
So now we have

$$\begin{aligned}
\frac{\beta\left(\omega_{t+1}^i\right)^{-\sigma_i}}{\left(\omega_t^i\right)^{-\sigma_i}} &= \frac{\beta 2^{-\sigma_A}}{2^{-\sigma_A}} = \frac{\beta 1^{-\sigma_B}}{1^{-\sigma_B}} = \frac{p_{t+1}}{p_t} \\
\beta &= \frac{p_{t+1}}{p_t}
\end{aligned}$$

Next, to determine relative prices, we normalize the price of period-0 consumption good to 1. That is, choose $p_0 = 1$.
Then,

$$\begin{aligned}
\frac{p_1}{p_0} &= p_1 = \beta \\
\frac{p_2}{p_1} &= \frac{p_2}{\beta} = \beta \\
p_2 &= \beta^2
\end{aligned}$$

Similarly,

$$p_t = \beta^t.$$

Verify the market clearing condition:

$$\begin{aligned}
c_t^A + c_t^B &= \omega_t^A + \omega_t^B \qquad t = 0, 1, 2, ... \\
2 + 1 &= 2 + 1
\end{aligned}$$

Verifying budget constraints:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t \omega_t^i &= \sum_{t=0}^{\infty} p_t c_t^i \\
\sum_{t=0}^{\infty} \beta^t 2 &= \sum_{t=0}^{\infty} \beta^t 2 \qquad \text{for A,} \\
\sum_{t=0}^{\infty} \beta^t 1 &= \sum_{t=0}^{\infty} \beta^t 1 \qquad \text{for B.}
\end{aligned}$$

So, the competitive equilibrium is:

$$\begin{aligned}
c_t^A &= \omega_t^A = 2 \\
c_t^B &= \omega_t^B = 1 \\
p_t &= \beta^t.
\end{aligned}$$

To find the marginal utility of wealth λ^i, for each consumer we can use the FOC:

$$\begin{aligned}
\beta^t u'\left(c_t^A\right) &= \lambda^A p_t \\
\lambda^A &= \frac{\beta^t \left(c_t^A\right)^{-\sigma_A}}{\beta^t} \\
\lambda^A &= 2^{-\sigma_A}.
\end{aligned}$$

Similarly,

$$\lambda^B = 1^{-\sigma_B}.$$

(b) In this case, since the consumers receive differing endowments in different periods we suspect they may trade and so we guess that the competitive equilibrium allocation will be $c_t^i = c_{t+1}^i = c^i$. As we did above we can plug in our guess for consumption in the FOC , normalize $p_0 = 1$ and get

$$p_t = \beta^t.$$

Plugging in our guess and the prices into the budget constraint for each individual, we have

$$\begin{aligned}
&\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t \omega_t^i \\
\Rightarrow\quad &\sum_{t=0}^{\infty} \beta^t c^i = \sum_{t=0}^{\infty} \beta^t \omega^i \\
\Rightarrow\quad &\sum_{t=0}^{\infty} \beta^t c^i = \sum_{t=0}^{\infty} \beta^{2t} \omega_e^i + \beta \sum_{t=0}^{\infty} \beta^{2t} \omega_o^i \\
\Rightarrow\quad &c^i = \frac{1}{1+\beta}\omega_e^i + \frac{\beta}{1+\beta}\omega_o^i
\end{aligned}$$

where ω_i^e and ω_i^o means ω^i in even and odd period, respectively.

It is straightforward to check that this allocation satisfies the market clearing condition. Therefore, the competitive equilibrium for this economy is

$$\begin{aligned}
c_t^A &= \frac{2+\beta}{1+\beta} \\
c_t^B &= \frac{1+2\beta}{1+\beta} \\
p_t &= \beta^t.
\end{aligned}$$

To find the marginal utility of wealth λ^i, for each consumer we can use the FOC:

$$\begin{aligned}
\beta^t u'\left(c_t^A\right) &= \lambda^A p_t \\
\lambda^A &= \frac{\beta^t \left(\frac{2+\beta}{1+\beta}\right)^{-\sigma_A}}{\beta^t} \\
\lambda^A &= \left(\frac{2+\beta}{1+\beta}\right)^{-\sigma_A}.
\end{aligned}$$

Similarly,

$$\lambda^B = \left(\frac{1+2\beta}{1+\beta}\right)^{-\sigma_B}.$$

(c) We want to show that the competitive equilibrium allocation is of the form

$$c_t^i = \theta^i \bar{\omega}_t \qquad i = A, B.$$

The FOC:

$$\frac{\beta^{t+k} u'\left(c_{t+k}^i\right)}{\beta^t u'\left(c_t^i\right)} = \frac{p_{t+k}}{p_t}$$

Or

$$\frac{\beta^t u'\left(c_t^i\right)}{u'\left(c_0^i\right)} = \frac{p_t}{p_0}$$

So we have

$$\frac{\beta^t \left(\theta^i \bar{\omega}_t\right)^{-\sigma}}{\left(\theta^i \bar{\omega}_0\right)^{-\sigma}} = \frac{p_t}{p_0}$$

