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Problem Set 4	: Suggested Solutions	

1. (a) A consumer born in period t solves:

$$\max_{c_{1t}, c_{2,t+1}, m_{t+1}} c_{1t} + c_{2,t+1}$$

subject to

$$\begin{aligned} c_{1t} + q_t m_{t+1} &= 1 \\ c_{2,t+1} &= q_{t+1} m_{t+1} \\ m_{t+1} &\geq 0. \end{aligned}$$

Substituting in the budget constraints, the problem becomes:

$$\max_{m_{t+1}} (1 - q_t m_{t+1} + q_{t+1} m_{t+1})$$

The first order condition with respect to m_{t+1} gives us in equilibrium:

$$-q_t + q_{t+1} = 0 \Rightarrow q_t = q_{t+1} = q.$$

with

$$1 - q_t m_{t+1} \geq 0 \Rightarrow q_t = q \leq \frac{1}{m_{t+1}} = \frac{1}{M}$$

and

$$q_{t+1} m_{t+1} \geq 0 \Rightarrow q_{t+1} = q \geq 0$$

where we have used the fact that in equilibrium $m_{t+1} = M$.
And so we have

$$\begin{aligned} c_{1t} &= 1 - qM \\ c_{2,t+1} &= qM \qquad \forall t \geq 0, \end{aligned}$$

which satisfy the market clearing conditions.
And so we have a competitive equilibrium where

$$\begin{aligned} c_{2,0} &= q_0 M = qM, \\ c_{1t} &= 1 - qM \\ c_{2,t+1} &= qM \qquad \forall t \geq 0, \end{aligned}$$

with prices

$$q_t = q \qquad \text{for all } t \geq 0$$

where

$$q \in [0, \frac{1}{M}].$$

(b) The competitive equilibrium allocation of consumption would be Pareto optimal if each young could give their endowment to the old in each period.
To see this:
If $q = 0$, we have:

$$\begin{aligned} c_{2,0} &= 0 \\ c_{1t} &= 1 \qquad \forall t \geq 0 \\ c_{2,t+1} &= 0 \qquad \forall t \geq 0. \end{aligned}$$

If $q \in \left(0, \frac{1}{M}\right)$,we have:

$$\begin{aligned} c_{2,0} &< 1 \\ c_{1t} &< 1 \qquad \forall t \geq 0 \\ c_{2,t+1} &< 1 \qquad \forall t \geq 0, \end{aligned}$$

with

$$c_{1t} + c_{2,t+1} = 1 \qquad \forall t \geq 0.$$

If $q = \frac{1}{M}$,we have:

$$\begin{aligned} c_{2,0} &= 1 \\ c_{1t} &= 0 \qquad \forall t \geq 0 \\ c_{2,t+1} &= 1 \qquad \forall t \geq 0. \end{aligned}$$

As we can see, all generation born at time $t \geq 0$ receive 1 unit of consumption in each of the three cases. However, the initial old receive the maximum amount possible (i.e. one unit) only when we have $q = \frac{1}{M}$.

2. (a) The consumer's problem is:

$$\max_{\{c_{1t}, c_{2,t+1}, s_t\}} u\left(c_{1t}\right) + \beta u\left(c_{2,t+1}\right)$$

s.t.

$$\begin{aligned} w_t &= c_{1t} + s_t \\ c_{2,t+1} &= R_{t+1} s_t + \lambda w_{t+1} \end{aligned}$$

We substitute in for c_{1t} and $c_{2,t+1}$ and take the f.o.c. w.r.t. s_t:

$$-u'(w_t - s_t) + \beta u'(R_{t+1}s_t + \lambda w_{t+1}) R_{t+1} = 0$$

Assuming that prices are competitively determined, we have

$$w_{t+1} = (1-a) k_{t+1}^a (1+\lambda)^{-a}$$

and

$$R_{t+1} = a k_{t+1}^{a-1} (1+\lambda)^{1-a}.$$

Moreover, given log utility and the fact that

$$s_t = k_{t+1}$$

in equilibrium, we have:

