Suggested Solutions to Homework #3
Econ 511b (Part I), Spring 2004

1. **Consider an exchange economy with two (types of) consumers. Type-A consumers comprise fraction λ of the economy's population and type-B consumers comprise fraction $1-\lambda$ of the economy's population. Each consumer has (constant) endowment ω in each period. A consumer of type i has preferences over consumption streams of the form $\sum_{t=0}^{\infty} \beta_i^t \log(c_t)$. Assume that $1 > \beta_A > \beta_B > 0$: type-A consumers are more patient than type-B consumers. Consumers trade a risk-free bond in each period. There is no restriction on borrowing except for a no-Ponzi-game condition. Each consumer has zero assets in period 0.**

   (a) Carefully define a sequential competitive equilibrium for this economy.

   A sequential competitive equilibrium for the economy $\{u_A, u_B, \omega\}$, is a sequence $\{c_{it}^*\}_{t=0}^{\infty}, \{a_{i,t+1}^*\}_{t=0}^{\infty}, \{q_t^*\}_{t=0}^{\infty}$ (where q_t^* means price of Arrow security) for $i = A, B$ such that

   (1) For $i = A, B$,

$$
\begin{aligned}
\left\{c_{it}^*, a_{i,t+1}^*\right\}_{t=0}^{\infty} &= \arg\max \sum_{t=0}^{\infty} \beta_i^t \log(c_{it}) \\
& s.t. \\
& c_{it} + q_t^* a_{i,t+1} = a_{i,t} + \omega \\
& \lim_{t\to\infty} a_{i,t+1} \left(\prod_{t=0}^{\infty} q_t \right) \geq 0 \\
& a_{i,0} = 0, c_{it} \geq 0
\end{aligned}
$$

   (2) $\lambda c_{At}^* + (1-\lambda) c_{Bt}^* = \omega \;\; for \;\; t = 0, 1, 2...$

   (3) $\lambda a_{A,t+1}^* + (1-\lambda) a_{B,t+1}^* = 0 \;\; for \;\; t = 0, 1, 2...$

   (b) Carefully define a recursive competitive equilibrium for this economy.

   A Recursive Competitive Equilibrium for the economy $\{u_A, u_B, \omega\}$ is a set of functions:

$$
\begin{aligned}
price\ function &: q(A) \\
policy\ function &: a_i' = g_i(a_i, A) \\
value\ functions &: v_i(a_i, A) \\
transition\ function &: A' = G(A)
\end{aligned}
$$

   such that:

   (1) For $i = A, B, a_i' = g_i(a_i, A)$ and $v_i(a_i, A)$ solves

$$
\begin{aligned}
v_i(a_i, A) &= \max_{\{c_i, a_i'\}} \log(c_i) + \beta_i v_i\left(a_i', A'\right) \\
& s.t. \\
& c_i + q(A) a_i' = a_i + \omega \\
& A' = G(A)
\end{aligned}
$$

(2) Consistency:

$$\begin{aligned} G(A) &= g_A(A, A) \\ -\frac{\lambda}{1-\lambda} G(A) &= g_B\left(-\frac{\lambda}{1-\lambda} A, A\right) \end{aligned}$$

(c) Show that this economy has no steady state: in particular, show that the type-B agents become poorer and poorer over time and consume zero in the limit.

To solve this problem, we first get Euler equation. In recursive formulation, consumers solve

$$\begin{aligned} v_i(a_i, A) &= \max_{\{c_i, a_i'\}} \log\left(a_i + \omega - q(A) a_i'\right) + \beta_i v_i\left(a_i', A'\right) \\ & \quad s.t. \\ & \quad A' = G(A) \end{aligned}$$

Solve for F.O.C. and use envelope condition, we get the Euler equation:

$$\beta_i \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = q(A)$$

or equivalently,

$$\beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} = \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

Now we can see that $c_{i,t+1} \neq c_{i,t}$ $(\forall i, \forall t)$. Suppose not, without loss of generality let $c_{A,t+1} = c_{A,t}$. By feasibility condition, we know that $c_{B,t+1} = c_{B,t}$. Plug into the equation we get $\beta_A = \beta_A$, a contradiction. As a result, there cannot be any steady state in this economy.

