

Econ 510a (second half)
Yale University
Fall 2006
Prof. Tony Smith

HOMWORK #4

This homework assignment is due at 5PM on Friday, December 1 in Marnix Amand's mailbox.

1. Consider a two-period exchange economy with two types of consumers of equal measure. Each consumer maximizes $u(c_0^i) + \beta E[u(c_1^i)]$, where c_t^i , $i = 1, 2$, is the consumption of a type- i consumer in period t . In period 0, type- i consumers are endowed with ω_0^i units of the (nonstorable) consumption good. Endowments in period 1 are random: with probability π_j , $j = 1, 2$, a type- i consumer receives ω_{1j}^i units of the consumption good in period 1. In period 0, consumers trade Arrow securities whose payoffs depend on the state of the world in period 1.
 - (a) Carefully define a competitive equilibrium for this economy. Are markets complete? Explain why or why not.
 - (b) Suppose that $\bar{\omega}_0 = \bar{\omega}_{11} = \bar{\omega}_{12}$, where $\bar{\omega}_0 \equiv \sum_{i=1}^2 \omega_0^i$ and $\bar{\omega}_{1j} \equiv \sum_{i=1}^2 \omega_{1j}^i$. Prove an aggregation theorem for this economy: that is, show that redistributions of the period-0 endowments do not affect the equilibrium prices of the Arrow securities. (Note that a redistribution leaves the aggregate endowment unchanged.) In addition, characterize the equilibrium consumption allocation as much as possible.
 - (c) Now suppose that $\bar{\omega}_0 \neq \bar{\omega}_{11} \neq \bar{\omega}_{12}$. Prove an aggregation theorem for this economy under the assumption that u has a constant elasticity of intertemporal substitution equal to σ^{-1} . In addition, characterize the equilibrium consumption allocation as much as possible.
 - (d) Now suppose that in period 0 consumers cannot trade Arrow securities but instead can trade only a riskfree bond (i.e., a sure claim to one unit of the consumption good in period 1). Carefully define a competitive equilibrium for this economy.
 - (e) For the economy in part (d), set $\omega_0^1 = \omega_0^2$, $\omega_{11}^1 = \omega_{12}^2$, $\omega_{12}^1 = \omega_{11}^2$, and $\pi_1 = \pi_2 = 1/2$, but do not assume a functional form for u . Find the equilibrium consumption allocation and the equilibrium bond price. How does the bond price compare to the one in part (c)? Explain.

Solution (a) As full set of Arrow securities are traded, markets are complete. A competitive equilibrium for this economy is defined by $\{c_0^i, c_{1j}^i, q_j, a_j^i\}_{i,j=1,2}$ where

(1) For each i , $\{c_0^i, c_{1j}^i, a_j^i\}_{j=1,2}$ solve

$$\begin{aligned} \max u(c_0) + \beta \mathbb{E}[u(c_1)] \\ \text{s.t. } c_0^i + \sum_{j=1}^2 q_j a_j^i &= w_0^i \\ c_{1j}^i &= a_j^i + w_{1j}^i \quad j = 1, 2 \end{aligned}$$

(2) market clearing:

good market: $c_0^1 + c_0^2 = w_0^1 + w_0^2$ and $c_{1j}^1 = w_{1j}^1 + w_{1j}^2$ for $j = 1, 2$.

asset market: $a_j^1 + a_j^2 = 0$ for $j = 1, 2$.

(b) We assume here that aggregate endowment is constant across time and states : $W = \bar{w}_0 = \bar{w}_{11} = \bar{w}_{12}$. The FOCs for the consumer i 's problem are:

$$\begin{aligned} a_1^i : -u'(c_0^i)q_1 + \beta\pi_1 u'(c_{11}^i) &= 0 \\ a_2^i : -u'(c_0^i)q_2 + \beta\pi_2 u'(c_{12}^i) &= 0 \end{aligned}$$

and then

$$(1) \quad q_j = \frac{\beta\pi_j u'(c_{1j}^i)}{u'(c_0^i)} \quad \text{for } i, j = 1, 2$$

But using the concavity of u , we get that $\frac{u'(c_{1j}^i)}{u'(c_0^i)} = 1$ for $i, j = 1, 2$. In fact, suppose there is some j for which

$$\frac{u'(c_{1j}^1)}{u'(c_0^1)} = \frac{u'(c_{1j}^2)}{u'(c_0^2)} > 1$$

Then, given the concavity of u , it would imply that $c_{1j}^1 < c_0^1$ and $c_{1j}^2 < c_0^2$, and thus at equilibrium we would have,

$$\bar{w}_{1j} = c_{1j}^1 + c_{1j}^2 < c_0^1 + c_0^2 = \bar{w}_0$$

which is a contradiction (given that aggregate endowments are equal). Therefore the Arrow securities prices are defined by

$$q_j = \beta\pi_j \quad \text{for } j = 1, 2$$

and so they are in fact independent of individual endowments.

