Instructor	: Tony Smith	November 10, 2005
T.A.	: Carolina Silva	
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Suggested Solutions	: Problem Set 3	

1. (a) We have the following problem:

$$\max_{\{c_t,a_{t+1}\}} \sum_{t\geq 0} \beta^t u(c_t)$$

$$s.t \;\; a_0 \;\text{ given}$$

$$c_t + a_{t+1} = Ra_t + w \quad \forall t \geq 0$$

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^t} \geq 0$$

Then, the transversality condition (TC) for this problem is

$$\lim_{t\to\infty} \beta^t u'(Ra_t^* + w - a_{t+1}^*)Ra_t^* = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \beta^t u'(c_t^*)Ra_t^* = 0$$

Now we have to prove that the TC plus the Euler eq. imply the nPg restriction. The Euler eq. is:

$$\begin{aligned}
& -\beta^{t-1}u'(c_{t-1}) + \beta^t u'(c_t)R = 0 && \forall t \geq 0 \\
& \Leftrightarrow u'(c_{t-1}) = \beta R u'(c_t) && \forall t \geq 0 \\
& \Rightarrow u'(c_0) = \beta^{t+1}R^{t+1}u'(c_{t+1}) && \forall t \geq 0 \qquad (1.a)\\
& \Rightarrow \frac{a_{t+1}}{R^t} = R\beta^{t+1}a_{t+1}\frac{u'(c_{t+1})}{u'(c_0)} && \forall t \geq 0 \\
& \Rightarrow \lim_{t\to\infty} \frac{a_{t+1}}{R^t} = \frac{1}{u'(c_0)} \lim_{t\to\infty} R\beta^{t+1}a_{t+1}u'(c_{t+1}) = 0 && \text{(because of TC)}
\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \frac{a_{t+1}}{R^t} = 0, \quad \text{and thus the nPg restriction is met.}$$

(b) We want to prove that a sequence $\{a_t^*\}$ that satisfies the TC and the Euler eq. maximizes the problem stated in part (a). Thus, letting $c_t^* = Ra_t^* + w - a_{t+1}^*$, this sequence is such that:

$$\lim_{t\to\infty} \beta^t u'(c_t^*)Ra_t^* = 0 \tag{1}$$

$$-\beta^t u'(c_t^*) + \beta^{t+1}Ru'(c_{t+1}^*) = 0 \tag{2}$$

Let $\{a_t\}$ be a feasible sequence (ie a_0 is the given one, it satisfies the budget constraints every period, and it met the nPg restriction), defining c_t as $Ra_t + w - a_{t+1}$, we want to prove that

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [u(c_t^*) - u(c_t)] \geq 0$$

Defining $A_T = \sum_{t=0}^{T} \beta^t[u(c_t^*) - u(c_t)]$, we have that:

$$\begin{aligned}
A_T &\geq \sum_{t=0}^{T} \beta^t[u'(c_t^*)R(a_t^* - a_t) - u'(c_t^*)(a_{t+1}^* - a_{t+1})] && \text{(by concavity of } u\text{)} \\
&= \sum_{t=0}^{T-1} \beta^t(a_{t+1}^* - a_{t+1}) \underbrace{[-u'(c_t^*) + \beta R u'(c_{t+1}^*)]}_{=0 \text{ by } (2)} + Ru'(c_0^*) \underbrace{(a_0^* - a_0)}_{=0} - \beta^T u'(c_T^*)(a_{T+1}^* - a_{T+1}) \\
&= -\beta^T u'(c_T^*)(a_{T+1}^* - a_{T+1}) \\
&= \beta^{T+1} R u'(c_{T+1}^*)(-a_{T+1}^* + a_{T+1}) && \text{(by (2))}
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{T\to\infty} A_T &\geq \lim_{T\to\infty} \beta^{T+1} R u'(c_{T+1}^*)(-a_{T+1}^* + a_{T+1}) \\
&= \lim_{T\to\infty} \beta^{T+1} R u'(c_{T+1}^*) a_{T+1} && \text{(by (1))} \\
&= \lim_{T\to\infty} \beta^{T+1} R^{T+1} u'(c_{T+1}^*) \frac{a_{T+1}}{R^T} \\
&= \lim_{T\to\infty} u'(c_0^*) \frac{a_{T+1}}{R^T} && \text{(by (1.a) of part (a))} \\
&= 0 && \text{(because } \{a_t\} \text{ satisfies nPg)}
\end{aligned}$$

which is what we wanted to show.

