Econ 510a (second half)
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Solutions for Homework #4

Question 1

(a) A Recursive Competitive Equilibrium for the economy is a set of functions:

$$\begin{aligned} \textit{price function} &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ \textit{policy function} &: k' = g\left(k, \overline{k}\right) \\ \textit{value function} &: v\left(k, \overline{k}\right) \\ \textit{transition function} &: \overline{k}' = H\left(\overline{k}\right) \end{aligned}$$

such that:
(1) $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) = & \max_{\{c,k'\}} u(c) + \beta v\left(k', \overline{k}'\right) \\ & s.t. \\ & c + k' = r\left(\overline{k}\right)k + (1-\delta)k + w\left(\overline{k}\right) \\ & \overline{k}' = H\left(\overline{k}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= F_1\left(\overline{k}, 1, \overline{k}\right) = \alpha A \overline{k}^{\alpha+\gamma-1} \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1, \overline{k}\right) = (1-\alpha) A \overline{k}^{\alpha+\gamma} \end{aligned}$$

(3) Consistency:

$$H\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

Solve for consumer's problem in the normal way, we get the Euler equation:

$$\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(r\left(\overline{k}_{t+1}\right)+1-\delta\right) = 1 \Leftrightarrow$$
$$\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(\alpha A\overline{k}_{t+1}^{\alpha+\gamma-1}+1-\delta\right) = 1$$

Aggregate production is equal to $G\left(\overline{k},l\right) \equiv F\left(\overline{k},l,\overline{k}\right) = A\overline{k}^{a+\gamma}l^{1-a}$ (which we get if we impose the equilibrium condition $k = \overline{k}$) and thus aggregate consumption in every period is equal to:

$$\overline{c}_t = G\left(\overline{k}_t,l\right)+(1-\delta)\overline{k}_t-\overline{k}_{t+1}$$

Since all agents are identical, in equilibrium it will be the case that $\overline{c}_t = c_t$ for every t (you can think of this as each individual's consumption will equal per capita consumption).

If we substitute this in the individual's Euler equation we derived above, we get:

$$\frac{\beta u'(\overline{c}')}{u'(\overline{c})}\left(\alpha A\overline{k}'^{\alpha+\gamma-1}+1-\delta\right) = 1 \Leftrightarrow$$
$$\frac{\beta u'\left(G\left(\overline{k}',l\right)+(1-\delta)\overline{k}'-\overline{k}''\right)}{u'\left(G\left(\overline{k},l\right)+(1-\delta)\overline{k}-\overline{k}'\right)}\left(\alpha A\overline{k}'^{\alpha+\gamma-1}+1-\delta\right) = 1 \Leftrightarrow$$
$$\beta u'\left(G\left(\overline{k}',l\right)+(1-\delta)\overline{k}'-\overline{k}''\right)\left(\alpha A\overline{k}'^{\alpha+\gamma-1}+1-\delta\right) = u'\left(G\left(\overline{k},l\right)+(1-\delta)\overline{k}-\overline{k}'\right)$$

which is a second-order difference equation that governs the evolution of the economy's aggregates.

(b) The recursive formulation of the planning problem is

$$\begin{aligned} v\left(\overline{k}\right) &= \max_{\{c,k'\}} u(c)+\beta v\left(\overline{k}'\right) \\ & s.t. \\ & c+k' = A\overline{k}^{\alpha+\gamma}+(1-\delta)\overline{k} \end{aligned}$$

The Euler equation is

$$\frac{\beta u'(\overline{c}')}{u'(\overline{c})}\left(A(\alpha+\gamma)\overline{k}'^{\alpha+\gamma-1}+1-\delta\right) = 1$$

If we compare the above Euler equation to the one we found in the competitive equilibrium case we can see that they are different. Since planner's problem gives us the Pareto optimal allocation, the equilibrium in the competitive case cannot be Pareto optimal. The intuition behind this result is that the firms in the competitive equilibrium do not internalize the externality in the production, whereas the planner is able to do so.

