Econ 510a (second half)
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Solutions for Homework #5

Question 1

a) A recursive competitive equilibrium for the neoclassical growth model with valued leisure is a set of functions:

$$\begin{aligned}
\textit{price function} &: r\left(\overline{k}\right), w\left(\overline{k}\right)\\
\textit{policy function} &: k' = g_k\left(k, \overline{k}\right), l = g_l\left(k, \overline{k}\right)\\
\textit{value function} &: v\left(k, \overline{k}\right)\\
\textit{aggregate state} &: \overline{k}' = G\left(\overline{k}\right), \overline{l} = \overline{l}(\overline{k})
\end{aligned}$$

such that:

(1) Given $\overline{k}' = G\left(\overline{k}\right)$, $k' = g_k\left(k, \overline{k}\right), l = g_l\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned}
v\left(k, \overline{k}\right) = &\max_{\{c,l,k'\}} U(c,l) + \beta v\left(k', \overline{k}'\right)\\
&s.t.\\
&c + k' = r\left(\overline{k}\right)k + (1-\delta)k + w\left(\overline{k}\right)(L-l)\\
&\overline{k}' = G\left(\overline{k}\right)
\end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned}
r\left(\overline{k}\right) &= F_1\left(\overline{k}, L - \overline{l}(\overline{k})\right)\\
w\left(\overline{k}\right) &= F_2\left(\overline{k}, L - \overline{l}(\overline{k})\right)
\end{aligned}$$

(3) Consistency:

$$\begin{aligned}
G\left(\overline{k}\right) &= g_k\left(\overline{k}, \overline{k}\right)\\
\overline{l}(\overline{k}) &= g_l\left(\overline{k}, \overline{k}\right)
\end{aligned}$$

b) Solve a typical consumer's problem

$$
\begin{aligned}
v\left(k,\overline{k}\right) =& \max_{\{l,k'\}} U\left(r\left(\overline{k}\right)k+(1-\delta)k+w\left(\overline{k}\right)(L-l)-k',l\right)+\beta v\left(k',\overline{k}'\right)\\
& s.t.\\
& \overline{k}' = G\left(\overline{k}\right)
\end{aligned}
$$

We get the F.O.C. as

$$
\begin{aligned}
\{l\} &: U_1(c,l)w\left(\overline{k}\right) = U_2(c,l)\\
\{k'\} &: U_1(c,l) = \beta v_1\left(k',\overline{k}'\right)
\end{aligned}
$$

Use the envelope condition

$$
v_1\left(k,\overline{k}\right) = U_1(c,l)\left(r\left(\overline{k}\right)+(1-\delta)\right)
$$

We get the optimality condition:

$$
\begin{aligned}
\{l_t\} &: U_1(c_t,l_t)w\left(\overline{k}_t\right) = U_2(c_t,l_t)\\
\{k_{t+1}\} &: U_1(c_t,l_t) = \beta U_1(c_{t+1},l_{t+1})\left(r\left(\overline{k}_{t+1}\right)+(1-\delta)\right)
\end{aligned}
$$

Correspondingly, the functional F.O.C. is

$$
\begin{aligned}
\{l_t\} &: U_1\left(c_t,g_l\left(k_t,\overline{k}_t\right)\right)w\left(\overline{k}_t\right) = U_2\left(c_t,g_l\left(k_t,\overline{k}_t\right)\right)\\
\{k_{t+1}\} &: U_1\left(c_t,g_l\left(k_t,\overline{k}_t\right)\right) = \beta U_1\left(c_{t+1},g_l\left(k_{t+1},\overline{k}_{t+1}\right)\right)\left(r\left(G\left(\overline{k}_t\right)\right)+(1-\delta)\right)
\end{aligned}
$$

where $c_t = r\left(\overline{k}_t\right)k_t+(1-\delta)k_t+w\left(\overline{k}_t\right)\left(L-g_l\left(k_t,\overline{k}_t\right)\right)-g_k\left(k_t,\overline{k}_t\right)$,
$r\left(\overline{k}\right) = F_1\left(\overline{k},L-\overline{l}\right) = F_1\left(\overline{k},L-g_l\left(\overline{k},\overline{k}\right)\right)$, and
$w\left(\overline{k}\right) = F_2\left(\overline{k},L-\overline{l}\right) = F_2\left(\overline{k},L-g_l\left(\overline{k},\overline{k}\right)\right)$.