$$\begin{aligned}
\frac{\beta a k_{t+1}^{a-1}(1+\lambda)^{1-a}}{a k_{t+1}^{a-1} k_{t+1} (1+\lambda)^{1-a} + \lambda (1-a) k_{t+1}^a (1+\lambda)^{-a}} &= \frac{1}{(1-a) k_t^a (1+\lambda)^{-a} - k_{t+1}} \\
\frac{\beta a k_{t+1}^{a-1}(1+\lambda)}{a k_{t+1}^a (1+\lambda) + \lambda(1-a) k_{t+1}^a} &= \frac{1}{(1-a) k_t^a (1+\lambda)^{-a} - k_{t+1}} \\
\frac{\beta a (1+\lambda)}{a k_{t+1} (1+\lambda) + \lambda(1-a) k_{t+1}} &= \frac{1}{(1-a) k_t^a (1+\lambda)^{-a} - k_{t+1}} \\
\beta a (1+\lambda)(1-a) k_t^a (1+\lambda)^{-a} - \beta a (1+\lambda) k_{t+1} &= k_{t+1} (a(1+\lambda) + \lambda(1-a)) \\
k_{t+1} (a + a\lambda + \lambda - a\lambda + \beta a (1+\lambda)) &= \beta a (1-a)(1+\lambda)^{1-a} k_t^a \Leftrightarrow \\
k_{t+1} &= \frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} k_t^a
\end{aligned}$$

And so the steady state level of capital is:

$$\overline{k} = \left(\frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} \right)^{\frac{1}{1-a}}.$$

(b) The steady state is dynamically efficient if:

$$\begin{aligned}
f'\left(\overline{k}\right) &> 1 \Longrightarrow \\
a \left(\frac{\beta a (1-a)(1+\lambda)^{1-a}}{a + \lambda + \beta a (1+\lambda)} \right)^{\frac{a-1}{1-a}} &> 1 \Leftrightarrow \\
a \frac{a + \lambda + \beta a (1+\lambda)}{\beta a (1-a)(1+\lambda)^{1-a}} &> 1 \Leftrightarrow \\
\frac{a + \lambda + \beta a (1+\lambda)}{\beta (1-a)(1+\lambda)^{1-a}} &> 1
\end{aligned}$$

Let's take the extreme case where $\lambda = 1$. Then the steady state level of capital will be:

$$\begin{aligned} \frac{a+1+2\beta a}{\beta(1-a)2^{1-a}} &> 1 \Leftrightarrow \\ a+1+2\beta a &> \beta(1-a)2^{1-a} \end{aligned}$$

Since

$$\beta < 1, (1-a) < 1 \quad \text{and} \quad 2^{1-a} < 1,$$

it will be the case that the l.h.s is less than the r.h.s. (since the r.h.s. is clearly larger than 1). Therefore, when $\lambda = 1$, the steady state level of capital is dynamically efficient for *any* value of a and β. If we define

$$h(\lambda) \equiv \frac{a+\lambda+\beta a(1+\lambda)}{\beta(1-a)(1+\lambda)^{1-a}},$$

and notice that $h(\cdot)$ is a continuous function, it becomes apparent that there will be some "cutoff" value $\overline{\lambda} \in (0,1)$, above which the $h(\lambda) > 1$, for all a and β (and thus the steady state level of capital is dynamically efficient).

The intuition behind this result is that the higher the labor income the old generation receives, the less they need to save when they are young and thus the capital stock will be lower as well. Thus the probability that the steady state level of capital exceeds the golden rule, will also be lower.

3. The agent's problem

$$\begin{aligned} &\max_{c_{1,t}, c_{2,t+1}, s_t} \log(c_{1,t}) + \beta \log(c_{2,t+1}) \\ &s.t. \\ &c_{1,t} + s_t + d = w_t \\ &c_{2,t+1} = R_{t+1}s_t + d. \end{aligned}$$

Plugging in the budget constraints, the first-order condition with respect to s_t is:

$$\begin{aligned} \frac{1}{w_t - s_t - d} &= \frac{\beta R_{t+1}}{R_{t+1}s_t + d} \\ R_{t+1}s_t + d &= \beta R_{t+1}(w_t - s_t - d) \\ s_t + \frac{d}{R_{t+1}} &= \beta(w_t - s_t - d) \\ s_t &= \frac{\beta}{1+\beta}(w_t - d) - \frac{1}{1+\beta}\frac{d}{R_{t+1}}. \end{aligned}$$

At the steady state we have:

$$\begin{aligned} K^* &= \frac{\beta}{1+\beta}(w_t - d) - \frac{1}{1+\beta}\frac{d}{R_{t+1}} \\ &= \frac{\beta}{1+\beta}(w(K) - d) - \frac{1}{1+\beta}\frac{d}{R(K)}, \end{aligned}$$

where the time subscript has been dropped for convenience.
Setting $K = g(d)$:

$$g(d) = \frac{\beta}{1+\beta}\left(w(g(d)) - d\right) - \frac{1}{1+\beta}\frac{d}{R(g(d))}.$$