We start to prove the convergence property of consumption path. First, we want to show that $\{c_{At}\}_{t=0}^{\infty}$ $(\{c_{Bt}\}_{t=0}^{\infty})$ is an increasing (decreasing) sequence. We already know that $c_{A,t+1} \neq c_{A,t}$ $(\forall t)$. Now suppose that $c_{A,t+1} < c_{A,t}$ for some t. By the feasibility condition, we know that $c_{B,t+1} > c_{B,t}$. From the strict concavity of felicity function, we have

$$\begin{aligned} & \frac{u'(c_{A,t+1})}{u'(c_{A,t})} > 1 > \frac{u'(c_{B,t+1})}{u'(c_{B,t})} \\ \Rightarrow \quad & \beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} > \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})} \end{aligned}$$

which contradicts Euler equation.

Since bounded monotone sequence has a limit, we have $c_{At} \to \overline{c}$ for $t \to \infty$. But we have shown that the economy has no steady state, so c_{At} can converge to nowhere but the boundary, i.e. $c_{At} \to \omega$ and $c_{Bt} \to 0$.

2. **Consider an infinite-horizon one-sector growth model with an externality in production. Leisure is not valued and the (representative) consumer has time-separable preferences with discount factor $\beta \in (0,1)$. Consumers own the factors of production. Capital depreciates at rate δ. There is a large number of identical firms each of which has the following production technology:**

$$F\left(k,l,\overline{k}\right) = Ak^{\alpha}l^{1-\alpha}\overline{k}^{\gamma}$$

where k is the capital rented by the firm, $\overline{k}$ is the aggregate capital stock, and the parameters α and γ satisfy $0 < \gamma < 1-\alpha$ and $\alpha \in (0,1)$. Thus there is a productive externality from the rest of the economy: a higher aggregate capital stock increases the firm's productivity. A typical (small) firm takes the aggregate capital stock as given when choosing its inputs.

(a) **Define a recursive competitive equilibrium for this economy. Be clear about which variables consumers and firms take as given when they solve their optimization problems. Find the competitive equilibrium steady-state aggregate capital stock as a function of primitives.**

A Recursive Competitive Equilibrium for the economy is a set of functions:

$$\begin{aligned}
\textit{price function} &: \ r\left(\overline{k}\right), w\left(\overline{k}\right) \\
\textit{policy function} &: \ k^{'} = g\left(k,\overline{k}\right) \\
\textit{value function} &: \ v\left(k,\overline{k}\right) \\
\textit{transition function} &: \ \overline{k}^{'} = G\left(\overline{k}\right)
\end{aligned}$$

such that:

(1) $k^{'} = g\left(k,\overline{k}\right)$ and $v\left(k,\overline{k}\right)$ solves consumer's problem:

$$\begin{aligned}
v\left(k,\overline{k}\right) &= \max_{\left\{c,k'\right\}} u\left(c\right) + \beta v\left(k^{'},\overline{k}^{'}\right) \\
& \quad s.t. \\
& \quad c + k^{'} = r\left(\overline{k}\right)k + (1-\delta)k + w\left(\overline{k}\right) \\
& \quad \overline{k}^{'} = G\left(\overline{k}\right)
\end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned}
r\left(\overline{k}\right) &= F_1\left(\overline{k},1,\overline{k}\right) = \alpha A\overline{k}^{\alpha+\gamma-1} \\
w\left(\overline{k}\right) &= F_2\left(\overline{k},1,\overline{k}\right) = (1-\alpha)A\overline{k}^{\alpha+\gamma}
\end{aligned}$$