c) Impose the equilibrium conditions $k_t = \overline{k}_t, l_t = \overline{l}_t$, we have

$$
\begin{aligned}
\{l_t\} &: U_1\left(\overline{c}_t,\overline{l}_t\right)F_2\left(\overline{k}_t,L-\overline{l}_t\right) = U_2\left(\overline{c}_t,\overline{l}_t\right)\\
\{k_{t+1}\} &: U_1\left(\overline{c}_t,\overline{l}_t\right) = \beta U_1\left(\overline{c}_{t+1},\overline{l}_{t+1}\right)\left(F_1\left(\overline{k}_t,L-\overline{l}_t\right)+(1-\delta)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{c}_t &= F_1\left(\overline{k}_t,L-\overline{l}_t\right)\overline{k}_t+(1-\delta)\overline{k}_t+F_2\left(\overline{k}_t,L-\overline{l}_t\right)\left(L-\overline{l}_t\right)-\overline{k}_{t+1}\\
&= F\left(\overline{k}_t,L-\overline{l}_t\right)+(1-\delta)\overline{k}_t-\overline{k}_{t+1}
\end{aligned}
$$

which are identical to the first-order conditions associated with the planning problem for this economy.

Question 2

a) You’re on own your own here..

b) Given the transition matrix:

$$P = \begin{pmatrix} 0.98 & 0.019 & 0.001 \\ 0.019 & 0.98 & 0.001 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

we can calculate the invariant distribution π according to the formula:

$$\pi' = \pi' P$$

and we get that:

$$\pi_1 = 0.4995$$
$$\pi_2 = 0.4995$$
$$\pi_3 = 0.001$$

The unconditional expected value of the dividend is:

$$E(d) = \pi_1 d_1 + \pi_2 d_2 + \pi_3 d_3 = 0.9995$$

The unconditional variance is:

$$Var(d) = \sum_{i=1}^{3} \pi_i (d_i - E(d))^2 = 0.000649$$

and therefore the coefficient of variation of the dividend is:

$$\frac{\sqrt{Var(d)}}{E(d)} = \frac{0.025482}{0.9995} = 0.025495$$

which is quite close to 0.02.

c) The prices of the nine (3^2) Arrow securities are:

$$
\begin{aligned}
q_{11} &= \frac{\beta u'(d_1)}{u'(d_1)}\pi_{11} = \beta\pi_{11} = 0.97804 \\
q_{12} &= \frac{\beta u'(d_2)}{u'(d_1)}\pi_{12} = \beta\left(\frac{d_2}{d_1}\right)^{-\sigma}\pi_{12} = 0.020542 \\
q_{13} &= \frac{\beta u'(d_3)}{u'(d_1)}\pi_{13} = \beta\left(\frac{d_3}{d_1}\right)^{-\sigma}\pi_{13} = 0.003834 \\
q_{21} &= \frac{\beta u'(d_1)}{u'(d_2)}\pi_{21} = \beta\left(\frac{d_1}{d_2}\right)^{-\sigma}\pi_{21} = 0.017504 \\
q_{22} &= \frac{\beta u'(d_2)}{u'(d_2)}\pi_{22} = \beta\pi_{22} = 0.97804 \\
q_{23} &= \frac{\beta u'(d_3)}{u'(d_2)}\pi_{23} = \beta\left(\frac{d_3}{d_2}\right)^{-\sigma}\pi_{23} = 0.003834 \\
q_{31} &= \frac{\beta u'(d_1)}{u'(d_3)}\pi_{31} = \beta\left(\frac{d_1}{d_3}\right)^{-\sigma}\pi_{31} = 0.119906 \\
q_{32} &= \frac{\beta u'(d_2)}{u'(d_3)}\pi_{32} = \beta\left(\frac{d_2}{d_3}\right)^{-\sigma}\pi_{32} = 0.129894 \\
q_{33} &= \frac{\beta u'(d_3)}{u'(d_3)}\pi_{33} = 0
\end{aligned}
$$

d) The price of a risk free bond in each of the 3 states is:

$$
\begin{aligned}
q_1 &= q_{11} + q_{12} + q_{13} = 1.00242 \\
q_2 &= q_{21} + q_{22} + q_{23} = 0.999378 \\
q_3 &= q_{31} + q_{32} + q_{33} = 0.2498
\end{aligned}
$$

