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Suggested Solutions	: Problem Set 5	

1. (a) A sequential competitive equilibrium for the economy is a sequence $\{c_t^*, a_t^*, q_t^*\}_{t\geq 0}$ such that

(1) $\{c_t^*, a_t^*\}_{t\geq 0}$ solves the consumer's problem:

$$\max \sum_{t\geq 0} \beta^t \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma}$$

$$s.t. \quad c_t + q_t^* a_{t+1} = w_t + a_t \qquad \forall t \geq 0$$

$$a_0 = 0, \quad nPg: \lim_{t\to\infty} a_{t+1} \left(\prod_{j=0}^{t} q_j^* \right) \geq 0$$

(2)markets clearing

Assets market: $a_t^* = 0 \quad \forall t \geq 0.$

Goods market: $c_t^* = w_t \quad \forall t \geq 0.$

(b) The dynamic programming problem faced by the consumer is (under the conjecture of $q_t = q \;\; \forall t$):

$$v(a_t, c_{t-1}, w_t) = \max_{c_t, a_{t+1}} \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma} + \beta v(a_{t+1}, c_t, g w_t)$$

$$s.t. \quad c_t + q a_{t+1} = w_t + a_t$$

(c) Let's derive the Euler equation for the consumer's problem. Let $u(c_t, c_{t-1}) \equiv \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma}$, then the FOC with respect to a_{t+1} is:

$$u_1(c_t, c_{t-1}) q + \beta v_2(a_{t+1}, c_t, g w_t) q = \beta v_1(a_{t+1}, c_t, g w_t) \tag{1}$$

and the envelope conditions,

$$a_t: \qquad v_1(a_t, c_{t-1}, w_t) = u_1(c_t, c_{t-1}) + \beta v_2(a_{t+1}, c_t, g w_t) \tag{2}$$

$$c_{t-1}: \qquad v_2(a_t, c_{t-1}, w_t) = u_2(c_t, c_{t-1}) \tag{3}$$

Eqs. (2) and (3) imply

$$v_1(a_t, c_{t-1}, w_t) = u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t) \tag{4}$$

and finally, replacing eqs. (3) and (4) in (1), we get the following Euler eq.

$$u_1(c_t, c_{t-1}) q + \beta q u_2(c_{t+1}, c_t) = \beta u_1(c_{t+1}, c_t) + \beta^2 u_2(c_{t+2}, c_{t+1}) \tag{5}$$

and using the particular functional form for u we get

$$q\left((c_t - \lambda c_{t-1})^{-\sigma} - \beta\lambda (c_{t+1} - \lambda c_t)^{-\sigma} \right) = \beta (c_{t+1} - \lambda c_t)^{-\sigma} + \beta^2 \lambda (c_{t+2} - \lambda c_{t+1})^{-\sigma}$$

(d) From the Euler eq. we get that

$$q = \frac{\beta(c_{t+1} - \lambda c_t)^{-\sigma} + \beta^2\lambda(c_{t+2} - \lambda c_{t+1})^{-\sigma}}{(c_t - \lambda c_{t-1})^{-\sigma} - \beta\lambda(c_{t+1} - \lambda c_t)^{-\sigma}} \tag{6}$$

Now, in equilibrium we must have $c_t = w_t$, imposing this in (6) we get that the equilibrium bond price is

$$q = \frac{\beta(gw_{t+1} - \lambda w_t)^{-\sigma} + \beta^2\lambda(g^2 w_{t+2} - \lambda g w_{t+1})^{-\sigma}}{(w_t - \frac{\lambda}{g} w_{t-1})^{-\sigma} - \beta\lambda(gw_{t+1} - \lambda w_t)^{-\sigma}} = \beta g^{-\sigma}$$

which is in fact independent of t, thus our conjecture was correct. The effects of the different parameters are:

$\uparrow \beta \Rightarrow \uparrow q$: more patience will lead to a higher demand for savings, and thus to a higher bond price.

$\uparrow g \Rightarrow \downarrow q$: the higher the rate of growth of endowment, the less the need for savings, the lower the asset price.

$\uparrow \sigma \Rightarrow \downarrow q$: the higher the coefficient of risk aversion, the lower the bond price.

Changes in λ does not affect q, this is only due to the special utility function used here.

2. (a) Yes, markets are complete in this economy. As the number if linearly independent returns is equal to the number of possible states, ie, span$\begin{bmatrix} 1 & d_L \\ 1 & d_H \end{bmatrix} = \mathbb{R}^2$, agents are able to trade in all goods independently.