2. (a) Let's compute the competitive equilibrium. The consumer's problem is

$$\max_{c_0, c_1, b_1} u(c_0) + \beta u(c_1)$$

$$s.t. \; c_0 + q_0 b_1 = w_0 + \pi_0$$

$$c_1 = w_1 + b_1 + \pi_1$$

which implies the FOC

$$q_0 = \beta \frac{u'(w_1 + b_1 + \pi_1)}{u'(w_0 + \pi_0 - q_0 b_1)}$$

The firm's problem is now

$$\max_{n_0, n_1, k_1} f(k_0, n_0) - w_0 n_0 - k_1 + q_0(f(k_1, n_1) - w_1 n_1)$$

which has the following FOC: $w_0 = f_2(k_0, n_0)$, $w_1 = f_2(k_1, n_1)$ and $q_0 = 1/f_1(k_1, n_1)$. Now, the market clearing conditions are $n_0 = n_1 = 1$ and $b_1 = 0$, and thus profits are such that:

$$\pi_0 = f(k_0, 1) - w_0 - k_1 \quad \text{and} \quad \pi_1 = f(k_1, 1) - w_1$$

The consumer's and firm's FOC imply that

$$q_0 = \beta \frac{u'(w_1 + b_1 + \pi_1)}{u'(w_0 + \pi_0 - q_0 b_1)} = \frac{1}{f_1(k_1, n_1)}$$

using the other FOCs of the firm, the market clearing conditions and the formulas for the profits, this implies that

$$q_0 = \beta \frac{u'(f(k_1, 1))}{u'(f(k_0, 1) - k_1)} = \frac{1}{f_1(k_1, 1)}$$

thus, the equation defined by the second equality defines implicitly the equilibrium k_1^* and then $q_0^* = 1/f_1(k_1^*, 1)$.
Now, in the competitive equilibrium where capital is managed by consumers, the problem they solve is

$$\max_{k_1} u(r_0 k_0 + w_0 - k_1) + \beta u(r_1 k_1 + w_1)$$

which implies the FOC

$$-u'(r_0 k_0 + w_0 - k_1) + \beta r_1 u'(r_1 k_1 + w_1) = 0$$

and as the production function exhibits CRS (and using clearing mkt. conditions), this is equivalent to

$$u'(f(k_0, 1) - k_1) = \beta r_1 u'(f(k_1, 1))$$

which is exactly the same equation we got in the first part (that consumption allocations for both problems are the same is direct).

(b) Now we have a market for shares, and the consumer's problem becomes:

$$\max_{c_0, c_1, b_1, s_1} \quad u(c_0) + \beta u(c_1)$$

$$s.t. \;\; c_0 + q_0 b_1 + p_0 s_1 = w_0 + p_0 + \pi_0$$

$$c_1 = w_1 + b_1 + s_1 \pi_1$$

which implies the FOC's

$$p_0 = \beta \pi_1 \frac{u'(c_1)}{u'(c_0)} \quad \text{and} \quad q_0 = \beta \frac{u'(c_1)}{u'(c_0)}$$

Note that then $p_0 = q_0 \pi_1$, which is a no arbitrage condition. In fact, suppose $p_0 < q_0 \pi_1$ then an agent could buy a share at a price of p_0, which entitles him to π_1 units of consumption good the next period, and then go to the bonds market and sell this promise of π_1 units of consumption good in the next period at a price of $q_0 \pi_1$, which is larger than what he spent originally; thus, using this method agents can easily get rich. What basically happens here is that the bonds and share markets are offering the same finantial objects, and that is why, as we will see, the equilibrium is the same if we have both or only one of these markets.
The firm's problem has not changed, so we still have the first order condition $q_0 = 1/f_1(k_1, 1)$, and thus in equilibrium we must have

$$q_0 = \beta \frac{u'(c_1)}{u'(c_0)} = \frac{1}{f_1(k_1, 1)}$$

and thus when replacing the equilibrium conditions $s_1 = 1$ and $b_1 = 0$, and the wages determined by the firm's FOCs, we get

$$q_0 = \beta \frac{u'(f(k_1, 1))}{u'(f(k_0, 1) - k_1)} = \frac{1}{f_1(k_1, 1)}$$

which is the same equation we found in part (a). Call k_1^* the solution to the second equality, then

$$p_0 = \frac{(f(k_1^*, 1) - f_2(k_1^*, 1))}{f_1(k_1^*, 1)}$$

(c) The competitive allocation is still going to be Pareto optimal. In this setting, consumers take profits as given when solving their problem, thus the income tax is equivalent to reduce the level of this exogenous variable, but as proceeds are returned to consumers, in equilibrium the level of this variable is going to be same as in part (a), and therefore optimal allocations are going to be same too.