The price of a perpetual claim to the tree’s dividends in each of the 3 states of the world is:

$$
\begin{aligned}
p_1 &= q_{11}(d_1 + p_1) + q_{12}(d_2 + p_2) + q_{13}(d_3 + p_3) \\
p_2 &= q_{21}(d_1 + p_1) + q_{22}(d_2 + p_2) + q_{23}(d_3 + p_3) \\
p_3 &= q_{31}(d_1 + p_1) + q_{32}(d_2 + p_2) + q_{33}(d_3 + p_3)
\end{aligned}
$$

Substituting in for the prices of the Arrow securities and the dividends, we get a system of 3 equations and 3 unknowns and after solving it we get:

$$
\begin{aligned}
p_1 &= 504.616 \\
p_2 &= 467.9896 \\
p_3 &= 121.5451
\end{aligned}
$$

e) The unconditional expected value of the risk-free rate of return is:

$$
r^f = \pi_1\left(\frac{1}{q_1} - 1\right) + \pi_2\left(\frac{1}{q_2} - 1\right) + \pi_3\left(\frac{1}{q_3} - 1\right) = 0.002108
$$

We know that the rate of return on the tree, given state i today and state j tomorrow, is:

$$r_{ij}^e \equiv \frac{p_j + d_j}{p_i} - 1$$

Therefore the expected rate of return, in each of the 3 states is:

$$r_1^e = \pi_{11}\left(\frac{p_1 + d_1}{p_1} - 1\right) + \pi_{12}\left(\frac{p_2 + d_2}{p_1} - 1\right) + \pi_{13}\left(\frac{p_3 + d_3}{p_1} - 1\right) = -0.000119$$
$$r_2^e = \pi_{21}\left(\frac{p_1 + d_1}{p_2} - 1\right) + \pi_{22}\left(\frac{p_2 + d_2}{p_2} - 1\right) + \pi_{23}\left(\frac{p_3 + d_3}{p_2} - 1\right) = 0.002841$$
$$r_3^e = \pi_{31}\left(\frac{p_1 + d_1}{p_3} - 1\right) + \pi_{32}\left(\frac{p_2 + d_2}{p_3} - 1\right) + \pi_{33}\left(\frac{p_3 + d_3}{p_3} - 1\right) = 3.00923$$

Finally the unconditional expected rate of return of the tree is:

$$r^e = \pi_1 r_1^e + \pi_2 r_2^e + \pi_3 r_3^e = 0.004369$$

Therefore the equity premium is:

$$r^e - r^f = 0.004369 - 0.002108 = 0.002261$$

or 0.2261% which is considerably less than 1.5%, which is its value in quarterly U.S. data. Only with a coefficient of risk aversion of approximately 6.8, can we match the observed value.

f) The price of an Arrow security that pays 1 unit of the consumption good in state j, two periods from today, is given by (assuming we're in state i today):

$$q'_{ij} = \frac{\beta^2 u'(d_j)}{u'(d_i)} \Pr(s_{t+2} = s_j | s_t = s_i)$$

We know that:

$$\Pr(s_{t+2} = s_j | s_t = s_i) = \sum_{k=1}^{n} P_{ik} \cdot P_{kj} =$$
$$= [P^2]_{ij}$$

where $[P^2]_{ij}$ denotes the $(i,j)^{th}$ entry of the matrix P^2 which is equal to:

$$P^2 = \begin{pmatrix} 0.96126 & 0.03774 & 0.001 \\ 0.03774 & 0.96126 & 0.001 \\ 0.4995 & 0.4995 & 0.001 \end{pmatrix}$$

