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Suggested Solutions	: Problem Set 4	

1. (a) Given the equivalence of competitive equilibrium and central planning problem here, we can conjecture the transition function of aggregate state as $\bar{k}' = sA\bar{k}^{\alpha}$. Now we can define a recursive competitive equilibrium for this economy explicitly.

A Recursive Competitive Equilibrium for this economy is a set of functions:

$$\begin{aligned} price\ function &: \ r\left(\bar{k}\right), w\left(\bar{k}\right) \\ policy\ function &: \ k' = g\left(k, \bar{k}\right) \\ value\ function &: \ v\left(k, \bar{k}\right) \\ transition\ function &: \ \bar{k}' = sA\bar{k}^{\alpha} \end{aligned}$$

such that:

(1) $k' = g\left(k, \bar{k}\right)$ and $v\left(k, \bar{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \bar{k}\right) &= \max_{\{c,k'\}} \ln(c) + \beta v\left(k', \bar{k}'\right) \\ & \quad s.t. \\ & \quad c + k' = r\left(\bar{k}\right)k + w\left(\bar{k}\right) \\ & \quad \bar{k}' = sA\bar{k}^{\alpha} \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\bar{k}\right) &= F_1\left(\bar{k}, 1\right) = \alpha A\bar{k}^{\alpha-1} \\ w\left(\bar{k}\right) &= F_2\left(\bar{k}, 1\right) = (1-\alpha)A\bar{k}^{\alpha} \end{aligned}$$

(3) Consistency:

$$sA\bar{k}^{\alpha} = g\left(\bar{k}, \bar{k}\right)$$

We solve for the recursive competitive equilibrium by the "guess and verify" method. We conjecture that the value function is of the following form:

$$v\left(k, \bar{k}\right) = a + b\ln(k + d\bar{k}) + e\ln(\bar{k})$$

Now we start to solve for the solution by plugging in our conjecture. Bellman equation becomes

$$v\left(k, \bar{k}\right) = \max_{\{k'\}} \ln\left(\alpha A\bar{k}^{\alpha-1}k + (1-\alpha)A\bar{k}^{\alpha} - k'\right) + \beta\left(a + b\ln(k' + d\bar{k}') + e\ln(\bar{k}')\right)$$

$$s.t. \quad \bar{k}' = sA\bar{k}^{\alpha}$$

or

$$v\left(k, \bar{k}\right) = \max_{\{k'\}} \ln\left(\alpha A\bar{k}^{\alpha-1}k + (1-\alpha)A\bar{k}^{\alpha} - k'\right) + \beta\left(a + b\ln(k' + dsA\bar{k}^{\alpha}) + e\ln(sA\bar{k}^{\alpha})\right)$$

Take F.O.C. with respect to k', we have

$$\frac{1}{\alpha A\bar{k}^{\alpha-1}k+(1-\alpha)A\bar{k}^{\alpha}-k'}=\frac{\beta b}{k'+dsA\bar{k}^{\alpha}}\Rightarrow k'+dsA\bar{k}^{\alpha}=\beta b\left(\alpha A\bar{k}^{\alpha-1}k+(1-\alpha)A\bar{k}^{\alpha}-k'\right)$$
$$\Rightarrow k'=\frac{\beta b}{1+\beta b}\alpha A\bar{k}^{\alpha-1}k+\frac{\beta b(1-\alpha)-ds}{1+\beta b}A\bar{k}^{\alpha}$$

