Econ 510a (second half)
Prof: Tony Smith
TA: Theodore Papageorgiou
Fall 2004
Yale University
Dept. of Economics

Solutions for Homework #3

Question 1

(a) The recursive formulation for the planning problem is

$$\begin{aligned} v(k) &= \max_{\{c,l,k'\}} U(c,l) + \beta v(k') \\ & s.t. \\ & c + k' = F(k, L-l) + (1-\delta)k \end{aligned}$$

or

$$v(k) = \max_{\{l,k'\}} U(F(k, L-l) + (1-\delta)k - k', l) + \beta v(k')$$

From the way we write it, we can see that the state variable is k, and control variables are l, k'.

(b) The F.O.C. is

$$\begin{aligned} \{l\} &: U_1(c,l)F_2(k, L-l) = U_2(c,l) \\ \{k'\} &: U_1(c,l) = \beta v'(k') \end{aligned}$$

From Envelope Theorem, we have

$$v'(k) = U_1(c,l)(F_1(k, L-l) + (1-\delta))$$

Iterate forward for one period, it becomes

$$v'(k') = U_1(c', l')(F_1(k', L-l') + (1-\delta))$$

Plug it into F.O.C., we get the final optimality conditions:

$$\begin{aligned} \{l_t\} &: U_1(c_t, l_t)F_2(k_t, L-l_t) = U_2(c_t, l_t) \\ \{k_{t+1}\} &: U_1(c_t, l_t) = \beta U_1(c_{t+1}, l_{t+1})(F_1(k_{t+1}, L-l_{t+1}) + (1-\delta)) \end{aligned}$$

(c) In steady state, the optimality condition becomes

$$\left\{\overline{l}^*\right\} : U_1\left(\overline{c}^*, \overline{l}^*\right)F_2(\overline{k}^*, L - \overline{l}^*) = U_2\left(\overline{c}^*, \overline{l}^*\right)$$
$$\left\{\overline{k}^*\right\} : F_1(\overline{k}^*, L - \overline{l}^*) = \frac{1}{\beta} - (1-\delta)$$

where $\overline{c}^* = F(\overline{k}^*, L - \overline{l}^*) - \delta\overline{k}^*$. We can see that $\overline{k}^*$ and $\overline{l}^*$ depends on both β, δ, production technology $F(k_t, n_t)$, and utility function $U(c_t, l_t)$.

In a growth model without valued leisure, the steady state is determined by the equation

$$F_1(\overline{k}^*, L) = \frac{1}{\beta} - (1-\delta)$$

which does not depend on the utility function $U(c_t, l_t)$.

Now let's compare two models. First, in the model with leisure choice we add an additional equation which states that the marginal rate of substitution between consumption and leisure must equal to the marginal rate of transformation. Second, the equation about $\overline{k}^*$ is the same, except that the level of steady-state leisure is different. Third, for the equation $F_1(\overline{k}^*, L - \overline{l}^*) = \frac{1}{\beta} - (1-\delta)$, due to the difference in the steady-state leisure level, the steady-state capital stock is also different. For example, if $F_{12} > 0$ as in the case of Cobb-Douglas production function, the capital stock in the model with leisure will be lower than that without leisure choice (since $L - \overline{l}^* < L$).

(d) With $F(k,n) = k^a n^{1-a}$ and $u(c,l) = \frac{(c^\theta l^{1-\theta})^{1-\sigma} - 1}{1-\sigma}$, the steady state conditions become:

$$\left\{\overline{l}^*\right\} : (1-\alpha)\overline{k}^{*a}\overline{n}^{*-a} = \frac{c(1-\theta)}{l\theta}$$
$$\left\{\overline{k}^*\right\} : a\left(\frac{\overline{n}^*}{\overline{k}^*}\right)^{1-a} = \frac{1}{\beta} - (1-\delta)$$

which leads to, if we normalize $L = 1$:

$$\overline{l}^* = \frac{a\beta(1-\theta)\delta - (1-\theta)(1-\beta(1-\delta))}{a\beta(1-\theta)\delta - (1-a\theta)(1-\beta(1-\delta))}$$
$$\overline{n}^* = 1 - \overline{l}^*$$
$$\overline{k}^* = \frac{(1-\beta(1-\delta))(a\theta - \theta)}{a\beta(1-\theta)\delta - (1-a\theta)(1-\beta(1-\delta))}$$
$$\overline{c}^* = \overline{k}^{*a}\overline{n}^{*1-a} - \delta\overline{k}^*$$

Question 2

(a) A competitive equilibrium is a set of sequences $\{c_t^*\}_{t=0}^\infty, \{b_t^*\}_{t=0}^\infty, \{q_t^*\}_{t=0}^\infty$ such that:

1. $\{c_t^*, b_{t+1}^*\}_{t=0}^\infty = \underset{\{c_t, b_{t+1}\}_{t=0}^\infty}{\arg\max} \left\{E_0 \sum_{t=0}^\infty \beta^t \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma}\right\}$

s.t.

$$c_t + q_t b_{t+1} = b_t + w_t$$
$$c_t \geq 0, \forall t; b_0 = 0$$
$$\lim_{t\to\infty} b_{t+1}\left(\prod_{j=0}^{t} q_j\right) \geq 0$$