(3) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k},\overline{k}\right)$$

Solve for consumer's problem in the normal way, we solve for the Euler equation as

$$\frac{\beta u^{'}\left(c_{t+1}\right)}{u^{'}\left(c_t\right)}\left(r\left(\overline{k}_{t+1}\right) + 1 - \delta\right) = 1$$

In steady state, we have

$$
\begin{aligned}
r\left(\overline{k}^{c}\right)+1-\delta &= \frac{1}{\beta} \\
\Rightarrow \quad \alpha A\left(\overline{k}^{c}\right)^{\alpha+\gamma-1} &= \frac{1}{\beta}-1+\delta \\
\Rightarrow \quad \overline{k}^{c} &= \left(\frac{\frac{1}{\beta}-1+\delta}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}
\end{aligned}
$$

where $\overline{k}^{c}$ represents the competitive equilibrium steady-state aggregate capital stock.

(b) **Write the planning problem for this economy in recursive form. The planner internalizes the externality in production: his production technology is**

$$F\left(\overline{k}, l, \overline{k}\right) = A\overline{k}^{\alpha+\gamma} l^{1-\alpha}$$

Find the steady-state aggregate capital stock implied by the planning problem. Show that it is higher than the competitive equilibrium steady-state aggregate capital stock.

The recursive formulation of the planning problem is

$$
\begin{aligned}
v\left(\overline{k}\right) \quad = \quad & \max_{\{c,k'\}} u(c)+\beta v\left(\overline{k}'\right) \\
& s.t. \\
& c+k' = A\overline{k}^{\alpha+\gamma}+(1-\delta)\overline{k}
\end{aligned}
$$

The Euler equation is

$$\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_{t}\right)}\left(A\left(\alpha+\gamma\right)\overline{k}_{t+1}^{\alpha+\gamma-1}+1-\delta\right)=1$$

In steady state, we have

$$
\begin{aligned}
A\left(\alpha+\gamma\right)\left(\overline{k}^{o}\right)^{\alpha+\gamma-1}+1-\delta &= \frac{1}{\beta} \\
\Rightarrow \quad \overline{k}^{o} &= \left(\frac{\frac{1}{\beta}-1+\delta}{\left(\alpha+\gamma\right)A}\right)^{\frac{1}{\alpha+\gamma-1}} \\
> \quad \overline{k}^{c} &= \left(\frac{\frac{1}{\beta}-1+\delta}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}} \qquad (\text{since } \alpha<\alpha+\gamma<1)
\end{aligned}
$$

where $\overline{k}^{o}$ represents the optimal steady-state aggregate capital stock.

(c) **Now introduce a government into the competitive equilibrium that you defined in part (a). The government subsidizes investment expenditures at a proportional rate τ and finances these subsidies by means**

of a lump-sum tax on consumers. The investment subsidy is constant across time but the lump-sum tax varies over time so as to balance the government's budget in every period. Define a recursive competitive equilibrium for this economy.

A Recursive Competitive Equilibrium for the economy with taxation is a set of functions:

$$\begin{aligned}
\textit{price function} &: \; r\left(\overline{k}\right), w\left(\overline{k}\right) \\
\textit{policy function} &: \; k' = g\left(k, \overline{k}\right) \\
\textit{value function} &: \; v\left(k, \overline{k}\right) \\
\textit{taxation function} &: \; T\left(\overline{k}\right) \\
\textit{transition function} &: \; \overline{k}' = G\left(\overline{k}\right)
\end{aligned}$$

such that:

(1) $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned}
v\left(k, \overline{k}\right) &= \max_{\{c,k'\}} u(c) + \beta v\left(k', \overline{k}'\right) \\
& s.t. \\
& c + (1-\tau)\left(k' - (1-\delta)k\right) = r\left(\overline{k}\right)k + w\left(\overline{k}\right) - T\left(\overline{k}\right) \\
& \overline{k}' = G\left(\overline{k}\right)
\end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned}
R\left(\overline{k}\right) &= F_1\left(\overline{k}, 1, \overline{k}\right) = \alpha A \overline{k}^{\alpha+\gamma-1} \\
w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1, \overline{k}\right) = (1-\alpha) A \overline{k}^{\alpha+\gamma}
\end{aligned}$$

(3) Government balances budget in each period:

$$T\left(\overline{k}\right) = \tau\left(\overline{k}' - (1-\delta)\overline{k}\right)$$

(3) Consistency of transition function:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

(d) For what subsidy rate τ is the competitive equilibrium steady-state aggregate capital stock the same as the steady-state aggregate capital stock in the planning problem?