Now we can calculate the prices of the 9 two-state Arrow securities:

$$q'_{11} = \frac{\beta^2 u'(d_1)}{u'(d_1)}\pi'_{11} = \beta^2\pi'_{11} = 0.957419$$
$$q'_{12} = \frac{\beta^2 u'(d_2)}{u'(d_1)}\pi'_{12} = \beta^2\left(\frac{d_2}{d_1}\right)^{-\sigma}\pi'_{12} = 0.04072$$
$$q'_{13} = \frac{\beta^2 u'(d_3)}{u'(d_1)}\pi'_{13} = \beta^2\left(\frac{d_3}{d_1}\right)^{-\sigma}\pi'_{13} = 0.004141$$
$$q'_{21} = \frac{\beta^2 u'(d_1)}{u'(d_2)}\pi'_{21} = \beta^2\left(\frac{d_1}{d_2}\right)^{-\sigma}\pi'_{21} = 0.034699$$
$$q'_{22} = \frac{\beta^2 u'(d_2)}{u'(d_2)}\pi'_{22} = \beta^2\pi'_{22} = 0.957419$$
$$q'_{23} = \frac{\beta^2 u'(d_3)}{u'(d_2)}\pi'_{23} = \beta^2\left(\frac{d_3}{d_2}\right)^{-\sigma}\pi'_{23} = 0.003826$$
$$q'_{31} = \frac{\beta^2 u'(d_1)}{u'(d_3)}\pi'_{31} = \beta^2\left(\frac{d_1}{d_3}\right)^{-\sigma}\pi'_{31} = 0.119546$$
$$q'_{32} = \frac{\beta^2 u'(d_2)}{u'(d_3)}\pi'_{32} = \beta^2\left(\frac{d_2}{d_3}\right)^{-\sigma}\pi'_{32} = 0.129504$$
$$q'_{33} = \frac{\beta^2 u'(d_3)}{u'(d_3)}\pi'_{33} = 0$$

where $\pi'_{ij} = \Pr(s_{t+2} = s_j | s_t = s_i)$. Therefore, the price of a risk free bond in each of the 3 states, two periods from today, is:

$$q'_1 = q'_{11} + q'_{12} + q'_{13} = 1.00228$$
$$q'_2 = q'_{21} + q'_{22} + q'_{23} = 0.995944$$
$$q'_3 = q'_{31} + q'_{32} + q'_{33} = 0.24905$$

and thus the unconditional expected value of the risk-free rate of return is on a two-period risk free bond is:

$$r_2 = \pi_1\left(\frac{1}{q'_1} - 1\right) + \pi_2\left(\frac{1}{q'_2} - 1\right) + \pi_3\left(\frac{1}{q'_3} - 1\right) = 0.003913$$

Moreover:

$$(1 + r_2)^{1/2} - 1 = 0.001955$$

Which is quite close to $r_1 = 0.002108$, the unconditional expected value of the rate of return on a one-period risk-free bond. Thus the term structure is fairly flat between these two interest rates.

Question 3

(a) Again, can't help you here.

(b) If normalize the labor supply to equal 1, the production function is given by:

$$F(k_t, \gamma^t) = \gamma^t F\left(\frac{k_t}{\gamma^t}, 1\right)$$

In the decentralized version of the model, the consumer's problem is given by:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + k_{t+1} = (1-\tau)\left(r\left(\overline{k}_t\right) - \delta\right)k_t + k_t + w\left(\overline{k}_t\right) + T\left(\overline{k}_t\right)$$

where rental rate of capital and labor are determined competitively:

$$\begin{aligned} r\left(\overline{k}_t\right) &= F_1\left(\frac{\overline{k}_t}{\gamma^t}, 1\right) \\ w\left(\overline{k}_t\right) &= \gamma^t F_1\left(\frac{\overline{k}_t}{\gamma^t}, 1\right)\left(-\frac{\overline{k}_t}{\gamma^t}\right) + F\left(\frac{\overline{k}_t}{\gamma^t}, 1\right)\gamma^t \end{aligned}$$

we observe that the rate of return of capital is constant over time (since $\overline{k}_t$ also grows at rate γ), whereas the wage rate per unit of labor is growing over time.