(c) A Recursive Competitive Equilibrium for the economy with taxation is a set of functions:

$$\begin{aligned} \textit{price function} &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ \textit{policy function} &: k' = g\left(k, \overline{k}\right) \\ \textit{value function} &: v\left(k, \overline{k}\right) \\ \textit{taxation function} &: T\left(\overline{k}\right) \\ \textit{transition function} &: \overline{k}' = H\left(\overline{k}\right) \end{aligned}$$

such that:
(1) $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) = &\max_{\{c,k'\}} u(c) + \beta v\left(k', \overline{k}'\right) \\ &s.t. \\ &c + (1-\tau)(k' - (1-\delta)k) = r\left(\overline{k}\right)k + w\left(\overline{k}\right) - T\left(\overline{k}\right) \\ &\overline{k}' = H\left(\overline{k}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} R\left(\overline{k}\right) &= F_1\left(\overline{k}, 1, \overline{k}\right) = \alpha A \overline{k}^{\alpha+\gamma-1} \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1, \overline{k}\right) = (1-\alpha) A \overline{k}^{\alpha+\gamma} \end{aligned}$$

(3) Government balances budget in each period:

$$T\left(\overline{k}\right) = \tau\left(\overline{k}' - (1-\delta)\overline{k}\right)$$

(3) Consistency of transition function:

$$H\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

(d) We can solve for F.O.C. in the normal way as

$$(1-\tau)u'(c) = \beta v_1\left(k', \overline{k}'\right)$$

Envelope condition gives us

$$v_1\left(k, \overline{k}\right) = u'(c)\left(r\left(\overline{k}\right) + (1-\tau)(1-\delta)\right)$$

The Euler equation is

$$\begin{aligned} &\frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{1}{1-\tau}\left(r\left(\overline{k}\right) + (1-\tau)(1-\delta)\right) = 1 \\ \Rightarrow\ &\frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{1}{1-\tau}\left(\alpha A \overline{k}^{\alpha+\gamma-1} + (1-\tau)(1-\delta)\right) = 1 \end{aligned}$$

In steady state, we have

$$\frac{\beta}{1-\tau}\left(\alpha A\overline{k}^{\alpha+\gamma-1} + (1-\tau)(1-\delta)\right) = 1$$

$$\Rightarrow \alpha A\overline{k}^{\alpha+\gamma-1} = (1-\tau)\left(\frac{1}{\beta} - (1-\delta)\right)$$

$$\Rightarrow \overline{k}^{\tau} = \left(\frac{(1-\tau)\left(\frac{1}{\beta} - (1-\delta)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}$$

where $\overline{k}^{\tau}$ represents the steady-state aggregate capital stock in economy with taxation.
From the Euler equation of the planner's problem we can find the steady state:

$$A(\alpha+\gamma)\left(\overline{k}^{o}\right)^{\alpha+\gamma-1} + 1 - \delta = \frac{1}{\beta}$$

$$\Rightarrow \overline{k}^{o} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{(\alpha+\gamma)A}\right)^{\frac{1}{\alpha+\gamma-1}}$$

where $\overline{k}^{o}$ represents the optimal steady-state aggregate capital stock.
Now set $\overline{k}^{\tau} = \overline{k}^{o}$, we have

$$\overline{k}^{\tau} = \overline{k}^{o}$$

$$\Rightarrow \left(\frac{(1-\tau)\left(\frac{1}{\beta} - (1-\delta)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{(\alpha+\gamma)A}\right)^{\frac{1}{\alpha+\gamma-1}}$$

$$\Rightarrow \frac{1-\tau}{\alpha} = \frac{1}{\alpha+\gamma}$$

$$\Rightarrow \tau = \frac{\gamma}{\alpha+\gamma}$$

Therefore, for $\tau = \frac{\gamma}{\alpha+\gamma}$ the competitive equilibrium steady-state aggregate capital stock the same as the steady-state aggregate capital stock in the planning problem.