- (c) Now aggregate endowments are not equal but the felicity function is a CES. Replacing this functional form in equation (1) obtained in part (a) we get that

$$q_j = \frac{\beta\pi_j(c_{1j}^1)^{-\sigma}}{(c_0^1)^{-\sigma}} = \frac{\beta\pi_j(c_{1j}^2)^{-\sigma}}{(c_0^2)^{-\sigma}}$$

then, the second equality implies that $\frac{c_0^1}{c_{1j}^1} = \frac{c_0^2}{c_{1j}^2}$ for $j = 1, 2$. Imposing the equilibrium conditions in this equality, we get that for each j

$$(2) \quad \frac{\bar{w}_0 - c_0^2}{\bar{w}_{1j} - c_{1j}^2} = \frac{c_0^2}{c_{1j}^2} \Rightarrow c_0^2 = \frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^2$$

Replacing this in the equation for the Arrow securities prices we get that

$$q_j = \frac{\beta\pi_j(c_{1j}^2)^{-\sigma}}{\left(\frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^2\right)^{-\sigma}} = \beta\pi_j \left(\frac{\bar{w}_0}{\bar{w}_{1j}}\right)^\sigma$$

and thus the Arrow securities prices only depend on the aggregate endowment. Also, using (2) and the FOCs we get

$$c_0^i = \frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^i \quad \text{for } i, j = 1, 2.$$

- (d) Now there are no Arrow securities available, consumers can only trade a risk free bond. A competitive equilibrium for this economy is composed of $\{c_0^i, c_{1j}^i, q, a^i\}_{i,j=1,2}$ where

(1) For each i , $\{c_0^i, c_{1j}^i, a^i\}_{j=1,2}$ solve

$$\max u(c_0) + \beta\mathbb{E}[u(c_1^i)]$$

$$s.t. \quad c_0^i + qa^i = w_0^i$$

$$c_{1j}^i = a^i + w_{1j}^i \quad j = 1, 2$$

(2) market clearing:

good market: $c_0^1 + c_0^2 = w_0^1 + w_0^2$ and $c_{1j}^1 = w_{1j}^1 + w_{1j}^2$ for $j = 1, 2$.

asset market: $a^1 + a^2 = 0$ for $j = 1, 2$.

- (e) We now have an economy as the one in part (d) with no functional form for u and where $w_0^1 = w_0^2 \equiv w_0$, $w_{11}^1 = w_{12}^2 \equiv w_1$, $w_{12}^1 = w_{11}^2 \equiv w_2$ and $\pi_1 = \pi_2 = 0.5$. The consumer i 's FOC is given by:

$$a_i : \quad -u'(c_0^i)q + \frac{1}{2}\beta u'(c_{11}^i) + \frac{1}{2}\beta u'(c_{12}^i) = 0$$

which implies

$$(3) \quad q = \beta \frac{u'(c_{11}^i) + u'(c_{12}^i)}{2u'(c_0^i)}$$

Given the symmetry in endowments and probabilities, we make the following guess:

$$c_0^1 = c_0^2 = w_0, \quad c_{11}^1 = c_{12}^2 \quad \text{and} \quad c_{12}^1 = c_{11}^2$$

First, plugging this guess in the budget constraints, we get that $a^1 = a^2 = 0$. Then, we need that $c_{11}^1 = c_{12}^2 = w_1$ and $c_{12}^1 = c_{11}^2 = w_2$ (ie, everybody consumes his own endowment). Finally we also need both FOCs to be satisfied, and thus, we need the RHS of (3) to be independent of i . Replacing our guess in (3) we get

$$\text{for } i = 1, \quad (3) \Rightarrow q = \beta \frac{u'(w_1) + u'(w_2)}{2u'(w_0)}$$

and

$$\text{for } i = 2, \quad (3) \Rightarrow q = \beta \frac{u'(w_2) + u'(w_1)}{2u'(w_0)}$$

and thus the FOCs are satisfied. Summarizing, we have that the equilibrium allocations are

$$c_0^1 = c_0^2 = w_0, \quad c_{11}^1 = c_{12}^2 = w_1 \quad \text{and} \quad c_{12}^1 = c_{11}^2 = w_2$$

and the bond price is given by

$$q = \beta \frac{u'(w_2) + u'(w_1)}{2u'(w_0)}$$

and thus it is the sum of the Arrow securities prices found in part (c), using the fact that $\pi_1 = \pi_2 = 0.5$ (no arbitrage condition!).