Plug back into Bellman equation, we have

$$\begin{aligned}
v\left(k,\bar{k}\right) &\equiv \ln\left(\frac{1}{1+\beta b}\alpha A\bar{k}^{\alpha-1}k+\frac{(1-\alpha)+ds}{1+\beta b}A\bar{k}^{\alpha}\right)\\
&\quad+\beta\left(a+b\ln(\frac{\beta b}{1+\beta b}\alpha A\bar{k}^{\alpha-1}k+\frac{\beta b(1-\alpha+ds)}{1+\beta b}A\bar{k}^{\alpha})+e\ln(sA\bar{k}^{\alpha})\right)\\
&\equiv \left[(1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right)+\beta a+\beta b\ln(\beta b)+\beta e\ln(sA)\right]+\\
&\quad(1+\beta b)\ln\left(k+\frac{(1-\alpha)+ds}{\alpha}\bar{k}\right)+\left[(1+\beta b)(\alpha-1)+\alpha\beta e\right]\ln\left(\bar{k}\right)
\end{aligned}$$

Since the $LHS=a+b\ln(k+d\bar{k})+e\ln(\bar{k})$, we have the following equations:

$$\begin{aligned}
(1)\ a &= (1+\beta b)\ln\left(\frac{\alpha A}{1+\beta b}\right)+\beta a+\beta b\ln(\beta b)+\beta e\ln(sA)\\
(2)\ b &= (1+\beta b)\\
(3)\ d &= \frac{(1-\alpha)+ds}{\alpha}\\
(4)\ e &= (1+\beta b)(\alpha-1)+\alpha\beta e
\end{aligned}$$

Solve for these four equations, we get the solutions

$$\begin{aligned}
a &= \frac{1}{(1-\beta)^2}\left[\ln\left(\alpha A(1-\beta)\right)+\beta\ln\left(\frac{\beta}{1-\beta}\right)+\frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(sA)\right]\\
b &= \frac{1}{1-\beta}\\
d &= \frac{1-\alpha}{\alpha-s}\\
e &= \frac{\alpha-1}{(1-\alpha\beta)(1-\beta)}
\end{aligned}$$

This gives us the form of value function. Plug these coefficients into F.O.C., we get

$$\begin{aligned}
k' &= \frac{\beta b}{1+\beta b}\alpha A\bar{k}^{\alpha-1}k+\frac{\beta b(1-\alpha)-ds}{1+\beta b}A\bar{k}^{\alpha}\\
&= \alpha\beta A\bar{k}^{\alpha-1}k+\frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\bar{k}^{\alpha}
\end{aligned}$$

Therefore, we have our value function and policy function as

$$\begin{aligned}
v\left(k,\bar{k}\right) &= a+b\ln(k+d\bar{k})+e\ln(\bar{k})\\
g(k,\bar{k}) &= \alpha\beta A\bar{k}^{\alpha-1}k+\frac{(1-\alpha)(\alpha\beta-s)}{\alpha-s}A\bar{k}^{\alpha}
\end{aligned}$$

with coefficients as defined above.

(b) Imposing consistency condition, we have

$$
\begin{aligned}
G(\bar{k}) = g(\bar{k},\bar{k}) &\Rightarrow sA\bar{k}^{\alpha} = \alpha\beta A\bar{k}^{\alpha-1}\bar{k} + \frac{(1-\alpha)(\alpha\beta - s)}{\alpha - s}A\bar{k}^{\alpha} \\
&\Rightarrow (\alpha - s)s = (\alpha - s)\alpha\beta + (1-\alpha)(\alpha\beta - s) \\
&\Rightarrow s^2 - (1+\alpha\beta)s + \alpha\beta = 0 \\
&\Rightarrow s_1 = \alpha\beta \text{ and } s_2 = 1
\end{aligned}
$$

Consider the context of this economic problem, obviously only $s = \alpha\beta$ will be a solution to this problem. Consequently the resulting recursive competitive equilibrium is

$$
\begin{aligned}
v\left(k,\bar{k}\right) &= a + b\ln(k + d\bar{k}) + e\ln(\bar{k}) \\
g(k,\bar{k}) &= \alpha\beta A\bar{k}^{\alpha-1}k \\
G(\bar{k}) &= \alpha\beta A\bar{k}^{\alpha}
\end{aligned}
$$

with coefficients as

$$
\begin{aligned}
a &= \frac{1}{(1-\beta)^2}\left[\ln\left(\alpha A(1-\beta)\right) + \beta\ln\left(\frac{\beta}{1-\beta}\right) + \frac{(\alpha-1)\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right] \\
b &= \frac{1}{1-\beta} \\
d &= \frac{1-\alpha}{\alpha(1-\beta)} \\
e &= \frac{\alpha - 1}{(1-\alpha\beta)(1-\beta)}
\end{aligned}
$$