Question 2

(a) Given the equivalence of competitive equilibrium and central planning problem here, we can conjecture the transition function of aggregate state as $\overline{k}' = sA\overline{k}^{\alpha}$. Now we can define a recursive competitive equilibrium for this economy explicitly.

A Recursive Competitive Equilibrium for this economy is a set of functions:

$$
\begin{aligned}
\textit{price function} &: r(\overline{k}), w(\overline{k}) \\
\textit{policy function} &: k' = g(k, \overline{k}) \\
\textit{value function} &: v(k, \overline{k}) \\
\textit{transition function} &: \overline{k}' = sA\overline{k}^{\alpha}
\end{aligned}
$$

such that:
(1) $k' = g(k, \overline{k})$ and $v(k, \overline{k})$ solves consumer's problem:

$$
\begin{aligned}
v(k, \overline{k}) = &\max_{\{c,k'\}} \ln(c) + \beta v(k', \overline{k}') \\
&s.t. \\
&c + k' = r(\overline{k})k + w(\overline{k}) \\
&\overline{k}' = sA\overline{k}^{\alpha}
\end{aligned}
$$

(2) Price is competitively determined:

$$
\begin{aligned}
r(\overline{k}) &= F_1(\overline{k}, 1) = \alpha A\overline{k}^{\alpha-1} \\
w(\overline{k}) &= F_2(\overline{k}, 1) = (1-\alpha)A\overline{k}^{\alpha}
\end{aligned}
$$

(3) Consistency:

$$
sA\overline{k}^{\alpha} = g(\overline{k}, \overline{k})
$$

We solve for the recursive competitive equilibrium by the "guess and verify" method. We conjecture that the value function is of the following form:

$$
v(k, \overline{k}) = a + b\ln(k + d\overline{k}) + e\ln(\overline{k})
$$

Now we start to solve for the solution by plugging in our conjecture. Bellman equation becomes

$$
\begin{aligned}
v(k, \overline{k}) = &\max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k + (1-\alpha)A\overline{k}^{\alpha} - k'\right) \\
&+ \beta\left(a + b\ln(k' + d\overline{k}') + e\ln(\overline{k}')\right) \\
&s.t. \\
&\overline{k}' = sA\overline{k}^{\alpha}
\end{aligned}
$$

or

$$
\begin{aligned}
v(k, \overline{k}) = &\max_{\{k'\}} \ln\left(\alpha A\overline{k}^{\alpha-1}k + (1-\alpha)A\overline{k}^{\alpha} - k'\right) \\
&+ \beta\left(a + b\ln(k' + dsA\overline{k}^{\alpha}) + e\ln(sA\overline{k}^{\alpha})\right)
\end{aligned}
$$

Take F.O.C. with respect to k', we have

$$\frac{1}{\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'} = \frac{\beta b}{k'+dsA\overline{k}^{\alpha}}$$

$$\Rightarrow k'+dsA\overline{k}^{\alpha} = \beta b\left(\alpha A\overline{k}^{\alpha-1}k+(1-\alpha)A\overline{k}^{\alpha}-k'\right)$$

$$\Rightarrow k' = \frac{\beta b}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k+\frac{\beta b(1-\alpha)-ds}{1+\beta b}A\overline{k}^{\alpha}$$

Plug back into Bellman equation, we have

$$\begin{aligned}
v\left(k,\overline{k}\right) &\equiv \ln\left(\frac{1}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k+\frac{(1-\alpha)+ds}{1+\beta b}A\overline{k}^{\alpha}\right) \\
&\quad +\beta\left(a+b\ln(\frac{\beta b}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k+\frac{\beta b(1-\alpha+ds)}{1+\beta b}A\overline{k}^{\alpha})+e\ln(sA\overline{k}^{\alpha})\right) \\
&\equiv \left[(1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right)+\beta a+\beta b\ln(\beta b)+\beta e\ln(sA)\right]+ \\
&\quad (1+\beta b)\ln\left(k+\frac{(1-\alpha)+ds}{\alpha}\overline{k}\right)+[(1+\beta b)(\alpha-1)+\alpha\beta e]\ln\left(\overline{k}\right)
\end{aligned}$$