We can solve for F.O.C. in the normal way as

$$(1-\tau)u'(c) = \beta v_1\left(k', \overline{k}'\right)$$

Envelope condition gives us

$$v_1\left(k, \overline{k}\right) = u'(c)\left(r\left(\overline{k}\right) + (1-\tau)(1-\delta)\right)$$

The Euler equation is

$$
\begin{aligned}
&\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(1-\tau\right)\left(r\left(\overline{k}\right)+\left(1-\tau\right)\left(1-\delta\right)\right)=1\\
\Rightarrow\ &\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\frac{1}{1-\tau}\left(\alpha A\overline{k}^{\alpha+\gamma-1}+\left(1-\tau\right)\left(1-\delta\right)\right)=1
\end{aligned}
$$

In steady state, we have

$$
\begin{aligned}
&\frac{\beta}{1-\tau}\left(\alpha A\overline{k}^{\alpha+\gamma-1}+\left(1-\tau\right)\left(1-\delta\right)\right)=1\\
\Rightarrow\ &\alpha A\overline{k}^{\alpha+\gamma-1}=\left(1-\tau\right)\left(\frac{1}{\beta}-\left(1-\delta\right)\right)\\
\Rightarrow\ &\overline{k}^{\tau}=\left(\frac{\left(1-\tau\right)\left(\frac{1}{\beta}-\left(1-\delta\right)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}
\end{aligned}
$$

where $\overline{k}^{\tau}$ represents the steady-state aggregate capital stock in economy with taxation. Now set $\overline{k}^{\tau}=\overline{k}^{o}$, we have

$$
\begin{aligned}
&\overline{k}^{\tau}=\overline{k}^{o}\\
\Rightarrow\ &\left(\frac{\left(1-\tau\right)\left(\frac{1}{\beta}-\left(1-\delta\right)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}=\left(\frac{\frac{1}{\beta}-1+\delta}{\left(\alpha+\gamma\right)A}\right)^{\frac{1}{\alpha+\gamma-1}}\\
\Rightarrow\ &\frac{1-\tau}{\alpha}=\frac{1}{\alpha+\gamma}\\
\Rightarrow\ &\tau=\frac{\gamma}{\alpha+\gamma}
\end{aligned}
$$

Therefore, for $\tau=\frac{\gamma}{\alpha+\gamma}$ the competitive equilibrium steady-state aggregate capital stock the same as the steady-state aggregate capital stock in the planning problem.

3. **Consider a neoclassical growth model with logarithmic utility, Cobb-Douglas production, full depreciation of the capital stock in one period, and inelastic labor supply (leisure is not valued). In this problem, you will solve explicitly for the recursive competitive equilibrium of this economy (assuming that the economy is decentralized in the same manner as in the second problem). Carefully define a sequential competitive equilibrium for this economy.**

   (a) **Suppose that aggregate capital evolves according to $k'=G\left(\overline{k}\right)=sf\left(\overline{k},1\right)$, where f is the economy's production function. (Why is this a reasonable conjecture?) Find explicit formulas for the value function $v(k,\overline{k})$ and the decision rule $k'=g\left(k,\overline{k}\right)$ of a "small" (or typical) consumer who takes the law of motion of aggregate capital as given. The functions v and g depend on s as well as on primitives of technology and preferences.**