Now let's check whether $G(\bar{k})$ solves the central planning problem.
The recursive formulation of central planning problem is

$$
v\left(\bar{k}\right) = \max_{k'} \ln\left(A\bar{k}^{\alpha} - \bar{k}'\right) + \beta v\left(\bar{k}'\right)
$$

Guess that $v\left(\bar{k}\right) = a + b\ln\bar{k}$. Plug into Bellman equation, we have

$$
v\left(\bar{k}\right) = \max_{k'} \ln\left(A\bar{k}^{\alpha} - \bar{k}'\right) + \beta\left(a + b\ln\bar{k}'\right)
$$

F.O.C. is

$$
\frac{1}{A\bar{k}^{\alpha} - \bar{k}'} = \beta b\frac{1}{\bar{k}'} \quad\Rightarrow\quad \bar{k}' = \frac{\beta b}{1+\beta b}A\bar{k}^{\alpha}
$$

Plug back into Bellman equation, we have

$$
\begin{aligned}
& v\left(\bar{k}\right) = \max_{k'} \ln\left(A\bar{k}^{\alpha} - \bar{k}'\right) + \beta\left(a + b\ln\bar{k}'\right) \\
\Rightarrow\quad & a + b\ln\bar{k} = \ln\left(A\bar{k}^{\alpha} - \frac{\beta b}{1+\beta b}A\bar{k}^{\alpha}\right) + \beta\left(a + b\ln\left(\frac{\beta b}{1+\beta b}A\bar{k}^{\alpha}\right)\right) \\
\Rightarrow\quad & a + b\ln\bar{k} = \left[(1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b\ln(\frac{\beta b}{1+\beta b})\right] + \alpha(1+\beta b)\ln\bar{k}
\end{aligned}
$$

Again since this is an identity, we have

$$
a = (1+\beta b)\ln A + \ln\left(\frac{1}{1+\beta b}\right) + \beta a + \beta b\ln(\frac{\beta b}{1+\beta b})
$$

and

$$b = \alpha\left(1 + \beta b\right)$$

Solve for this, we get

$$\begin{aligned} a &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta\right) \\ b &= \frac{\alpha}{1-\alpha\beta} \end{aligned}$$

Therefore, the solution for central planning problem is

$$\begin{aligned} v\left(\bar{k}\right) &= \frac{1}{1-\beta}\left(\frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta\right) + \frac{\alpha}{1-\alpha\beta}\ln\bar{k} \\ \bar{k}' &= \alpha\beta A\bar{k}^{\alpha} \end{aligned}$$

This is the same aggregate state evolution law as in the recursive competitive equilibrium.

(c) Take derivative with respect to $\bar{k}$, we have

$$g\left(k,\bar{k}\right) = \alpha\beta A\bar{k}^{\alpha-1}k \quad \Rightarrow \quad g_2\left(k,\bar{k}\right) = \alpha\beta\left(\alpha-1\right)A\bar{k}^{\alpha-2}k < 0$$

and

$$v\left(k,\bar{k}\right) = a + b\ln(k+d\bar{k}) + e\ln(\bar{k}) \Rightarrow v_2\left(k,\bar{k}\right) = \frac{bd}{k+d\bar{k}} + \frac{e}{\bar{k}}$$

$$\Rightarrow v_2\left(k,\bar{k}\right) = \frac{\alpha(1-\alpha)(\bar{k}-k)}{\left(\alpha\left(1-\beta\right)k + \left(1-\alpha\right)\bar{k}\right)\left(1-\alpha\beta\right)\bar{k}}$$

$$\Rightarrow v_2\left(k,\bar{k}\right)\begin{cases} >0 & \text{if } \bar{k} > k \\ =0 & \text{if } \bar{k} = k \\ <0 & \text{if } \bar{k} < k \end{cases}$$

Note that here we do not consider equilibrium behavior, therefore the sign is ambiguous.