Since the $LHS = a+b\ln(k+d\overline{k})+e\ln(\overline{k})$, we have the following equations:

$$(1)\ a = (1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right)+\beta a+\beta b\ln(\beta b)+\beta e\ln(sA)$$

$$(2)\ b = (1+\beta b)$$

$$(3)\ d = \frac{(1-\alpha)+ds}{\alpha}$$

$$(4)\ e = (1+\beta b)(\alpha-1)+\alpha\beta e$$

Solve for these four equations, we get the solutions

$$a = \frac{1}{(1-\beta)^2}\left[\ln(\alpha A(1-\beta))+\beta\ln\left(\frac{\beta}{1-\beta}\right)+\frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(sA)\right]$$

$$b = \frac{1}{1-\beta}$$

$$d = \frac{1-\alpha}{\alpha-s}$$

$$e = \frac{\alpha-1}{(1-\alpha\beta)(1-\beta)}$$

This gives us the form of value function. Plug these coefficients into F.O.C., we get

$$\begin{aligned}
k' &= \frac{\beta b}{1+\beta b}\alpha A\overline{k}^{\alpha-1}k+\frac{\beta b(1-\alpha)-ds}{1+\beta b}A\overline{k}^{\alpha} \\
&= \alpha\beta A\overline{k}^{\alpha-1}k+\frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\overline{k}^{\alpha}
\end{aligned}$$

Therefore, we have our value function and policy function as

$$v\left(k,\overline{k}\right) = a + b\ln(k + d\overline{k}) + e\ln(\overline{k})$$
$$g(k,\overline{k}) = \alpha\beta A\overline{k}^{\alpha-1}k + \frac{(1-\alpha)(\alpha\beta - s)}{\alpha - s}A\overline{k}^{\alpha}$$

with coefficients as defined above.

(b) Imposing consistency condition, we have

$$\begin{aligned}
& G(\overline{k}) = g(\overline{k},\overline{k}) \\
\Rightarrow\ & sA\overline{k}^{\alpha} = \alpha\beta A\overline{k}^{\alpha-1}\overline{k} + \frac{(1-\alpha)(\alpha\beta - s)}{\alpha - s}A\overline{k}^{\alpha} \\
\Rightarrow\ & (\alpha - s)s = (\alpha - s)\alpha\beta + (1-\alpha)(\alpha\beta - s) \\
\Rightarrow\ & s^2 - (1+\alpha\beta)s + \alpha\beta = 0 \\
\Rightarrow\ & s_1 = \alpha\beta \text{ and } s_2 = 1
\end{aligned}$$

Consider the context of this economic problem, obviously only $s = \alpha\beta$ will be a solution to this problem. Consequently the resulting recursive competitive equilibrium is

$$\begin{aligned}
v\left(k,\overline{k}\right) &= a + b\ln(k + d\overline{k}) + e\ln(\overline{k}) \\
g(k,\overline{k}) &= \alpha\beta A\overline{k}^{\alpha-1}k \\
G(\overline{k}) &= \alpha\beta A\overline{k}^{\alpha}
\end{aligned}$$

with coefficients as

$$\begin{aligned}
a &= \frac{1}{(1-\beta)^2}\left[\ln(\alpha A(1-\beta)) + \beta\ln\left(\frac{\beta}{1-\beta}\right) + \frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right] \\
b &= \frac{1}{1-\beta} \\
d &= \frac{1-\alpha}{\alpha(1-\beta)} \\
e &= \frac{\alpha - 1}{(1-\alpha\beta)(1-\beta)}
\end{aligned}$$

Now let's check whether $G(\overline{k})$ solves the central planning problem.
The recursive formulation of central planning problem is

$$v\left(\overline{k}\right) = \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}'\right) + \beta v\left(\overline{k}'\right)$$

Guess that $v\left(\overline{k}\right) = a + b\ln\overline{k}$. Plug into Bellman equation, we have

$$v\left(\overline{k}\right) = \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}'\right) + \beta\left(a + b\ln\overline{k}'\right)$$