2.

$$b_t^* = 0, \forall t \text{ (bonds market clearing)}$$

3.

$$c_t^* = w_t, \forall t \text{ (goods market clearing)}$$

(b) To simplify notation, we conjecture that in equilibrium the bond price is constant across time (we will check this conjecture later). Now the recursive formulation of the consumer's problem is

$$v(b_t, c_{t-1}, \omega_t) = \max_{\{c_t, b_{t+1}\}} \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma} + \beta v(b_{t+1}, c_t, \omega_{t+1})$$
$$s.t.$$
$$c_t + q b_{t+1} = b_t + \omega_t$$

or equivalently,

$$v(b_t, c_{t-1}, \omega_t) = \max_{\{b_{t+1}\}} \frac{(a_t + \omega_t - q b_{t+1} - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma} + \beta v(b_{t+1}, b_t + \omega_t - q b_{t+1}, \omega_{t+1})$$

Note the choice of aggregate state variable here. In principle we should include the triple aggregate state $(A, \overline{\varpi}_{t-1}, \overline{\varpi}_t)$ into our state variable. But here we know that $A = 0$, since it is a representative agent economy. And as long as we know about one value in the pair $(\overline{\varpi}_{t-1}, \overline{\varpi}_t)$, we can deduce the other from the constant growth rate g. Therefore, we need only one aggregate endowment (either $\overline{\varpi}_{t-1}$ or $\overline{\varpi}_t$) as our aggregate state variable. For example, we could choose $\overline{\varpi}_{t-1}$ and the bond price would be $q_t = q(\overline{\varpi}_{t-1})$. To save notation further, we can even write $q_t = q(\omega_{t-1})$, since this is a representative agent exchange economy $(\omega_{t-1} = \overline{\varpi}_{t-1})$ and we cannot change (either individual or aggregate) endowment anyway. Furthermore, due to the special utility function here, we can conjecture that the bond price is constant across time and check it later. So after a long chain of reasoning, we choose $q_t = q$ and only include the individual triple state (b_t, c_{t-1}, ω_t) into our recursive formulation.

(c) F.O.C. for this problem is

$$b_{t+1} : (u_1(c_t, c_{t-1}) + \beta v_2(t+1))q = \beta v_1(t+1)$$

where $v_1(t+1)$ and $v_2(t+1)$ are partial derivatives of $v(b_{t+1}, c_t, \omega_{t+1})$.

The envelope condition is

$$\begin{aligned} b_t &: v_1(t) = u_1(c_t, c_{t-1}) + \beta v_2(t+1) \\ c_{t-1} &: v_2(t) = u_2(c_t, c_{t-1}) \end{aligned}$$

Solve for this, we get

$$\begin{aligned} v_1(t) &= u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t) \\ v_2(t) &= u_2(c_t, c_{t-1}) \end{aligned}$$

Iterate forward for one period and plug into F.O.C., we get the Euler Equation

$$\begin{aligned} (u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t))q_{t+1} &= \beta(u_1(c_{t+1}, c_t) + \beta u_2(c_{t+2}, c_{t+1})) \\ \Rightarrow q &= \frac{\beta(u_1(c_{t+1}, c_t) + \beta u_2(c_{t+2}, c_{t+1}))}{u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t)} \\ \Rightarrow q &= \frac{\beta((c_{t+1} - \lambda c_t)^{-\sigma} - \beta\lambda(c_{t+2} - \lambda c_{t+1})^{-\sigma})}{(c_t - \lambda c_{t-1})^{-\sigma} - \beta\lambda(c_{t+1} - \lambda c_t)^{-\sigma}} \end{aligned}$$

Notice the similarity with normal Euler equation: it is the marginal rate of substitution between consumption c_{t+1} and c_t. The difference is the involvement of two period felicity function, which is due to the ”habit persistence”.

(d) In equilibrium, we must have $c_t = \omega_t$. Plug into the Euler Equation, we get the equilibrium bond price as

$$\begin{aligned} q &= \frac{\beta(u_1(\omega_{t+1}, \omega_t) + \beta u_2(\omega_{t+2}, \omega_{t+1}))}{u_1(\omega_t, \omega_{t-1}) + \beta u_2(\omega_{t+1}, \omega_t)} \\ &= \frac{\beta((\omega_{t+1} - \lambda\omega_t)^{-\sigma} - \beta\lambda(\omega_{t+2} - \lambda\omega_{t+1})^{-\sigma})}{(\omega_t - \lambda\omega_{t-1})^{-\sigma} - \beta\lambda(\omega_{t+1} - \lambda\omega_t)^{-\sigma}} \\ &= \frac{\beta((g\omega_t - \lambda\omega_t)^{-\sigma} - \beta\lambda(g^2\omega_t - \lambda g\omega_t)^{-\sigma})}{(\omega_t - \frac{\lambda}{g}\omega_t)^{-\sigma} - \beta\lambda(g\omega_t - \lambda\omega_t)^{-\sigma}} \\ &= \beta g^{-\sigma} \end{aligned}$$

This verifies our conjecture that $q_t = q$. The result is quite intuitive: (a) the more patient ($\beta \uparrow$) the individuals are, the higher the demand for savings, and the higher the asset price will be; (b) the higher of the growth rate of endowment ($g \uparrow$), the less need for saving, the lower the asset price.