2. Define the function F as $F(k,n,\bar{k}) = Ak^{\alpha}n^{1-\alpha}\bar{k}^{\gamma}$, and then

$$f(k,n,\bar{k}) = F(k,n,\bar{k}) + (1-\delta)k$$

We will use this notation all through this question.

(a) A sequential competitive equilibrium for this economy is a sequence $\{R_t^*, w_t^*, c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0}$ such that

(1)Given $\{R_t^*, w_t^*\}_{t\geq 0}$, $\{c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0}$ solves the consumer's problem:

$$\{c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0} = \text{argmax} \sum_{t\geq 0} \beta^t u(c_t)$$

$$s.t. \quad c_t + k_{t+1} = k_t R_t^* + w_t^* n_t$$

$$k_0 \quad \text{given}$$

$$n_t \leq 1, \quad \forall t \geq 0$$

$$\lim_{t\to\infty} \frac{k_{t+1}}{\prod_{j=0}^t R_t^*}$$

(2)$\{k_{t+1}^*, n_t^*\}_{t\geq 0}$ solves the firm's problem:

$$(k_t^*, n_t^*) = \arg\max_{k_t, n_t} F(k_t, n_t, k_t^*) + (1-\delta)k_t - R_t^* k_t - w_t^* n_t \quad \forall t$$

(3)Market clearing:

$$\text{labor market: } n_t = 1 \quad \forall t$$

$$\text{good's market: } c_t^* + k_{t+1}^* = F(k_t^*, 1, k_t^*) + (1-\delta)k_t^* \quad \forall t$$

(b) A Recursive Competitive Equilibrium for the economy is a set of functions:

$$\begin{aligned} price\ function &: r\left(\bar{k}\right), w\left(\bar{k}\right) \\ policy\ function &: k' = g\left(k, \bar{k}\right) \\ value\ function &: v\left(k, \bar{k}\right) \\ transition\ function &: \bar{k}' = G\left(\bar{k}\right) \end{aligned}$$

such that:

(1) $k' = g\left(k, \bar{k}\right)$ and $v\left(k, \bar{k}\right)$ solves consumer's problem:

$$\begin{aligned} v(k, \bar{k}) &= \max_{\{c, k'\}} u(c) + \beta v(k', \bar{k}') \\ & s.t. \\ & c + k' = r\left(\bar{k}\right) k + w\left(\bar{k}\right) \\ & \bar{k}' = G\left(\bar{k}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r(\bar{k}) &= f_1(\bar{k}, 1, \bar{k}) = \alpha A \bar{k}^{\alpha+\gamma-1} + (1-\delta) \\ w(\bar{k}) &= f_2(\bar{k}, 1, \bar{k}) = (1-\alpha) A \bar{k}^{\alpha+\gamma} \end{aligned}$$

(3) Consistency:

$$G(\bar{k}) = g(\bar{k}, \bar{k})$$

(c) Solve for consumer's problem in the normal way, we get the Euler equation:

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} r(\bar{k}_{t+1}) = 1 \quad \Leftrightarrow \quad \frac{\beta u'(c_{t+1})}{u'(c_t)} (\alpha A \bar{k}_{t+1}^{\alpha+\gamma-1} + 1 - \delta) = 1$$

Imposing the equilibrium condition $k = \bar{k}$ and $n = 1$, we get that aggregate consumption in every period is equal to:

$$\bar{c}_t = F(\bar{k}_t, 1, \bar{k}_t) + (1-\delta)\bar{k}_t - \bar{k}_{t+1}$$

Since all agents are identical, in equilibrium it will be the case that $\bar{c}_t = c_t$ for every t (you can think of this as each individual's consumption will equal per capita consumption).