Totally differentiating with respect to d, we have:

$$\begin{aligned} g'(d) &= \frac{\beta}{1+\beta}\big(w'(g(d))g'(d) - 1\big) - \frac{1}{1+\beta}\left[\frac{R(g(d)) - dR'(g(d))g'(d)}{R(g(d))^2}\right] \\ &= \left[\frac{\beta}{1+\beta}w'(g(d)) + \frac{1}{1+\beta}\frac{dR'(g(d))}{R(g(d))^2}\right]g'(d) - \left[\frac{\beta}{1+\beta} + \frac{1}{1+\beta}\frac{1}{R(g(d))}\right]. \end{aligned}$$

And so:

$$g'(d) = \frac{-\left[\frac{\beta}{1+\beta} + \frac{1}{1+\beta}\frac{1}{R(g(d))}\right]}{1 - \left[\frac{\beta}{1+\beta}w'(g(d)) + \frac{1}{1+\beta}\frac{dR'(g(d))}{R(g(d))^2}\right]}. \tag{1}$$

We know

$$\begin{aligned} w(K) &= (1-\alpha)K^\alpha N^{-\alpha} \\ &= (1-\alpha)K^\alpha \end{aligned}$$

in equilibrium. And so

$$w'(K) = \alpha(1-\alpha)K^{\alpha-1} > 0.$$

Similarly,

$$\begin{aligned} R(K) &= \alpha K^\alpha N^{1-\alpha} + (1-\delta) \\ &= \alpha K^\alpha + (1-\delta) \end{aligned}$$

in equilibrium. And so

$$R'(K) = \alpha(\alpha-1)K^{\alpha-2} < 0.$$

As we cannot sign the numerator in [1], in general we need to have

$$g'(d) < 0 \Leftrightarrow \left(1 - \left[\frac{\beta}{1+\beta}w'(g(d)) + \frac{1}{1+\beta}\frac{dR'(g(d))}{R(g(d))^2}\right]\right) > 0$$

for an increase in d to reduce the steady-state aggregate capital stock.
Is an increase in d (from $d = 0$) Pareto improving?
Clearly it is for the initial old.
Now let's consider the steady state lifetime utility of a typical generation born at time $t \geq 0$:

$$u(w(g(d)) - g(d) - d) + \beta u(R(g(d))g(d) + d)$$

where we have plugged in $g(d)$ for K.
Differentiating with respect to d and letting $c_i, i = 1, 2$, denote the steady state consumption of the young and old, respectively, we have:

$$\begin{aligned}
& u'(c_1)(w'(g(d))g'(d) - g'(d) - 1) + \beta u'(c_2)(R'(g(d))g'(d)g(d) + g'(d)R(d) + 1) \\
= \; & u'(c_1)(w'g' - g' - 1) + \beta u'(c_2)(R'g'g + g'R + 1) \\
= \; & g' \underbrace{\left(-u'(c_1) + \beta R u'(c_2)\right)}_{= 0 \quad \text{(this is the f.o.c. for savings when young)}} + u'(c_1)(w'g' - 1) + \beta u'(c_2)(R'g'g + 1) \\
= \; & u'(c_1)(w'g' - 1) + \beta u'(c_2)(R'g'g + 1) \\
= \; & u'(c_1)\left[\left(\alpha(1-\alpha)K^{\alpha-1}\right)g' - 1\right] - \beta u'(c_2)\left[\left(\alpha(\alpha-1)K^{\alpha-2}\right)g'K - 1\right] \\
& \quad \text{where we plugged in the expressions for } w' \text{ and } R' \text{and used } g(d) = K \\
= \; & \left(\left(\alpha(1-\alpha)K^{\alpha-1}\right)g' - 1\right)\left(u'(c_1) - \beta u'(c_2)\right).
\end{aligned}$$

We know

$$u'(c_1) = \beta R u'(c_2).$$

So

$$\begin{aligned}
u'(c_1) - \beta u'(c_2) &= \beta R u'(c_2) - \beta u'(c_2) \\
&= \beta\left(R - 1\right) u'(c_2) \\
&< 0 \quad \text{if} \;\; R < 1.
\end{aligned}$$

That is, the expression will be less than zero, if the steady state with is dynamically inefficient. So if $R < 1$,

$$\left(\left(\alpha(1-\alpha)K^{\alpha-1}\right)g' - 1\right)\left(u'(c_1) - \beta u'(c_2)\right) > 0.$$

[Recall that $g' < 0$ if $d = 0$.]
In other words, if the economy starts in a dynamically inefficient steady state with $d = 0$ (i.e. no social security), then a small increase in d improves the welfare of all generations.
In the initial steady state is dynamically efficient, then increasing d from zero improves the welfare of the initial old, but lowers the welfare of subsequent generations.