3. (a) A Competitive Equilibrium with date-0 trading for the economy $\{u_1, u_2\}$, $\{\omega_{it}\}_{t=0}^{\infty}$ is a vector of prices $\{p_t\}_{t=0}^{\infty}$ and a vector of quantities $\{c_{it}^*\}_{t=0}^{\infty}$ for $i = 1, 2$ such that
(1) For $i = 1, 2$

$$\{c_{it}^*\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right)$$

$$s.t. \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it}$$

(2) $c_{1t} + c_{2t} = \omega_{1t} + \omega_{2t}$ for $t = 0, 1, 2...$

(b) The FOC for consumer i is

$$\frac{\beta^j u'\left(c_{i,t+j}\right)}{u'\left(c_{i,t}\right)} = \frac{p_{t+j}}{p_t} \text{ for } \forall t, j$$

This together with budget constraint and market clearing condition determines the competitive equilibrium. Here there are two ways to solve for the equilibrium. One way is to solve for the system of simultaneous equations; another way is to make a guess of solution and check the feasibility for each equations. Due to the special structure of the model, here it is easier to proceed with the second way. Now guess that $c_{it} = c_i$ for $\forall t$. Replacing into the FOC and normalizing $p_0 = 1$, we get

$$p_t = \beta^t$$

Now, plugging into the budget constraint for each individual, we have

$$\sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it} \quad \Rightarrow \quad \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^t \omega_i$$
$$\Rightarrow \quad c_i = \omega_i$$

It is easy to check that it satisfies the market clearing condition. Therefore, the competitive equilibrium for this economy is

$$c_{1t} = \omega_1 = 2, \quad c_{2t} = \omega_2 = 1 \quad \text{and} \quad p_t = \beta^t$$

(c) Again guess that $c_{it} = c_i$ for $\forall t$. Replacing these into the FOC and normalizing $p_0 = 1$ we have

$$p_t = \beta^t$$

Plugging into the budget constraint for each individual, we have

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it} \quad & \Rightarrow \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^t \omega_i \\
& \Rightarrow \sum_{t=0}^{\infty} \beta^t c_i = \sum_{t=0}^{\infty} \beta^{2t} \omega_i^e + \beta \sum_{t=0}^{\infty} \beta^{2t} \omega_i^o \\
& \Rightarrow c_i = \frac{1}{1+\beta} \omega_i^e + \frac{\beta}{1+\beta} \omega_i^o
\end{aligned}$$

where ω_i^e and ω_i^o means ω_i in even and odd period, respectively.
It is easy to check that it satisfies the market clearing condition. Therefore, the competitive equilibrium for this economy is

$$c_{1t} = \frac{2+\beta}{1+\beta}, \quad c_{2t} = \frac{1+2\beta}{1+\beta} \quad \text{and} \quad p_t = \beta^t$$

(d) Now guess the equilibrium as

$$\begin{aligned}
c_{1t} &= c_1^o \text{ for } t \text{ odd} \\
c_{1t} &= c_1^e \text{ for } t \text{ even} \\
c_{2t} &= c_2^o \text{ for } t \text{ odd} \\
c_{2t} &= c_2^e \text{ for } t \text{ even} \\
p_t &= \beta^{t-1} p^o \text{ for } t \text{ odd} \\
p_t &= \beta^t \text{ for } t \text{ even}
\end{aligned}$$

Replacing into the FOC we get

$$\begin{aligned}
\frac{\beta^j u'(c_{i,t+j})}{u'(c_{i,t})} = \frac{p_{t+j}}{p_t} & \Rightarrow \frac{\beta u'(c_i^o)}{u'(c_i^e)} = p^o \\
& \Rightarrow \beta \left(\frac{c_1^e}{c_1^o} \right)^{\sigma} = \beta \left(\frac{c_2^e}{c_2^o} \right)^{\sigma} = p^o
\end{aligned} \tag{1}$$