If we substitute this in the individual's Euler equation we derived above, we get:

$$\frac{\beta u'(\bar{c}')}{u'(\bar{c})} (\alpha A \bar{k}'^{\alpha+\gamma-1} + 1 - \delta) = 1$$

$$\Leftrightarrow \frac{\beta u'(F(\bar{k}', 1, \bar{k}') + (1-\delta)\bar{k}' - \bar{k}'')}{u'(F(\bar{k}, 1, \bar{k}) + (1-\delta)\bar{k} - \bar{k}')} (\alpha A \bar{k}'^{\alpha+\gamma-1} + 1 - \delta) = 1$$

$$\Leftrightarrow \beta u'(F(\bar{k}', 1, \bar{k}') + (1-\delta)\bar{k}' - \bar{k}'')(\alpha A \bar{k}'^{\alpha+\gamma-1} + 1 - \delta) = u'(F(\bar{k}, 1, \bar{k}) + (1-\delta)\bar{k} - \bar{k}')$$

which is a second-order difference equation that governs the evolution of the economy's aggregates.

(d) The recursive formulation of the planning problem is

$$\begin{aligned} v\left(\bar{k}\right) &= \max_{\{c,k'\}} u\left(c\right) + \beta v\left(\bar{k}'\right) \\ & \quad s.t. \\ & \quad c + k' = A\bar{k}^{\alpha+\gamma} + (1-\delta)\bar{k} \end{aligned}$$

The Euler equation is

$$\frac{\beta u'\left(\bar{c}'\right)}{u'\left(\bar{c}\right)} \left(A\left(\alpha+\gamma\right)\bar{k}'^{\alpha+\gamma-1} + 1 - \delta\right) = 1$$

If we compare the above Euler equation to the one we found in the competitive equilibrium case we can see that they are different. Since planner's problem gives us the Pareto optimal allocation, the equilibrium in the competitive case cannot be Pareto optimal. The intuition behind this result is that the firms in the competitive equilibrium do not internalize the externality in the production, whereas the planner is able to do so.

(e) A Recursive Competitive Equilibrium for the economy with taxation is a set of functions:

$$\begin{aligned} price\ function &: r(\bar{k}), w(\bar{k}) \\ policy\ function &: k' = g(k, \bar{k}) \\ value\ function &: v(k, \bar{k}) \\ taxation\ function &: T(\bar{k}) \\ transition\ function &: \bar{k}' = G(\bar{k}) \end{aligned}$$

such that:
(1) $k' = g\left(k, \bar{k}\right)$ and $v\left(k, \bar{k}\right)$ solves consumer's problem:

$$\begin{aligned}
v(k, \bar{k}) &= \max_{\{c,k'\}} u(c) + \beta v(k', \bar{k}') \\
& \quad s.t. \\
& \quad c + (1-\tau)(k' - (1-\delta)k) = R(\bar{k})k + w(\bar{k}) - T(\bar{k}) \\
& \quad \bar{k}' = G(\bar{k})
\end{aligned}$$

where $R(\bar{k}) = r(\bar{k}) - (1-\delta)$.
(2) Prices are competitively determined:

$$\begin{aligned}
r(\bar{k}) &= f_1(\bar{k}, 1, \bar{k}) = \alpha A \bar{k}^{\alpha+\gamma-1} + (1-\delta) \\
w(\bar{k}) &= f_2(\bar{k}, 1, \bar{k}) = (1-\alpha) A \bar{k}^{\alpha+\gamma}
\end{aligned}$$