Given the equivalence of competitive equilibrium and central planning problem here, we can conjecture the transition function of aggregate state as $\overline{k}^{'} = sA\overline{k}^{\alpha}$. Now we can define a recursive competitive equilibrium for this economy explicitly.
A Recursive Competitive Equilibrium for this economy is a set of functions:

$$\begin{aligned} price\ function &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ policy\ function &: k^{'} = g\left(k, \overline{k}\right) \\ value\ function &: v\left(k, \overline{k}\right) \\ transition\ function &: \overline{k}^{'} = sA\overline{k}^{\alpha} \end{aligned}$$

such that:
(1) $k^{'} = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) &= \max_{\{c,k'\}} \ln(c) + \beta v\left(k^{'}, \overline{k}^{'}\right) \\ & s.t. \\ & c + k^{'} = r\left(\overline{k}\right) k + w\left(\overline{k}\right) \\ & \overline{k}^{'} = sA\overline{k}^{\alpha} \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= F_1\left(\overline{k}, 1\right) = \alpha A\overline{k}^{\alpha-1} \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1\right) = (1-\alpha) A\overline{k}^{\alpha} \end{aligned}$$

(3) Consistency:

$$sA\overline{k}^{\alpha} = g\left(\overline{k}, \overline{k}\right)$$

We solve for the recursive competitive equilibrium by the "guess and verify" method. It turns out that the correct guess about value function is

$$v\left(k, \overline{k}\right) = a + b\ln(k + d\overline{k}) + e\ln(\overline{k})$$

To demonstrate why this could even be a correct guess in the first place, we motivate it by value function iteration. We will do two value function iterations, then we can find the form of guess.
We start from $v^0\left(k, \overline{k}\right) = 0$. Plug this into Bellman equation, we get

$$v^1\left(k, \overline{k}\right) = \max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1} k + (1-\alpha) A\overline{k}^{\alpha} - k^{'}\right)$$

Obviously the solution is $k^{'} = 0$. So we get the solution

$$\begin{aligned} v^1\left(k, \overline{k}\right) &= \ln\left(\alpha A\overline{k}^{\alpha-1} k + (1-\alpha) A\overline{k}^{\alpha} - k^{'}\right) \\ &= \ln(\alpha A) + \ln(k + \frac{1-\alpha}{\alpha}\overline{k}) + (\alpha - 1)\ln\left(\overline{k}\right) \end{aligned}$$

Iterate again, the Bellman equation becomes

$$\begin{aligned}
v^2\left(k,\overline{k}\right) &= \max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'\right)+ \\
&\quad \beta\left[\ln(\alpha A)+\ln(k'+\frac{1-\alpha}{\alpha}\overline{k}')+(\alpha-1)\ln\left(\overline{k}'\right)\right] \\
&= \max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'\right)+ \\
&\quad \beta\left[\ln(\alpha A)+\ln(k'+\frac{1-\alpha}{\alpha}sA\overline{k}^{\alpha})+(\alpha-1)\ln\left(sA\overline{k}^{\alpha}\right)\right]
\end{aligned}$$

Its F.O.C. gives us

$$\begin{aligned}
\frac{1}{\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'} &= \frac{\beta}{k'+\frac{1-\alpha}{\alpha}sA\overline{k}^{\alpha}} \\
\Rightarrow\ k' &= \frac{1}{1+\beta}\left[\alpha\beta A\overline{k}^{\alpha-1}k+\left(\beta-\frac{s}{\alpha}\right)(1-\alpha)A\overline{k}^{\alpha}\right]
\end{aligned}$$