(b) A competitive equilibrium for this economy if composed by $\{c_0, c_1^L, c_1^H, q^{tree}, q^{rf}, s, a\}$ such that,

(1)$\{c_0, c_1^L, c_1^H, s, a\}$ solve

$$\begin{aligned} &\max u(c_0) + \beta(\pi u(c_1^H) + (1-\pi)u(c_1^L)) \\ &s.t. \quad c_0 + q^{tree}s + q^{rf}a = 1 + q^{tree} \\ &\qquad c_1^H = a + d_H s \\ &\qquad c_1^L = a + d_L s \end{aligned}$$

where a are the holdings of the risk free bond, and s, the holdings of trees.

(2) market clearing:

good market: $c_0 = 1$ and $c_1^i = d_i$ for $i = L, H$.

asset market: $s = 1$ and $a = 0$.

The FOCs of the consumer problem are:

$$s: \quad -u'(c_0)q^{tree} + \beta\pi u'(c_1^H)d_H + \beta(1-\pi)u'(c_1^L)d_L = 0 \tag{1}$$

$$a: \quad -u'(c_0)q^{rf} + \beta\pi u'(c_1^H) + \beta(1-\pi)u'(c_1^L) = 0 \tag{2}$$

Using the clearing market conditions of the good market we get:

$$(1) \quad \Rightarrow \quad q^{tree} = \frac{\beta\pi u'(d_H)d_H + \beta(1-\pi)u'(d_L)d_L}{u'(1)}$$

and

$$(2) \quad \Rightarrow \quad q^{rf} = \frac{\beta\pi u'(d_H) + \beta(1-\pi)u'(d_L)}{u'(1)}$$

(c) Now the market for trees is shut down, and thus markets are no longer complete. So a competitive equilibrium for this economy is $\{c_0, c_1^L, c_1^H, q, a\}$ such that
(1)$\{c_0, c_1^L, c_1^H, a\}$ solve

$$\max u(c_0) + \beta(\pi u(c_1^H) + (1-\pi)u(c_1^L))$$

$$s.t. \quad c_0 + qa = 1$$

$$c_1^H = a + d_H$$

$$c_1^L = a + d_L$$

(2) market clearing:
good market: $c_0 = 1$ and $c_1^i = d_i$ for $i = L, H$.
asset market: $a = 0$.

The FOC of the consumer problem is:

$$a: \quad -u'(c_0)q + \beta\pi u'(c_1^H) + \beta(1-\pi)u'(c_1^L) = 0$$

and using the equilibrium conditions we get

$$q = \frac{\beta\pi u'(d_H) + \beta(1-\pi)u'(d_L)}{u'(1)}$$

which is exactly the price of the risk bond, q^{rf}, obtained in part (b). Usually prices change when going from complete to incomplete markets, but as here (and in last part) no assets are actually trade in equilibrium, the prices needed to induce this absence of trade, are the same.

3. (a) As full set of Arrow securities are traded, markets are complete. A competitive equilibrium for this economy is defined by $\{c_0^i, c_{1j}^i, q_j, a_j^i\}_{i,j=1,2}$ where
(1)For each i, $\{c_0^i, c_{1j}^i, a_j^i\}_{j=1,2}$ solve

$$\max u(c_0) + \beta\mathbb{E}[u(c_1^i)]$$

$$s.t. \quad c_0^i + \sum_{j=1}^{2} q_j a_j^i = w_0^i$$

$$c_{1j}^i = a_j^i + w_{1j}^i \quad j = 1, 2$$

(2) market clearing:
good market: $c_0^1 + c_0^2 = w_0^1 + w_0^2$ and $c_{1j}^1 = w_{1j}^1 + w_{1j}^2$ for $j = 1, 2$.
asset market: $a_j^1 + a_j^2 = 0$ for $j = 1, 2$.

(b) We assume here that aggregate endowment is constant across time and states : $W = \bar{w}_0 = \bar{w}_{11} = \bar{w}_{12}$. The FOCs for the consumer i's problem are:

$$a_1^i : -u'(c_0^i)q_1 + \beta\pi_1 u'(c_{11}^i) = 0$$

$$a_2^i : -u'(c_0^i)q_2 + \beta\pi_2 u'(c_{12}^i) = 0$$

and then

$$q_j = \frac{\beta\pi_j u'(c_{1j}^i)}{u'(c_0^i)} \quad \text{for} \quad i, j = 1, 2 \tag{1}$$

But using the concavity of u, we get that $\frac{u'(c_{1j}^i)}{u'(c_0^i)} = 1$ for $i, j = 1, 2$. In fact, suppose there is some j for which

$$\frac{u'(c_{1j}^1)}{u'(c_0^1)} = \frac{u'(c_{1j}^2)}{u'(c_0^2)} > 1$$

Then, given the concavity of u, it would imply that $c_{1j}^1 < c_0^1$ and $c_{1j}^2 < c_0^2$, and thus at equilibrium we would have,

$$\bar{w}_{1j} = c_{1j}^1 + c_{1j}^2 < c_0^1 + c_0^2 = \bar{w}_0$$

which is a contradiction (given that aggregate endowments are equal). Therefore the Arrow securities prices are defined by

$$q_j = \beta\pi_j \quad \text{for} \quad j = 1, 2$$

and so they are in fact independent of individual endowments.