Our guess must satisfy the budget constraint, that is:

$$\sum_{t \geq 0} \beta^{2t} c_i^e + \sum_{t \geq 0} \beta^{2t} p^o c_i^o = \sum_{t \geq 0} \beta^{2t} \omega_i^e + \sum_{t \geq 0} \beta^{2t} p^o \omega_i^o \qquad i = 1, 2$$

which implies

$$c_1^e + p^o c_1^o = 2(1+p^o) \quad \text{and} \quad c_2^e + p^o c_2^o = p^o \tag{2}$$

and it also must satisfy the the market clearing conditions:

$$c_1^e + c_2^e = 2 \quad \text{and} \quad c_1^o + c_2^o = 3 \tag{3}$$

The system of the eqs. in (2) and (3) does not have a unique solution, but we also need

$$\frac{c_1^e}{c_1^o} = \frac{c_2^e}{c_2^o} \qquad \text{(because of (1))} \tag{4}$$

and this last equation pins down the unique solution, which is:

$$c_1^e = \frac{4(1+p^o)}{2+3p^o} \quad \text{and} \quad c_1^o = \frac{6(1+p^o)}{2+3p^o}$$

and

$$c_2^e = \frac{2p^o}{2+3p^o} \quad \text{and} \quad c_2^o = \frac{3p^o}{2+3p^o}$$

which implies that

$$p^o = \beta\left(\frac{c_1^e}{c_1^o}\right)^\sigma = \beta\left(\frac{2}{3}\right)^\sigma$$

So the equilibrium price is

$$p_t = \left(\frac{2}{3}\right)^\sigma \beta^t \qquad \text{for } t \text{ odd}$$

$$p_t = \beta^t \qquad \text{for } t \text{ even}$$

and replacing p^o in the consumption allocations we get that

$$c_1^e = \frac{4(3^\sigma + \beta 2^\sigma)}{2\cdot 3^\sigma + 3\beta 2^\sigma} \quad \text{and} \quad c_1^o = \frac{6(3^\sigma + \beta 2^\sigma)}{2\cdot 3^\sigma + 3\beta 2^\sigma}$$

and

$$c_2^e = \frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} \quad \text{and} \quad c_2^o = \frac{3\beta 2^\sigma}{2\cdot 3^\sigma + 3\beta 2^\sigma}$$

(e) The social planning problem for this economy is the following:

$$\max_{\{c_{1t}\}_{t\geq 0},\{c_{2t}\}_{t\geq 0}} \alpha_1 \sum_{t=0}^{\infty} \beta^t u(c_{1t}) + \alpha_2 \sum_{t=0}^{\infty} \beta^t u(c_{2t})$$

$$s.t. \;\; c_{1t} + c_{2t} = \omega_{1t} + \omega_{2t} \quad \forall t \geq 0$$

Now, from the FOC of the consumers' problem (date-0 trading), we have that their marginal utilities of wealth λ_i (ie, the lagrange multipliers) for $i = 1, 2$ are

$$\lambda_i = \frac{\beta^t u'(c_{it})}{p_t} \quad \Rightarrow \quad \lambda_i = u'(c_{i0})/p_0 = c_{i0}^{-\sigma}/p_0$$

and we know that the Pareto weights that deliver the competitive equilibrium allocation are:

$$\alpha_i = \frac{1/\lambda_i}{1/\lambda_1 + 1/\lambda_2}$$

Then, for each pair of endowments these weights are:
for part (b),

$$\lambda_1 = 2^{-\sigma} \quad \text{and} \quad \lambda_2 = 1 \quad \Rightarrow \quad \alpha_1 = \frac{1}{1+2^{-\sigma}} \quad \text{and} \quad \alpha_2 = \frac{2^{-\sigma}}{1+2^{-\sigma}}$$

for part (c),

$$\lambda_1 = \left(\frac{2+\beta}{1+\beta}\right)^{-\sigma} \quad \text{and} \quad \lambda_2 = \left(\frac{1+2\beta}{1+\beta}\right)^{-\sigma}$$

and then

$$\alpha_1 = \frac{(1+2\beta)^{-\sigma}}{(1+2\beta)^{-\sigma} + (2+\beta)^{-\sigma}} \quad \text{and} \quad \alpha_2 = \frac{(2+\beta)^{-\sigma}}{(1+2\beta)^{-\sigma} + (2+\beta)^{-\sigma}}$$

and for part (d),

$$\lambda_1 = \left(\frac{4(3^\sigma + \beta 2^\sigma)}{2 \cdot 3^\sigma + 3\beta 2^\sigma} \right)^{-\sigma} \quad \text{and} \quad \lambda_2 = \left(\frac{\beta 2^{\sigma+1}}{2 \cdot 3^\sigma + 3\beta 2^\sigma} \right)^{-\sigma}$$

and then

$$\alpha_1 = \frac{(\beta 2^{\sigma+1})^{-\sigma}}{(\beta 2^{\sigma+1})^{-\sigma} + (4(3^\sigma + \beta 2^\sigma))^{-\sigma}} = 1 - \alpha_2$$