- 2.** Consider an exchange economy with two (types of) consumers. Type- A consumers comprise fraction λ of the economy's population and type- B consumers comprise fraction $1 - \lambda$ of the economy's population. Each consumer has (constant) endowment ω in each period. A consumer of type i has preferences over consumption streams of the form $\sum_{t=0}^{\infty} \beta_i^t u(c_t)$. Assume that $1 > \beta_A > \beta_B > 0$: type- A consumers are more patient than type- B consumers. Consumers trade a one-period riskfree bond in each period. There is no restriction on borrowing except for a no-Ponzi-game condition. Assume that each consumer has zero bonds in period 0.

(a) Carefully define a sequential competitive equilibrium for this economy.

(b) Show that this economy has no steady state: in particular, show that the type- B consumers become poorer and poorer over time and consume zero in the limit.

Solution (a) A sequential competitive equilibrium for the economy $\{u_A, u_B, \omega\}$, is a sequence $\{c_{it}^*\}_{t=0}^\infty, \{b_{i,t+1}^*\}_{t=0}^\infty, \{q_t^*\}_{t=0}^\infty$ (where q_t^* means price of Arrow security) for $i = A, B$ such that

(1) For $i = A, B$,

$$\begin{aligned} \{c_{it}^*, b_{i,t+1}^*\}_{t=0}^\infty &= \arg \max \sum_{t=0}^\infty \beta_i^t u(c_{it}) \\ &\text{s.t.} \\ c_{it} + q_t^* b_{i,t+1} &= b_{i,t} + \omega \\ \lim_{t \rightarrow \infty} b_{i,t+1} \left(\prod_{j=0}^t q_j \right) &\geq 0 \\ b_{i,0} = 0, c_{it} &\geq 0 \end{aligned}$$

(2) $\lambda c_{At}^* + (1 - \lambda) c_{Bt}^* = \omega$ for $t = 0, 1, 2, \dots$

(3) $\lambda b_{A,t+1}^* + (1 - \lambda) b_{B,t+1}^* = 0$ for $t = 0, 1, 2, \dots$

(b) To solve this problem, we first get Euler equation. We have

$$\max_{\{c_{it}, b_{i,t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned} c_{it} + q_t^* b_{i,t+1} &= b_{i,t} + \omega \\ \lim_{t \rightarrow \infty} b_{i,t+1} \left(\prod_{j=0}^t q_j \right) &\geq 0 \\ b_{i,0} = 0, c_{it} &\geq 0 \end{aligned}$$

If we substitute in for $c_{i,t}$ from the budget constraint we have:

$$\max_{\{b_{i,t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta_i^t u(b_{i,t} + \omega - q_t^* b_{i,t+1})$$

By taking the derivative w.r.t. $b_{i,t+1}$, we get the Euler equation:

$$\beta_i \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = q_t^*$$

or equivalently,

$$\beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} = \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

Now we can see that $c_{i,t+1} \neq c_{i,t}$ ($\forall i, \forall t$). Suppose not, without loss of generality let $c_{A,t+1} = c_{A,t}$. By feasibility condition, we know that $c_{B,t+1} = c_{B,t}$. Plug into the equation we get $\beta_A = \beta_B$, a contradiction. As a result, there cannot be any steady state in this economy.

We start to prove the convergence property of consumption path. First, we want to show that $\{c_{At}\}_{t=0}^{\infty}$ ($\{c_{Bt}\}_{t=0}^{\infty}$) is an increasing (decreasing) sequence. We already know that $c_{A,t+1} \neq c_{A,t}$ ($\forall t$). Now suppose that $c_{A,t+1} < c_{A,t}$ for some t . By the feasibility condition, we know that $c_{B,t+1} > c_{B,t}$. From the strict concavity of felicity function, we have

$$\begin{aligned} \frac{u'(c_{A,t+1})}{u'(c_{A,t})} &> 1 > \frac{u'(c_{B,t+1})}{u'(c_{B,t})} \\ \Rightarrow \beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} &> \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})} \end{aligned}$$

which contradicts Euler equation.

Since bounded monotone sequence has a limit, we have $c_{At} \rightarrow \bar{c}$ for $t \rightarrow \infty$. But we have shown that the economy has no steady state, so c_{At} can converge to nowhere but the boundary, i.e. $c_{At} \rightarrow \omega$ and $c_{Bt} \rightarrow 0$.