So the value function in this iteration becomes

$$\begin{aligned}
v^2\left(k,\overline{k}\right) &= \ln\left(\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'\right)+ \\
&\quad \beta\left[\ln(\alpha A)+\ln(k'+\frac{1-\alpha}{\alpha}sA\overline{k}^{\alpha})+(\alpha-1)\ln\left(sA\overline{k}^{\alpha}\right)\right] \\
&= \ln\left(\frac{1}{1+\beta}\alpha A\overline{k}^{\alpha-1}\left(k+(1+\frac{s}{\alpha})\frac{1-\alpha}{\alpha}\overline{k}\right)\right)+ \\
&\quad \beta\left[\ln(\alpha A)+\ln\left[\frac{1}{1+\beta}\alpha\beta A\overline{k}^{\alpha-1}(k+\left(1+\frac{s}{\alpha}\right)\frac{1-\alpha}{\alpha}\overline{k})\right]+(\alpha-1)\ln\left(sA\overline{k}^{\alpha}\right)\right] \\
&= \left[\ln\left(\frac{\alpha\beta A}{(1+\beta)^2}\right)+(1+\beta)\ln(\alpha A)+(\alpha-1)\ln(sA)\right]+ \\
&\quad (1+\beta)\ln\left(k+(1+\frac{s}{\alpha})\frac{1-\alpha}{\alpha}\overline{k}\right)+(\alpha-1)(1+\beta+\alpha\beta)\ln\left(\overline{k}\right)
\end{aligned}$$

Now it is easy to see that an educated guess should be in the form $v\left(k,\overline{k}\right) = a+b\ln(k+d\overline{k})+e\ln(\overline{k})$. In theory we can iterate this process until convergence. But here an easier way is to do "guess and check". Now we start to solve for the solution by plugging in our conjecture. Bellman equation becomes

$$\begin{aligned}
v\left(k,\overline{k}\right) &= \max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'\right) \\
&\quad +\beta\left(a+b\ln(k'+d\overline{k}')+e\ln(\overline{k}')\right) \\
&\quad s.t. \\
&\quad \overline{k}'=sA\overline{k}^{\alpha}
\end{aligned}$$

or

$$\begin{aligned}
v\left(k,\overline{k}\right) &= \max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k + (1-\alpha)A\overline{k}^{\alpha} - k'\right) \\
&\quad +\beta\left(a + b\ln(k' + dsA\overline{k}^{\alpha}) + e\ln(sA\overline{k}^{\alpha})\right)
\end{aligned}$$

Take F.O.C. with respect to k', we have

$$\begin{aligned}
\frac{1}{\alpha A\overline{k}^{\alpha-1}k + (1-\alpha)A\overline{k}^{\alpha} - k'} &= \frac{\beta b}{k' + dsA\overline{k}^{\alpha}} \\
\Rightarrow\ k' + dsA\overline{k}^{\alpha} &= \beta b\left(\alpha A\overline{k}^{\alpha-1}k + (1-\alpha)A\overline{k}^{\alpha} - k'\right) \\
\Rightarrow\ k' &= \frac{\beta b}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k + \frac{\beta b(1-\alpha) - ds}{1+\beta b}A\overline{k}^{\alpha}
\end{aligned}$$

Plug back into Bellman equation, we have

$$\begin{aligned}
v\left(k,\overline{k}\right) &\equiv \ln\left(\frac{1}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k + \frac{(1-\alpha)+ds}{1+\beta b}A\overline{k}^{\alpha}\right) \\
&\quad +\beta\left(a + b\ln\left(\frac{\beta b}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k + \frac{\beta b(1-\alpha+ds)}{1+\beta b}A\overline{k}^{\alpha}\right) + e\ln(sA\overline{k}^{\alpha})\right) \\
&\equiv \left[(1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right) + \beta a + \beta b\ln(\beta b) + \beta e\ln(sA)\right] + \\
&\quad (1+\beta b)\ln\left(k + \frac{(1-\alpha)+ds}{\alpha}\overline{k}\right) + \left[(1+\beta b)(\alpha-1) + \alpha\beta e\right]\ln\left(\overline{k}\right)
\end{aligned}$$

Since the $LHS = a + b\ln(k + d\overline{k}) + e\ln(\overline{k})$, we have the following equations:

$$\begin{aligned}
(1)\ a &= (1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right) + \beta a + \beta b\ln(\beta b) + \beta e\ln(sA) \\
(2)\ b &= (1+\beta b) \\
(3)\ d &= \frac{(1-\alpha)+ds}{\alpha} \\
(4)\ e &= (1+\beta b)(\alpha-1) + \alpha\beta e
\end{aligned}$$