F.O.C. is

$$
\begin{aligned}
\frac{1}{A\overline{k}^{\alpha} - \overline{k}'} &= \beta b \frac{1}{\overline{k}'} \\
\Rightarrow \overline{k}' &= \frac{\beta b}{1+\beta b} A \overline{k}^{\alpha}
\end{aligned}
$$

Plug back into Bellman equation, we have

$$
\begin{aligned}
& v(\overline{k}) = \max_{k'} \ln\left(A\overline{k}^{\alpha} - \overline{k}'\right) + \beta\left(a + b\ln \overline{k}'\right) \\
\Rightarrow\ & a + b\ln \overline{k} = \ln\left(A\overline{k}^{\alpha} - \frac{\beta b}{1+\beta b}A\overline{k}^{\alpha}\right) + \beta\left(a + b\ln\left(\frac{\beta b}{1+\beta b}A\overline{k}^{\alpha}\right)\right) \\
\Rightarrow\ & a + b\ln \overline{k} = \left[(1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b \ln\left(\frac{\beta b}{1+\beta b}\right)\right] + \alpha(1+\beta b)\ln \overline{k}
\end{aligned}
$$

Again since this is an identity, we have

$$
a = (1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b \ln\left(\frac{\beta b}{1+\beta b}\right)
$$

and

$$
b = \alpha(1+\beta b)
$$

Solve for this, we get

$$
\begin{aligned}
a &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta\right) \\
b &= \frac{\alpha}{1-\alpha\beta}
\end{aligned}
$$

Therefore, the solution for central planning problem is

$$
\begin{aligned}
v(\overline{k}) &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta\right) + \frac{\alpha}{1-\alpha\beta}\ln \overline{k} \\
\overline{k}' &= \alpha\beta A \overline{k}^{\alpha}
\end{aligned}
$$

This is the same aggregate state evolution law as in the recursive competitive equilibrium.
(c) Take derivative with respect to $\overline{k}$, we have

$$
\begin{aligned}
& g(k, \overline{k}) = \alpha\beta A \overline{k}^{\alpha-1} k \\
\Rightarrow\ & g_2(k, \overline{k}) = \alpha\beta(\alpha-1) A \overline{k}^{\alpha-2} k \\
& < 0
\end{aligned}
$$

and

$$
\begin{aligned}
v(k,\overline{k}) &= a + b\ln(k + d\overline{k}) + e\ln(\overline{k}) \\
\Rightarrow v_2(k,\overline{k}) &= \frac{bd}{k + d\overline{k}} + \frac{e}{\overline{k}} \\
\Rightarrow v_2(k,\overline{k}) &= \frac{\alpha(1-\alpha)(\overline{k}-k)}{\left(\alpha(1-\beta)k + (1-\alpha)\overline{k}\right)(1-\alpha\beta)\overline{k}} \\
\Rightarrow v_2(k,\overline{k}) &\left\{ \begin{array}{l} > 0 \ if\ \overline{k} > k \\ = 0 \ if\ \overline{k} = k \\ < 0 \ if\ \overline{k} < k \end{array} \right\}
\end{aligned}
$$

Note that here we do not consider equilibrium behavior, therefore the sign is ambiguous.

(d) Take derivative with respect to $\overline{k}$, we have

$$
\begin{aligned}
v(\overline{k},\overline{k}) &= a + b\ln(1+d) + (b+e)\ln(\overline{k}) \\
\Rightarrow \frac{dV}{d\overline{k}} &= \frac{b+e}{\overline{k}} \\
\Rightarrow \frac{dV}{d\overline{k}} &= \frac{\alpha}{1-\alpha\beta}\frac{1}{\overline{k}} > 0
\end{aligned}
$$

Question 3

(a) Given the transition matrix

$$
P = \begin{pmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{pmatrix}
$$

we can calculate the stationary distribution π according to the formula

$$
\begin{aligned}
\pi' &= \pi' P \\
\Rightarrow &\left\{ \begin{array}{l} \pi_1 = 0.95\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.05\pi_1 + 0.9\pi_2 \end{array} \right.
\end{aligned}
$$