(c) Now aggregate endowments are not equal but the felicity function is a CES. Replacing this functional form in equation (1) obtained in part (a) we get that

$$q_j = \frac{\beta\pi_j(c_{1j}^1)^{-\sigma}}{(c_0^1)^{-\sigma}} = \frac{\beta\pi_j(c_{1j}^2)^{-\sigma}}{(c_0^2)^{-\sigma}}$$

then, the second equality implies that $\frac{c_0^1}{c_{1j}^1} = \frac{c_0^2}{c_{1j}^2}$ for $j = 1, 2$. Imposing the equilibrium conditions in this equality, we get that for each j

$$\frac{\bar{w}_0 - c_0^2}{\bar{w}_{1j} - c_{1j}^2} = \frac{c_0^2}{c_{1j}^2} \quad \Rightarrow \quad c_0^2 = \frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^2 \tag{2}$$

Replacing this in the equation for the Arrow securities prices we get that

$$q_j = \frac{\beta\pi_j(c_{1j}^2)^{-\sigma}}{(\frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^2)^{-\sigma}} = \beta\pi_j \left(\frac{\bar{w}_0}{\bar{w}_{1j}}\right)^{\sigma}$$

and thus the Arrow securities prices only depend on the aggregate endowment. Also, using (2) and the FOCs we get

$$c_0^i = \frac{\bar{w}_0}{\bar{w}_{1j}} c_{1j}^i \quad \text{for} \quad i, j = 1, 2.$$

(d) Now there are no Arrow securities available, consumers can only trade a risk free bond. A competitive equilibrium for this economy is composed of $\{c_0^i, c_{1j}^i, q, a^i\}_{i,j=1,2}$ where

(1)For each i, $\{c_0^i, c_{1j}^i, a^i\}_{j=1,2}$ solve

$$\max u(c_0) + \beta\mathbb{E}[u(c_1^i)]$$

$$s.t. \quad c_0^i + qa^i = w_0^i$$

$$c_{1j}^i = a^i + w_{1j}^i \quad j = 1, 2$$

(2) market clearing:
good market: $c_0^1 + c_0^2 = w_0^1 + w_0^2$ and $c_{1j}^1 = w_{1j}^1 + w_{1j}^2$ for $j = 1, 2$.
asset market: $a^1 + a^2 = 0$ for $j = 1, 2$.

(e) We now have an economy as the one in part (d) with no functional form for u and where $w_0^1 = w_0^2 \equiv w_0$, $w_{11}^1 = w_{12}^2 \equiv w_1$, $w_{12}^1 = w_{11}^2 \equiv w_2$ and $\pi_1 = \pi_2 = 0.5$. The consumer i's FOC is given by:

$$a_i: \quad -u'(c_0^i)q + \frac{1}{2}\beta u'(c_{11}^i) + \frac{1}{2}\beta u'(c_{12}^i) = 0$$

which implies

$$q = \beta \frac{u'(c_{11}^i) + u'(c_{12}^i)}{2u'(c_0^i)} \tag{3}$$

Given the symmetry in endowments and probabilities, we make the following guess:

$$c_0^1 = c_0^2 = w_0, \quad c_{11}^1 = c_{12}^2 \quad \text{and} \quad c_{12}^1 = c_{11}^2$$

First, plugging this guess in the budget constraints, we get that $a^1 = a^2 = 0$. Then, we need that $c_{11}^1 = c_{12}^2 = w_1$ and $c_{12}^1 = c_{11}^2 = w_2$ (ie, everybody consumes his own endowment). Finally we also need both FOCs to be satisfied, and thus, we need the RHS of (3) to be independent of i. Replacing our guess in (3) we get

$$\text{for } i = 1, \quad (3) \Rightarrow q = \beta \frac{u'(w_1) + u'(w_2)}{2u'(w_0)}$$

and

$$\text{for } i = 2, \quad (3) \Rightarrow q = \beta \frac{u'(w_2) + u'(w_1)}{2u'(w_0)}$$

and thus the FOCs are satisfied. Summarizing, we have that the equilibrium allocations are

$$c_0^1 = c_0^2 = w_0, \quad c_{11}^1 = c_{12}^2 = w_1 \quad \text{and} \quad c_{12}^1 = c_{11}^2 = w_2$$

and the bond price is given by

$$q = \beta \frac{u'(w_2) + u'(w_1)}{2u'(w_0)}$$

and thus it is the sum of the Arrow securities prices found in part (c), using the fact that $\pi_1 = \pi_2 = 0.5$ (no arbitrage condition!).