Solve for these four equations, we get the solutions

$$\begin{aligned}
a &= \frac{1}{(1-\beta)^2}\left[\ln\left(\alpha A(1-\beta)\right) + \beta\ln\left(\frac{\beta}{1-\beta}\right) + \frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(sA)\right] \\
b &= \frac{1}{1-\beta} \\
d &= \frac{1-\alpha}{\alpha-s} \\
e &= \frac{\alpha-1}{(1-\alpha\beta)(1-\beta)}
\end{aligned}$$

This gives us the form of value function. Plug these coefficients into F.O.C., we get

$$\begin{aligned} k' &= \frac{\beta b}{1+\beta b}\alpha A\bar{k}^{\alpha-1}k + \frac{\beta b(1-\alpha)-ds}{1+\beta b}A\bar{k}^{\alpha} \\ &= \alpha\beta A\bar{k}^{\alpha-1}k + \frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\bar{k}^{\alpha} \end{aligned}$$

Therefore, we have our value function and policy function as

$$\begin{aligned} v\left(k,\bar{k}\right) &= a + b\ln(k+d\bar{k}) + e\ln(\bar{k}) \\ g(k,\bar{k}) &= \alpha\beta A\bar{k}^{\alpha-1}k + \frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\bar{k}^{\alpha} \end{aligned}$$

with coefficients as defined above.

(b) Find the competitive equilibrium value of s by imposing the consistency condition $G(k) = g(\bar{k},\bar{k})$. Verify that the resulting law of motion for aggregate capital solves the planning problem for this economy. Display v and g for the equilibrium value of s.

Imposing consistency condition, we have

$$\begin{aligned} & G(\bar{k}) = g(\bar{k},\bar{k}) \\ \Rightarrow\ & sA\bar{k}^{\alpha} = \alpha\beta A\bar{k}^{\alpha-1}\bar{k} + \frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\bar{k}^{\alpha} \\ \Rightarrow\ & (\alpha-s)s = (\alpha-s)\alpha\beta + (1-\alpha)(\alpha\beta-s) \\ \Rightarrow\ & s^2 - (1+\alpha\beta)s + \alpha\beta = 0 \\ \Rightarrow\ & s_1 = \alpha\beta \text{ and } s_2 = 1 \end{aligned}$$

Consider the context of this economic problem, obviously only $s = \alpha\beta$ will be a solution to this problem. Consequently the resulting recursive competitive equilibrium is

$$\begin{aligned} v\left(k,\bar{k}\right) &= a + b\ln(k+d\bar{k}) + e\ln(\bar{k}) \\ g(k,\bar{k}) &= \alpha\beta A\bar{k}^{\alpha-1}k \\ G(\bar{k}) &= \alpha\beta A\bar{k}^{\alpha} \end{aligned}$$

with coefficients as

$$\begin{aligned} a &= \frac{1}{(1-\beta)^2}\left[\ln(\alpha A(1-\beta)) + \beta\ln\left(\frac{\beta}{1-\beta}\right) + \frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right] \\ b &= \frac{1}{1-\beta} \\ d &= \frac{1-\alpha}{\alpha(1-\beta)} \\ e &= \frac{\alpha-1}{(1-\alpha\beta)(1-\beta)} \end{aligned}$$

Now let's check whether $G(\overline{k})$ solves the central planning problem.
The recursive formulation of central planning problem is

$$v\left(\overline{k}\right) = \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}^{'}\right) + \beta v\left(\overline{k}^{'}\right)$$

Guess that $v\left(\overline{k}\right) = a + b\ln\overline{k}$. Plug into Bellman equation, we have

$$v\left(\overline{k}\right) = \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}^{'}\right) + \beta\left(a + b\ln\overline{k}^{'}\right)$$