If we transform the problem the way it is done in the notes, such that for every variable x_t, $\widehat{x}_t \equiv \frac{x_t}{\gamma^t}$, we have:

$$\max_{\{\widehat{c}_t, \widehat{k}_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{\widehat{c}_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} (\beta\gamma^{1-\sigma})^t \frac{1-\gamma^{-t(1-\sigma)}}{1-\sigma}$$

s.t.

$$\widehat{c}_t + \gamma\widehat{k}_{t+1} = (1-\tau)\left(r\left(\widehat{\overline{k}_t}\right) - \delta\right)\widehat{k}_t + \widehat{k}_t + \frac{w\left(\widehat{\overline{k}_t}\right)}{\gamma^t} + \frac{T\left(\widehat{\overline{k}_t}\right)}{\gamma^t}$$

Substituting in for $\widehat{c}_t$ and taking the f.o.c. gives us:

$$(\widehat{c}_t)^{-\sigma}\gamma = \beta[(1-\tau)\left(r\left(\widehat{\overline{k}_t}\right) - \delta\right) + 1](\widehat{c}_{t+1})^{-\sigma}$$

In the long run, it will be the case that $\widehat{c}_t = \widehat{c}_{t+1}$, so the above equation becomes:

$$\gamma = \beta[(1-\tau)\left(r\left(\widehat{k}_t\right)-\delta\right)+1]$$

In an economy without any capital income taxation, we would similarly get:

$$\gamma = \left(1+r\left(\widehat{k}_t\right)-\delta\right)$$

Since γ is a known parameters it must be the case that:

$$r\left(\widehat{k}_t^{tax}\right) > r\left(\widehat{k}_t\right) \Rightarrow$$
$$\widehat{k}_t^{tax} < \widehat{k}_t$$

where $\widehat{k}_t^{tax}$ is the steady state capital stock per efficiency unit of labor in an economy where the government taxes capital income. Summarizing, we see that the growth rate of both economies is the same and equal to γ. The growth rate is invariant to the tax rate because after transforming the model (by guessing that everything grows at rate γ along the balanced growth path), it is apparent that the transformed economy converges to a steady state (using standard arguments for a model with no growth we went over in the first lectures). So, after undoing the transformation, one sees that the economy is converging to a balanced growth path with the conjectured growth rate. (If one had guessed the wrong growth rate for the balanced growth path, then the transformed economy would not be stationary and one could not use standard arguments to show that the transformed economy converges to a steady state.)

(c) If capital income taxation was suddenly eliminated, the relevant steady state level of capital would now be $\widehat{k}_t$. As we saw in part b), the long run rate of growth is going to remain the same, γ, but in the short run the capital stock has to grow in a rate higher than γ in order to converge to the new steady state. Once it converges, the growth rate of capital will be γ again.

Question 4

(b) For the case of the Ak model the interest is equal to A. Moreover labor does not participate in the production process. If we assume there is a tax on capital income (net depreciation), the consumer's problem can be written as:

$$v(k) = \max_{\{c,k'\}} \{u(c) + \beta v(k')\}$$

s.t.

$$c + k' = (1-\tau)(r-\delta)k + k + T$$

Since the rental rate of capital is competitively determined, in the Ak model it will be the case that $r = A$.

Solving the above problem in the usual way (substitute in for c, find the f.o.c., the envelope condition, update one period and plug in the f.o.c.) we derive the following Euler equation:

$$\frac{u'(c_{t+1})}{u'(c_t)}\beta[(1-\tau)(A-\delta)+1] = 1$$

Assuming CRRA utility, the above Euler equation becomes:

$$\frac{c_{t+1}}{c_t} = [\beta((1-\tau)(A-\delta)+1)]^{1/\sigma} \Rightarrow$$

We conjecture (as we did in class) that $\overline{k}_{t+1} = e^g\overline{k}_t$. and thus $c_{t+1} = e^g c_t$ Then the Euler equation becomes:

$$\begin{aligned}\frac{e^{g^\tau}c_t}{c_t} &= [\beta((1-\tau)(A-\delta)+1)]^{1/\sigma} \Leftrightarrow \\ e^{g^\tau} &= [\beta((1-\tau)(A-\delta)+1)]^{1/\sigma} \Leftrightarrow \\ g^\tau &= \sigma^{-1}[\log\beta + \log[(1-\tau)(A-\delta)+1]\end{aligned}$$

whereas the growth rate of an economy without a tax rate on capital income, would be (solving out in the same way we did so far):

$$g = \sigma^{-1}[\log\beta + \log(1+A-\delta)]$$

which is clearly higher.

(c) If capital income taxation was suddenly eliminated, the economy would immediately switch to the new growth rate g (and of course output would also grow at the same rate g).