(3) Government balances budget in each period:

$$T\left(\bar{k}\right) = \tau\left(\bar{k}' - (1-\delta)\,\bar{k}\right)$$

(3) Consistency of transition function:

$$G\left(\bar{k}\right) = g\left(\bar{k}, \bar{k}\right)$$

(f) We can solve for F.O.C. in the normal way as

$$(1-\tau)\,u'(c) = \beta v_1\left(k', \bar{k}'\right)$$

Envelope condition gives us

$$v_1(k, \bar{k}) = u'(c)(R(\bar{k}) + (1-\tau)(1-\delta))$$

The Euler equation is

$$\begin{aligned}
& \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{1}{1-\tau}(R(\bar{k}) + (1-\tau)(1-\delta)) = 1 \\
\Leftrightarrow \quad & \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{1}{1-\tau}(\alpha A \bar{k}^{\alpha+\gamma-1} + (1-\tau)(1-\delta)) = 1
\end{aligned}$$

In steady state, we have

$$\begin{aligned}
& \frac{\beta}{1-\tau}\left(\alpha A \bar{k}^{\alpha+\gamma-1} + (1-\tau)(1-\delta)\right) = 1 \\
\Rightarrow \quad & \alpha A \bar{k}^{\alpha+\gamma-1} = (1-\tau)\left(\frac{1}{\beta} - (1-\delta)\right) \\
\Rightarrow \quad & \bar{k}^{\tau} = \left(\frac{(1-\tau)\left(\frac{1}{\beta} - (1-\delta)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}
\end{aligned}$$

where $\bar{k}^{\tau}$ represents the steady-state aggregate capital stock in economy with taxation.

From the Euler equation of the planner's problem we can find the steady state:

$$A\left(\alpha+\gamma\right)\left(\bar{k}^{o}\right)^{\alpha+\gamma-1}+1-\delta=\frac{1}{\beta}\quad\Rightarrow\quad\bar{k}^{o}=\left(\frac{\frac{1}{\beta}-1+\delta}{\left(\alpha+\gamma\right)A}\right)^{\frac{1}{\alpha+\gamma-1}}$$

where $\bar{k}^{o}$ represents the optimal steady-state aggregate capital stock.
Now set $\bar{k}^{\tau}=\bar{k}^{o}$, we have

$$\begin{aligned}\bar{k}^{\tau}=\bar{k}^{o}\Rightarrow\left(\frac{(1-\tau)\left(\frac{1}{\beta}-(1-\delta)\right)}{\alpha A}\right)^{\frac{1}{\alpha+\gamma-1}}&=\left(\frac{\frac{1}{\beta}-1+\delta}{\left(\alpha+\gamma\right)A}\right)^{\frac{1}{\alpha+\gamma-1}}\\ \Rightarrow\frac{1-\tau}{\alpha}&=\frac{1}{\alpha+\gamma}\\ \Rightarrow\tau&=\frac{\gamma}{\alpha+\gamma}\end{aligned}$$

Therefore, for $\tau=\frac{\gamma}{\alpha+\gamma}$ the competitive equilibrium steady-state aggregate capital stock the same as the steady-state aggregate capital stock in the planning problem.

3. (a) A sequential competitive equilibrium for this economy is a sequence $\{R_t^*,w_t^*,c_t^*,k_{t+1}^*,n_t^*\}_{t\geq 0}$ such that
(1)Given $\{R_t^*,w_t^*\}_{t\geq 0}$, $\{c_t^*,k_{t+1}^*,n_t^*\}_{t\geq 0}$ solves the consumer's problem:

$$\{c_t^*,k_{t+1}^*,n_t^*\}_{t\geq 0}=\text{argmax}\sum_{t\geq 0}\beta^t u(c_t,L-n_t)$$

$$\begin{aligned}s.t.\quad c_t+k_{t+1}&=k_tR_t^*+w_t^*n_t\\ k_0&\quad\text{given}\\ n_t&\leq L\quad\forall t\\ \lim_{t\to\infty}&\frac{k_{t+1}}{\prod_{j=0}^{t}R_t^*}\end{aligned}$$

where L is the total endowment of time every period.
(2)$\{k_{t+1}^*,n_t^*\}_{t\geq 0}$ solves the firm's problem:

$$(k_t^*,n_t^*)=\arg\max_{k_t,n_t}F(k_t,n_t)+(1-\delta)k_t-R_t^*k_t-w_t^*n_t\quad\forall t$$