The solution to this equation is

$$
\pi_1 = 2\pi_2
$$

Imposing the condition that $\pi_1 + \pi_2 = 1$, the solution is

$$\begin{cases} \pi_1 = \frac{2}{3} \\ \pi_2 = \frac{1}{3} \end{cases}$$

Correspondingly, the long run expected value is

$$E(z) = \pi' z = \frac{2}{3} \times 0.9 + \frac{1}{3} \times 1.1 = \frac{29}{30}$$

(b) Form the dynamic programming problem as

$$v\left(\overline{k}_i, z_i\right) = \max_{\overline{k}' \in \{k_1, k_2\}} \ln\left(z_i \overline{k}_i^{\alpha} - \overline{k}'\right) + \beta\left(p_{i1} v\left(\overline{k}', z_1\right) + p_{i2} v\left(\overline{k}', z_2\right)\right)$$

Since (k_i, z_i) takes on only 4 values, we assume that the policy function takes the following form.

$$\begin{aligned} g(k_1, z_1) &= k_1 \\ g(k_2, z_1) &= k_1 \\ g(k_1, z_2) &= k_2 \\ g(k_2, z_2) &= k_2 \end{aligned}$$

We need to prove that this is true. The way we are going to proceed is that we are going to calculate the value function values associated with this policy function and then verify that they are indeed maximum.

Using the parameter values given in the problem we get:

$$\begin{aligned} k_{ss} &= 0.1719 \\ k_1 &= 0.1633 \\ k_2 &= 0.18047 \end{aligned}$$

Let's try out all 4 cases and substitute in for the assumed policy function. For

$(k_i, z_i) = (k_1, z_1)$ we have:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_1\right) + \beta\left(p_{11} v\left(\overline{k}_1, z_1\right) + p_{12} v\left(\overline{k}_1, z_2\right)\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= \ln(0.9 * 0.1633^{0.36} - 0.1633) + 0.9\left(0.95 v\left(\overline{k}_1, \overline{z}_1\right) + 0.05 v\left(\overline{k}_1, \overline{z}_2\right)\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= -1.1861 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= -8.18 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right) \end{aligned}$$

Similarly we get:

$$
\begin{aligned}
v(\bar{k}_1, \bar{z}_2) &= -0.93543 + 0.855v(\bar{k}_2, \bar{z}_1) + 0.045v(\bar{k}_2, \bar{z}_2) \\
v(\bar{k}_2, \bar{z}_1) &= -1.13134 + 0.855v(\bar{k}_1, \bar{z}_1) + 0.045v(\bar{k}_1, \bar{z}_2) \\
v(\bar{k}_2, \bar{z}_2) &= -0.8833 + 0.855v(\bar{k}_2, \bar{z}_1) + 0.045v(\bar{k}_2, \bar{z}_2) \Leftrightarrow \\
v(\bar{k}_2, \bar{z}_2) &= -0.9249 + 0.8953v(\bar{k}_2, \bar{z}_1)
\end{aligned}
$$

Essentially we have a system of 4 equations with 4 unknowns. Plugging in the first equation into the third we get:

$$v(\bar{k}_2, \bar{z}_1) = -8.12524 + 0.3103v(\bar{k}_1, \bar{z}_2)$$

Similarly, plugging in the last equation into the second one we get:

$$v(\bar{k}_1, \bar{z}_2) = 0.89381 + 0.89529v(\bar{k}_2, \bar{z}_1)$$

Solving out the above system of 2 equations and 2 unknowns, we get:

$$
\begin{aligned}
v(\bar{k}_2, \bar{z}_1) &= -10.8668 \\
v(\bar{k}_1, \bar{z}_2) &= -8.8351
\end{aligned}
$$

ubstituting into equations 1 and 4 from above we get:

$$
\begin{aligned}
v(\bar{k}_1, \bar{z}_1) &= -10.9215 \\
v(\bar{k}_2, \bar{z}_2) &= -10.6539
\end{aligned}
$$