Question 3

(a) A sequential competitive equilibrium for the economy $\{u_A, u_B, \omega\}$, is a sequence $\{c_{it}^*\}_{t=0}^{\infty}, \{b_{i,t+1}^*\}_{t=0}^{\infty}, \{q_t^*\}_{t=0}^{\infty}$ (where q_t^* means price of Arrow security) for $i = A, B$ such that

(1) For $i = A, B$,

$$\{c_{it}^*, b_{i,t+1}^*\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$
$$s.t.$$
$$c_{it} + q_t^* b_{i,t+1} = b_{i,t} + \omega$$
$$\lim_{t\to\infty} b_{i,t+1}\left(\prod_{j=0}^{t} q_j\right) \geq 0$$
$$b_{i,0} = 0, c_{it} \geq 0$$

(2) $\lambda c_{At}^* + (1-\lambda)c_{Bt}^* = \omega \textit{ for } t = 0,1,2\ldots$
(3) $\lambda b_{A,t+1}^* + (1-\lambda)b_{B,t+1}^* = 0 \textit{ for } t = 0,1,2\ldots$

(b) To solve this problem, we first get Euler equation. We have

$$\max_{\{c_{it}, b_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$c_{it} + q_t^* b_{i,t+1} = b_{i,t} + \omega$$
$$\lim_{t\to\infty} b_{i,t+1}\left(\prod_{j=0}^{t} q_j\right) \geq 0$$
$$b_{i,0} = 0, c_{it} \geq 0$$

If we substitute in for $c_{i,t}$ from the budget constraint we have:

$$\max_{\{b_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u(b_{i,t} + \omega - q_t^* b_{i,t+1})$$

By taking the derivative w.r.t. $b_{i,t+1}$, we get the Euler equation:

$$\beta_i \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = q_t^*$$

or equivalently,

$$\beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} = \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

Now we can see that $c_{i,t+1} \neq c_{i,t}$ $(\forall i, \forall t)$. Suppose not, without loss of generality let $c_{A,t+1} = c_{A,t}$. By feasibility condition, we know that $c_{B,t+1} = c_{B,t}$. Plug into the equation we get $\beta_A = \beta_B$, a contradiction. As a result, there cannot be any steady state in this economy.

We start to prove the convergence property of consumption path. First, we want to show that $\{c_{At}\}_{t=0}^{\infty}$ $(\{c_{Bt}\}_{t=0}^{\infty})$ is an increasing (decreasing) sequence. We already know that $c_{A,t+1} \neq c_{A,t}$ $(\forall t)$. Now suppose that $c_{A,t+1} < c_{A,t}$ for some t. By the feasibility condition, we know that $c_{B,t+1} > c_{B,t}$. From the strict concavity of felicity function, we have

$$\frac{u'(c_{A,t+1})}{u'(c_{A,t})} > 1 > \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$
$$\Rightarrow \beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} > \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

which contradicts Euler equation.

Since bounded monotone sequence has a limit, we have $c_{At} \to \overline{c}$ for $t \to \infty$. But we have shown that the economy has no steady state, so c_{At} can converge to nowhere but the boundary, i.e. $c_{At} \to \omega$ and $c_{Bt} \to 0$.

Alternatively some of you suggested doing the following:

From the first-order condition it will be the case that:

$$\frac{u'(c_{B,0})}{u'(c_{A,0})} \frac{u'(c_{A,t})}{u'(c_{B,t})} = \left(\frac{\beta_B}{\beta_A}\right)^t$$

If we take the limit on both sides we have:

$$\lim_{t\to\infty} \frac{u'(c_{B,0})}{u'(c_{A,0})} \frac{u'(c_{A,t})}{u'(c_{B,t})} = \lim_{t\to\infty} \left(\frac{\beta_B}{\beta_A}\right)^t \Leftrightarrow$$
$$\frac{u'(c_{B,0})}{u'(c_{A,0})} \lim_{t\to\infty} \frac{u'(c_{A,t})}{u'(c_{B,t})} = 0$$

since $\frac{u'(c_{B,0})}{u'(c_{A,0})}$ is a constant and $\beta_B < \beta_A$. Therefore we conclude that:

$$\lim_{t\to\infty} \frac{u'(c_{A,t})}{u'(c_{B,t})} = 0$$

For the above equation to hold it must be the case that either $\lim_{t\to\infty} u'(c_{A,t}) = 0$ or that $\lim_{t\to\infty} u'(c_{B,t}) = \infty$ (or both). The first case is impossible however, since that would imply that $\lim_{t\to\infty} c_{A,t} = \infty$, but we know that $c_{A,t}$ is bounded by the total aggregate endowment. Therefore, $\lim_{t\to\infty} u'(c_{B,t}) = \infty \Leftrightarrow \lim_{t\to\infty} c_{B,t} = 0$ (Inada condition).