(3)Market clearing:

$$c_t^*+k_{t+1}^*=F(k_t^*,n_t^*)+(1-\delta)k_t^*\quad\forall t$$

(b) A recursive competitive equilibrium for the neoclassical growth model with valued leisure is a set of functions:

$$\begin{aligned}price\ function&:\ r(\bar{k}),w(\bar{k})\\ policy\ function&:\ k'=g_k(k,\bar{k}),l=g_l(k,\bar{k})\\ value\ function&:\ v(k,\bar{k})\\ aggregate\ state&:\ \bar{k}'=G(\bar{k}),\bar{l}=\bar{l}(\bar{k})\end{aligned}$$

such that:
(1) Given $\bar{k}' = G(\bar{k})$; $k' = g_k\left(k, \bar{k}\right)$, $l = g_l\left(k, \bar{k}\right)$ and $v\left(k, \bar{k}\right)$ solve the consumer's problem:

$$\begin{aligned}
v\left(k, \bar{k}\right) &= \max_{\{c,l,k'\}} u\left(c, l\right) + \beta v\left(k', \bar{k}'\right) \\
& s.t. \\
& c + k' = r(\bar{k})k + w(\bar{k})(L - l) \\
& \bar{k}' = G(\bar{k})
\end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned}
r(\bar{k}) &= F_1(\bar{k}, L - \bar{l}(\bar{k})) + (1 - \delta) \\
w(\bar{k}) &= F_2(\bar{k}, L - \bar{l}(\bar{k}))
\end{aligned}$$

(3) Consistency:

$$\begin{aligned}
G\left(\bar{k}\right) &= g_k\left(\bar{k}, \bar{k}\right) \\
\bar{l}(\bar{k}) &= g_l\left(\bar{k}, \bar{k}\right)
\end{aligned}$$

(c) Solve a typical consumer's problem

$$\begin{aligned}
v(k, \bar{k}) &= \max_{\{l,k'\}} u(r(\bar{k})k + w(\bar{k})(L - l) - k', l) + \beta v(k', \bar{k}') \\
& s.t. \\
& \bar{k}' = G(\bar{k})
\end{aligned}$$

We get the F.O.C. as

$$\begin{aligned}
\{l\} &: u_1(c, l)w(\bar{k}) = u_2(c, l) \\
\{k'\} &: u_1(c, l) = \beta v_1(k', \bar{k}')
\end{aligned}$$

Use the envelope condition

$$v_1(k, \bar{k}) = u_1(c, l)r(\bar{k})$$

In this way, we get the optimality conditions:

$$\begin{aligned}
\{l_t\} &: u_1(c_t, l_t)w(\bar{k}_t) = u_2(c_t, l_t) \\
\{k_{t+1}\} &: u_1(c_t, l_t) = \beta u_1(c_{t+1}, l_{t+1})r(\bar{k}_{t+1})
\end{aligned}$$

Correspondingly, the functional F.O.C. is

$$\begin{aligned}
\{l_t\} &: u_1(c_t, g_l(k_t, \bar{k}_t))w(\bar{k}_t) = u_2(c_t, g_l(k_t, \bar{k}_t)) \\
\{k_{t+1}\} &: u_1(c_t, g_l(k_t, \bar{k}_t)) = \beta u_1(c_{t+1}, g_l(k_{t+1}, \bar{k}_{t+1}))r(G(\bar{k}_t))
\end{aligned}$$

where

$$
\begin{aligned}
c_t &= r(\bar{k}_t)k_t + w(\bar{k}_t)(L - g_l(k_t, \bar{k}_t)) - g_k(k_t, \bar{k}_t) \\
r(\bar{k}) &= F_1(\bar{k}, L - \bar{l}) + (1-\delta) = F_1(\bar{k}, L - g_l(\bar{k}, \bar{k})) + (1-\delta) \\
w(\bar{k}) &= F_2(\bar{k}, L - \bar{l}) = F_2(\bar{k}, L - g_l(\bar{k}, \bar{k}))
\end{aligned}
$$