F.O.C. is

$$\begin{aligned} \frac{1}{A\overline{k}^{\alpha} - \overline{k}^{'}} &= \beta b\frac{1}{\overline{k}^{'}} \\ \Rightarrow \quad \overline{k}^{'} &= \frac{\beta b}{1+\beta b}A\overline{k}^{\alpha} \end{aligned}$$

Plug back into Bellman equation, we have

$$\begin{aligned} v\left(\overline{k}\right) &= \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}^{'}\right) + \beta\left(a + b\ln\overline{k}^{'}\right) \\ \Rightarrow \quad a + b\ln\overline{k} &= \ln\left(A\overline{k}^{\alpha} - \frac{\beta b}{1+\beta b}A\overline{k}^{\alpha}\right) + \beta\left(a + b\ln\left(\frac{\beta b}{1+\beta b}A\overline{k}^{\alpha}\right)\right) \\ \Rightarrow \quad a + b\ln\overline{k} &= \left[(1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b\ln(\frac{\beta b}{1+\beta b})\right] + \alpha\left(1+\beta b\right)\ln\overline{k} \end{aligned}$$

Again since this is an identity, we have

$$a = (1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b\ln(\frac{\beta b}{1+\beta b})$$

and

$$b = \alpha\left(1+\beta b\right)$$

Solve for this, we get

$$\begin{aligned} a &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta\right) \\ b &= \frac{\alpha}{1-\alpha\beta} \end{aligned}$$

Therefore, the solution for central planning problem is

$$\begin{aligned} v\left(\overline{k}\right) &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta\right) + \frac{\alpha}{1-\alpha\beta}\ln\overline{k} \\ \overline{k}^{'} &= \alpha\beta A\overline{k}^{\alpha} \end{aligned}$$

This is the same aggregate state evolution law as in the recursive competitive equilibrium.

(c) How does an increase in aggregate capital affect the savings behavior and the (indirect) utility of a typical consumer (holding fixed the consumer's holdings of capital)?

Take derivative with respect to $\overline{k}$, we have

$$\begin{aligned} & g\left(k,\overline{k}\right) = \alpha\beta A\overline{k}^{\alpha-1}k \\ \Rightarrow \ & g_2\left(k,\overline{k}\right) = \alpha\beta\left(\alpha-1\right)A\overline{k}^{\alpha-2}k \\ < \ & 0 \end{aligned}$$

and

$$\begin{aligned} & v\left(k,\overline{k}\right) = a + b\ln(k+d\overline{k}) + e\ln(\overline{k}) \\ \Rightarrow \ & v_2\left(k,\overline{k}\right) = \frac{bd}{k+d\overline{k}} + \frac{e}{\overline{k}} \\ \Rightarrow \ & v_2\left(k,\overline{k}\right) = \frac{\alpha(1-\alpha)(\overline{k}-k)}{\left(\alpha\left(1-\beta\right)k+\left(1-\alpha\right)\overline{k}\right)\left(1-\alpha\beta\right)\overline{k}} \\ \Rightarrow \ & v_2\left(k,\overline{k}\right)\begin{Bmatrix} >0 \ if \ \overline{k}>k \\ =0 \ if \ \overline{k}=k \\ <0 \ if \ \overline{k}<k \end{Bmatrix} \end{aligned}$$

Note that here we do not consider equilibrium behavior, therefore the sign is ambiguous.

(d) How does the equilibrium utility of a typical consumer vary with aggregate capital (taking into account that the consumer's holdings of capital equal aggregate capital in equilibrium)?

Take derivative with respect to $\overline{k}$, we have

$$\begin{aligned} & v\left(\overline{k},\overline{k}\right) = a + b\ln\left(1+d\right) + \left(b+e\right)\ln(\overline{k}) \\ \Rightarrow \ & \frac{dV}{d\overline{k}} = \frac{b+e}{\overline{k}} \\ \Rightarrow \ & \frac{dV}{d\overline{k}} = \frac{\alpha}{1-\alpha\beta}\frac{1}{\overline{k}} > 0 \end{aligned}$$