If we once again normalize $p_0 = 1$, then

$$p_t = \beta^t \left(\frac{\bar{\omega}_t}{\bar{\omega}_0}\right)^{-\sigma}.$$

Plugging back into the budget constraint, we have:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t c_t^i &= \sum_{t=0}^{\infty} p_t \omega_t^i \\
\sum_{t=0}^{\infty} \left[\beta^t \left(\frac{\bar{\omega}_t}{\bar{\omega}_0}\right)^{-\sigma} \left(\theta^i \bar{\omega}_t\right)\right] &= \sum_{t=0}^{\infty} \left[\beta^t \left(\frac{\bar{\omega}_t}{\bar{\omega}_0}\right)^{-\sigma} \omega_t^i\right] \\
\frac{\theta^i}{\left(\bar{\omega}_0\right)^{-\sigma}} \sum_{t=0}^{\infty} \left[\beta^t \left(\bar{\omega}_t\right)^{1-\sigma}\right] &= \frac{1}{\left(\bar{\omega}_0\right)^{-\sigma}} \sum_{t=0}^{\infty} \left[\beta^t \left(\bar{\omega}_t\right)^{-\sigma} \omega_t^i\right] \\
\theta^i &= \frac{\sum_{t=0}^{\infty} \left[\beta^t \left(\bar{\omega}_t\right)^{-\sigma} \omega_t^i\right]}{\sum_{t=0}^{\infty} \left[\beta^t \left(\bar{\omega}_t\right)^{1-\sigma}\right]}.
\end{aligned}$$

It is straightforward to check the market clearing conditions.

(d) In this case $\omega_t^A = 2$, $\forall t$, and type-B consumers' endowment stream is $\{1, 2, 1, 2, 1, 2, ...\}$. From the previous section we know that:

$$c_t^i = \theta^i \bar{\omega}_t \qquad i = A, B$$

where

$$\begin{aligned}
\theta^i &= \frac{\sum\limits_{t=0}^{\infty}\left[\beta^t\left(\bar{\omega}_t\right)^{-\sigma}\omega_t^i\right]}{\sum\limits_{t=0}^{\infty}\left[\beta^t\left(\bar{\omega}_t\right)^{1-\sigma}\right]} \\
&= \frac{\sum\limits_{t=0}^{\infty}\beta^t\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^i\right]}{\sum\limits_{t=0}^{\infty}\beta^t} \\
&= \frac{\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^i\right]_{even}+\beta\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^i\right]_{odd}}{\sum\limits_{t=0}^{\infty}\beta^t}.
\end{aligned}$$

Note that

$$\begin{aligned}
\left(\bar{\omega}_t\right)_{even} &= 3, \qquad \left(\bar{\omega}_t\right)_{odd}=4 \\
\left[\left(\bar{\omega}_t\right)^{-1}\omega^A\right]_{even} &= \frac{2}{3}, \qquad \left[\left(\bar{\omega}_t\right)^{-1}\omega_t^A\right]_{odd}=\frac{2}{4}=\frac{1}{2} \\
\left[\left(\bar{\omega}_t\right)^{-1}\omega^B\right]_{even} &= \frac{1}{3}, \qquad \left[\left(\bar{\omega}_t\right)^{-1}\omega_t^B\right]_{odd}=\frac{2}{4}=\frac{1}{2}
\end{aligned}$$

So

$$\begin{aligned}
\theta^A &= \frac{\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^A\right]_{even}+\beta\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^A\right]_{odd}}{\sum\limits_{t=0}^{\infty}\beta^t} \\
&= \frac{\frac{2}{3(1-\beta^2)}+\frac{\beta}{2(1-\beta^2)}}{\frac{1}{(1-\beta)}} \\
&= \frac{4+3\beta}{6(1+\beta)}
\end{aligned}$$

And

$$\begin{aligned}
\theta^B &= \frac{\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^B\right]_{even}+\beta\sum\limits_{t=0}^{\infty}\beta^{2t}\left[\left(\bar{\omega}_t\right)^{-1}\omega_t^B\right]_{odd}}{\sum\limits_{t=0}^{\infty}\beta^t} \\
&= \frac{\frac{1}{3(1-\beta^2)}+\frac{\beta}{2(1-\beta^2)}}{\frac{1}{(1-\beta)}} \\
&= \frac{2+3\beta}{6(1+\beta)}
\end{aligned}$$

Therefore we have

$$c_t^A = \theta^A \bar{\omega}_t = \left\{ \begin{array}{cc} \frac{4+3\beta}{2(1+\beta)} & \text{if } t \text{ even,} \\ \frac{2(4+3\beta)}{3(1+\beta)} & \text{if } t \text{ odd.} \end{array} \right\}$$

$$c_t^B = \theta^B \bar{\omega}_t = \left\{ \begin{array}{cc} \frac{2+3\beta}{2(1+\beta)} & \text{if } t \text{ even,} \\ \frac{2(2+3\beta)}{3(1+\beta)} & \text{if } t \text{ odd.} \end{array} \right\}$$

with

$$\begin{aligned} p_t &= \beta^t \left(\frac{\bar{\omega}_t}{\bar{\omega}_0} \right)^{-\sigma} \\ &= \left\{ \begin{array}{cc} \beta^t & \text{if } t \text{ even,} \\ \frac{3\beta^t}{4} & \text{if } t \text{ odd.} \end{array} \right\}. \end{aligned}$$

To find the marginal utility of wealth λ^i, for each consumer we can use the FOC:

$$\begin{aligned} \beta^t u'\left(c_t^A\right) &= \lambda^A p_t \\ \lambda^A &= .\frac{\beta^t \left(c_t^A\right)^{-1}}{p_t} \\ \lambda^A &= \frac{2(1+\beta)}{4+3\beta}. \end{aligned}$$

Similarly

$$\lambda^B = \frac{2(1+\beta)}{2+3\beta}.$$