We now need to check that this decision rule is optimal. We will go about checking this changing the decision rule in each of the 4 cases and then calculating the resulting value function. For example, when we have $(\bar{k}_1, \bar{z}_1)$ we will assume that the planner chooses $\bar{k}_2$ instead of $\bar{k}_1$. In that case we would get:

$$
\begin{aligned}
v^{alt}(\bar{k}_1, \bar{z}_1) &= \ln(z_1 \bar{k}_1^{\alpha} - \bar{k}_2) + \beta(p_{11}v(\bar{k}_2, z_1) + p_{12}v(\bar{k}_2, z_2)) \Rightarrow \\
v^{alt}(\bar{k}_1, \bar{z}_1) &= \ln(0.9 * 0.1633^{0.36} - 0.18047) + 0.9(0.95 * (-10.8668) + 0.05 * (-10.6539)) \Leftrightarrow \\
v^{alt}(\bar{k}_1, \bar{z}_1) &= -11.0145 < -10.9215 = v(\bar{k}_1, \bar{z}_1)
\end{aligned}
$$

Similarly we get:

$$
\begin{aligned}
v^{alt}(\bar{k}_1, \bar{z}_2) &= -10.7126 < -8.8351 = v(\bar{k}_1, \bar{z}_2) \\
v^{alt}(\bar{k}_2, \bar{z}_1) &= -11.4859 < -10.8668 = v(\bar{k}_2, \bar{z}_1) \\
v^{alt}(\bar{k}_2, \bar{z}_2) &= -10.7581 < -10.6539 = v(\bar{k}_2, \bar{z}_2)
\end{aligned}
$$

and we see that in each case the original decision rule performs better. The rational we use to conclude that our decision rule is indeed optimal is the following: our guess about the decision rule implies a certain value for our value function. On the other hand changing the decision rule with the only other possible alternative for each case would imply a different value function. If this alternative value function was actually closer to the true value function (which is the optimal) then it would give a higher value than our guess. But that is not the case. Therefore our guess is closer to the optimal. You can think of this procedure as a value function iteration. Given however that for each decision rule there is only one other alternative, this implies that out value function is indeed optimal and that our guess is indeed the correct decision rule.

(c) Based on policy function $g(k_i, z_i)$ and transition matrix of z, the pair (k, z) follows a Markov process with the transition matrix

	(z_1, k_1)	(z_1, k_2)	(z_2, k_1)	(z_2, k_2)
(z_1, k_1)	0.95	0	0.05	0
(z_1, k_2)	0.95	0	0.05	0
(z_2, k_1)	0	0.1	0	0.9
(z_2, k_2)	0	0.1	0	0.9

Now we can calculate the stationary distribution either on the computer or by hand. The result is

$$p(k_1, z_1) = \frac{19}{30} = 0.63333$$
$$p(k_2, z_1) = \frac{1}{30} = 0.033333$$
$$p(k_1, z_2) = \frac{1}{30} = 0.033333$$
$$p(k_2, z_2) = \frac{3}{10} = 0.3$$

Using the stationary distribution, the long-run (or unconditional) expected values of the capital stock and of output are

$$Ek = p(k_1, z_1)k_1 + p(k_1, z_2)k_1 + p(k_2, z_1)k_2 + p(k_2, z_2)k_2 = \frac{2}{3}k_1 + \frac{1}{3}k_2 = 0.1747$$
$$Ey = p(k_1, z_1) * z_1 k_1^a + p(k_2, z_1) * z_1 k_2^a + p(k_1, z_2) * z_2 k_1^a + p(k_2, z_2) * z_2 k_2^a = 0.51031$$

(d) Depending on the realization of each simulation, the result will differ a little bit. For example, one possible result based on $T = 10000$ is

$$\frac{1}{T} \sum_{t=1}^{T} k_t = 0.1747$$
$$\frac{1}{T} \sum_{t=1}^{T} y_t = 0.4687$$