Alternatively some of you suggested doing the following:

From the first-order condition it will be the case that:

$$\frac{u'(c_{B,0})}{u'(c_{A,0})} \frac{u'(c_{A,t})}{u'(c_{B,t})} = \left(\frac{\beta_B}{\beta_A} \right)^t$$

If we take the limit on both sides we have:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{u'(c_{B,0})}{u'(c_{A,0})} \frac{u'(c_{A,t})}{u'(c_{B,t})} &= \lim_{t \rightarrow \infty} \left(\frac{\beta_B}{\beta_A} \right)^t \Leftrightarrow \\ \frac{u'(c_{B,0})}{u'(c_{A,0})} \lim_{t \rightarrow \infty} \frac{u'(c_{A,t})}{u'(c_{B,t})} &= 0 \end{aligned}$$

since $\frac{u'(c_{B,0})}{u'(c_{A,0})}$ is a constant and $\beta_B < \beta_A$. Therefore we conclude that:

$$\lim_{t \rightarrow \infty} \frac{u'(c_{A,t})}{u'(c_{B,t})} = 0$$

For the above equation to hold it must be the case that either $\lim_{t \rightarrow \infty} u'(c_{A,t}) = 0$ or that $\lim_{t \rightarrow \infty} u'(c_{B,t}) = \infty$ (or both). The first case is impossible however, since that would imply that $\lim_{t \rightarrow \infty} c_{A,t} = \infty$, but we know that $c_{A,t}$ is bounded by the total aggregate endowment. Therefore, $\lim_{t \rightarrow \infty} u'(c_{B,t}) = \infty \Leftrightarrow \lim_{t \rightarrow \infty} c_{B,t} = 0$ (Inada condition).

3. Consider an exchange economy with two infinitely-lived consumers with identical preferences given by:

$$E \left(\sum_{t=0}^{\infty} \beta^t \log(c_t) \right).$$

Both of the consumers have random endowments that depend on an (exogenous) sequence of state variables $\{s_t\}_{t=0}^{\infty}$. The s_t 's are statistically independent random variables with identical probability distributions. Specifically, for each t , $s_t = H$ with probability π and $s_t = L$ with probability $1 - \pi$, where π does not depend on time or on the previous realization of states. If $s_t = H$, then the first consumer's endowment is 2 and the second consumer's endowment is 1; if $s_t = L$, then the first consumer's endowment is 1 and the second consumer's endowment is 0. Markets are complete.

- (a) Carefully define a competitive equilibrium with date-0 trading for this economy. (Assume that consumers make decisions before observing the realization of the state in period 0.)
- (b) Determine the competitive equilibrium allocation in terms of primitives.
- (c) Determine the prices of the Arrow securities in terms of primitives.
- (d) Use your answer from part (c) to determine the average rate of return on a (one-period) riskfree bond in this economy.

Solution

1. A Competitive Equilibrium with date-0 trading for this economy is a vector of prices $\{p_{t,j}\}_{t=0}^{\infty}$ and a vector of quantities $\{c_{t,j}^i\}_{t=0}^{\infty}$ for $j = 1, 2$ and $j = H, L$ such that
- (1) $\{c_{t,j}^i\}_{t=0}^{\infty}$ for solves

$$\max \sum_{t=0}^{\infty} \beta^t [\pi \log(c_{t,H}^i) + (1 - \pi) \log(c_{t,L}^i)]$$

$$s.t. \sum_{t=0}^{\infty} \sum_{j=H,L} p_{t,j} c_{t,j}^i = \sum_{t=0}^{\infty} \sum_{j=H,L} p_{t,j} \omega_{t,j}^i$$

(2) Markets clear

$$\sum_{i=1,2} c_{t,j}^i = \sum_{i=1,2} \omega_{t,j}^i$$

2. Agents 1 and 2 have expected incomes each period of resp. $\pi + 1$ and π , with expected endowment at $2\pi + 1$. Ex ante, they will split actual aggregate consumption proportionably to their ratio of expected endowments:

$$C_H^1 = 3 \frac{\pi + 1}{2\pi + 1} \quad C_L^1 = 1 \frac{\pi + 1}{2\pi + 1} \quad C_H^2 = 3 \frac{\pi}{2\pi + 1} \quad C_L^2 = 1 \frac{\pi}{2\pi + 1}$$

3. Write down the FOCs for the sequential trading problem. Notice that there are four prices to be found (2 Arrow securities are for sale in each state).

If today is H

$$p_H = \pi\beta \quad p_L = 3(1 - \pi)\beta$$

If today is L

$$p_H = 1/3\beta\pi \quad p_L = (1 - \beta)\pi$$

4. An intuitive answer is $\dots 1/\beta$! If you want to go through the whole algebra, you can. The price of a risk-free bond in each state is the price of two arrow securities. Then by computing the average, you find the average riskless (gross) return.
4. Read the following writings by Robert E. Lucas, Jr. on the methodology of modern macroeconomics:
- (a) “What Economists Do” (available on the course web site).
 - (b) “Methods and Problems in Business Cycle Theory” (available on the course web site).
 - (c) Sections 1–4 and 6–7 of “Econometric Policy Evaluation: A Critique” (handed out in lecture).

Solution Do it !