(f) A Competitive Equilibrium with sequential trading for the economy $\{u_1, u_2\}, \{\omega_{it}\}_{t=0}^\infty$ is a sequence $\{c_{it}^*\}_{t=0}^\infty, \left\{a_{i,t+1}^*\right\}_{t=0}^\infty, \{R_t^*\}_{t=0}^\infty$ (where R_t^* means interest rate from t to $t+1$) for $i = 1, 2$ such that

(1) For $i = 1, 2$

$$\{c_{it}^*, a_{i,t+1}^*\}_{t=0}^\infty = \arg\max \sum_{t=0}^\infty \beta^t u(c_{it})$$

$$s.t. \;\; c_{it} + a_{i,t+1} = R_t^* a_{i,t} + \omega_{it}$$

$$\lim_{t\to\infty} a_{i,t+1} \left(\prod_{j=0}^{t+1} R_j^{-1} \right) \geq 0$$

$$a_{i,0} = 0, c_{it} \geq 0$$

(2) $c_{1t}^* + c_{2t}^* = \omega_{1t} + \omega_{2t}$ for $t = 0, 1, 2...$

(3) $a_{1t}^* + a_{2t}^* = 0$ for $t = 0, 1, 2...$

Now we start to solve for equilibrium interest rate and asset holdings for different examples. In each example, it is easy to see that

$$\begin{aligned} R_t &= \frac{p_{t-1}}{p_t} \\ a_{i,t+1} &= R_t a_{i,t} + \omega_{it} - c_{it} \end{aligned}$$

We start with part (b). Plugging in the solution, we have

$$R_t = \frac{1}{\beta} \quad \text{and} \quad a_{i,t} = 0 \quad \forall t \geq 0$$

In part (c), we have

$$\begin{aligned} R_t &= \frac{1}{\beta} \\ a_{i,0} &= 0 \\ a_{i,1} &= \omega_{i0} - c_{i0} \\ a_{i,2} &= R_2 a_{i,1} + \omega_{i1} - c_{i1} = \frac{1}{\beta}(\omega_{i0} - c_{i0}) + \omega_{i1} - c_{i1} \\ &\ldots \end{aligned}$$

Plugging in the equilibrium solution, we have

$$\begin{aligned} R_t &= \frac{1}{\beta} \\ a_1^o &= -a_2^o = \frac{\beta}{1+\beta} \quad \text{for } t = 1, 3, 5, \ldots \\ a_1^e &= -a_2^e = 0 \quad \text{for } t = 0, 2, 4, \ldots \end{aligned}$$

In part (d), we have

$$\begin{aligned}
R^o &= \frac{1}{p^o} = \left(\frac{2}{3}\right)^{-\sigma}\frac{1}{\beta} \qquad \text{for } t = 1,3,5,... \\
R^e &= \frac{p^o}{\beta^2} = \left(\frac{2}{3}\right)^{\sigma}\frac{1}{\beta} \qquad \text{for } t = 2,4,6,... \\
a_{i,0} &= 0 \\
a_{1,1} &= \omega_{10} - c_{10} = 2 - c_1^e = \frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} \\
a_{2,1} &= -a_{1,1} = -\frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} \\
a_{1,2} &= R_1 a_{1,1} + \omega_{11} - c_{11} = 0 \\
a_{2,2} &= -a_{1,2} = 0 \\
a_{1,3} &= R_2 a_{1,2} + \omega_{12} - c_{12} = \frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} \\
a_{2,3} &= -a_{1,3} = -\frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} \\
&\ \ ...
\end{aligned}$$

Therefore, the equilibrium interest rate and asset holdings are

$$\begin{aligned}
&R^o = \left(\frac{2}{3}\right)^{-\sigma}\frac{1}{\beta} && \text{for } t = 1,3,5,... \\
&R^e = \left(\frac{2}{3}\right)^{\sigma}\frac{1}{\beta} && \text{for } t = 2,4,6,... \\
&a_1^o = \frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} && \text{for } t = 1,3,5,... \\
&a_1^e = 0 && \text{for } t = 0,2,4,... \\
&a_2^o = -\frac{\beta 2^{\sigma+1}}{2\cdot 3^\sigma + 3\beta 2^\sigma} && \text{for } t = 1,3,5,... \\
&a_2^e = 0 && \text{for } t = 0,2,4,...
\end{aligned}$$