Impose the equilibrium conditions $k_t = \bar{k}_t, l_t = \bar{l}_t$, we have

$$
\{l_t\} : u_1\left(\bar{c}_t, \bar{l}_t\right) F_2\left(\bar{k}_t, L - \bar{l}_t\right) = u_2\left(\bar{c}_t, \bar{l}_t\right) \tag{1}
$$

$$
\{k_{t+1}\} : u_1\left(\bar{c}_t, \bar{l}_t\right) = \beta u_1\left(\bar{c}_{t+1}, \bar{l}_{t+1}\right)\left(F_1\left(\bar{k}_t, L - \bar{l}_t\right) + (1-\delta)\right) \tag{2}
$$

where

$$
\begin{aligned}
\bar{c}_t &= F_1\left(\bar{k}_t, L - \bar{l}_t\right)\bar{k}_t + (1-\delta)\bar{k}_t + F_2\left(\bar{k}_t, L - \bar{l}_t\right)\left(L - \bar{l}_t\right) - \bar{k}_{t+1} \\
&= F\left(\bar{k}_t, L - \bar{l}_t\right) + (1-\delta)\bar{k}_t - \bar{k}_{t+1}
\end{aligned} \tag{3}
$$

which are identical to the first-order conditions associated with the planning problem for this economy.

(d) Now we have to solve explicitly for the steady state values given the following functional forms:

$$
f(\bar{k}, \bar{n}) = \underbrace{\bar{k}^{\alpha}\bar{n}^{1-\alpha}}_{F(\bar{k},\bar{n})} + (1-\delta)\bar{k} \quad \text{and} \quad u(c, l) = \lambda \log(c) + (1-\lambda)\log(l)
$$

Let $\bar{k}$, $\bar{l}$ and $\bar{c}$ be the steady state values of capital, leisure and consumption, replacing these values and the functional forms in eqs.(1), (2) and (3) we get:

$$
\frac{\lambda}{\bar{c}}(1-\alpha)\bar{k}^{\alpha}(L - \bar{l})^{-\alpha} = \frac{1-\lambda}{\bar{l}} \tag{4}
$$

$$
\frac{\lambda}{\bar{c}} = \beta\frac{\lambda}{\bar{c}}(\alpha\bar{k}^{\alpha-1}(L - \bar{l})^{1-\alpha} + 1 - \delta) \tag{5}
$$

$$
\bar{c} = \bar{k}^{\alpha}(L - \bar{l})^{1-\alpha} - \delta\bar{k} \tag{6}
$$

Defining $C = (\beta^{-1} + \delta - 1)/\alpha$, these equations imply that the steady state values are,

$$
\bar{k} = \frac{L\lambda(1-\alpha)C^{\alpha/(\alpha-1)}}{C(1-\alpha\lambda) + \delta(\lambda-1)}
$$

and then

$$
\bar{l} = L - C^{1/(1-\alpha)}\bar{k}
$$

A final comment: in problems 2 and 3, we could have used instead the following consumer's budget constraint and firm's objective function,

BC:	$c_t + K_{t+1} = r_t K_t + w_t n_t + (1-\delta)K_t$
Obj:	$\max AK_t^{\alpha} n_t^{1-\alpha} \bar{K}_t^{\gamma} - r_t K_t - w_t n_t$

This formulation is also correct and delivers the same sequential competitive equilibrium (and thus if you used it you are going to receive full score too), but the one used in this solution is more consistent with the way in which the setups in problems 2 and 3 are described.