Econ 510a (second half)
Yale University
Fall 2006
Prof. Tony Smith

HOMEWORK #5

This homework assignment is due at 5PM on Friday, December 8 in Marnix Amand's mailbox.

1. **(a)** In the Mehra-Prescott model that we discussed in lecture on November 30, for what value of γ (the degree of risk aversion) does the model generate an average (annual) equity premium of 6%? (Set $\beta = 0.99$ and fix the other parameters at the values that we used in lecture.) For this degree of risk aversion, what is the average (annual) risk-free rate? Is it close to the historical average of 1%?

   (b) In the Mehra-Prescott model, set $\gamma = 1$ (log utility) and $\beta = 0.99$, but suppose that the high-growth state lasts longer (on average) than the low-growth state: set $\phi_{11} = 0.6$ (but keep $\phi_{22} = 0.43$). Recalibrate the values of μ and δ so that the average (annual) growth rate of per capita consumption is 0.018 and the standard deviation of the (annual) growth rate of per capita consumption is 0.036. Calculate the average equity premium and the average risk-free rate for this new parametrization.

   (c) In the Mehra-Prescott model with $\gamma = 1$, $\beta = 0.99$, and the parameters of the process for consumption growth set equal to their values from class (i.e., $\phi_{11} = \phi_{22} = 0.43$, $\mu = 0.018$, and $\delta = 0.036$), find the average rate of return on a *two-period* risk-free bond, i.e., a sure claim to one unit of the consumption good two periods from now.

   Let the average (net) rate of return on the two-period bond be given by r_2. Similarly, let r_1 be the average (net) rate of return on a one-period risk-free bond. The gross two-period return $1 + r_2$ can be decomposed into the product of two successive (gross) one-period returns $1 + \tilde{r}_2$, where $\tilde{r}_2 = (1 + r_2)^{1/2} - 1$. The *term structure of interest rates* is upward-sloping if $\tilde{r}_2 > r_1$; otherwise, it is downward-sloping. Which way does the term structure slope if the Mehra-Prescott model is calibrated as in part (a)?

Solution

1. Using the model and notations covered in class, the expected returns are:

$$r^f = \pi \left(\frac{1}{p_L^r} - 1 + \frac{1}{p_H^r} - 1 \right)$$

$$
\begin{aligned}
r^e &= \pi\left(\phi_{HH}\left(\frac{(1+p^e_H)\lambda_H}{p^e_H}-1\right)+\phi_{LH}\left(\frac{(1+p^e_H)\lambda_H}{p^e_L}-1\right)\right)\\
&\quad +(1-\pi)\left(\phi_{HL}\left(\frac{(1+p^e_L)\lambda_L}{p^e_H}-1\right)+\phi_{LL}\left(\frac{(1+p^e_L)\lambda_L}{p^e_L}-1\right)\right)
\end{aligned}
$$

which, after further simplification, and using the expressions covered in class for the prices of risk free bonds and equity iin each state:

$$
\begin{aligned}
-\beta(\phi_{LH}\lambda_H^{1-\gamma}+\phi_{LL}\lambda_L^{1-\gamma}) &= \beta\phi_{LH}\lambda_H^{1-\gamma}p^e_H+(\beta\phi_{LL}\lambda_L^{1-\gamma}-1)p^e_L\\
-\beta(\phi_{HL}\lambda_L^{1-\gamma}+\phi_{HH}\lambda_H^{1-\gamma}) &= (\beta\phi_{HH}\lambda_H^{1-\gamma}-1)p^e_H+\beta\phi_{HL}\lambda_L^{1-\gamma}p^e_L
\end{aligned}
$$

This, with Matlab, delivers

$$
\begin{aligned}
\gamma &= 17.304\\
r^f &= 0.1468
\end{aligned}
$$

Intuition: γ is much to high, and so is the risk free rate (14.7 % !). Obviously, the quantitative predictions of the Mehra-Prescott model are way off target.

2. First, we have to compute the new probability transition matrix:

$$
P = \begin{pmatrix} 0.6 & 0.57\\ 0.4 & 0.43 \end{pmatrix}
$$

which gives you the new unconditional probabilities (i.e. the invariant distribution):

$$
\pi = \begin{pmatrix} 0.59\\ 0.41 \end{pmatrix}
$$

And we have to recalibrate λ_1 and λ_2 such that:

$$
\begin{aligned}
1+0.018 &= \pi_1\lambda_1+\pi_2\lambda_2\\
0.036^2 &= \pi_1\lambda_1^2+\pi_2\lambda_2^2-(\pi_1\lambda_1+\pi_2\lambda_2)^2
\end{aligned}
$$

Which delivers:

$$
\begin{aligned}
\lambda_1 &= 1.061\\
\lambda_2 &= 0.975
\end{aligned}
$$

which gives us:

$$
\begin{aligned}
r^f &= 0.0270\\
r^e-r^f &= 0.0013
\end{aligned}
$$

Intuition: Again, the quantitative results are way off target. The risk free rate is ridiculously low (0.13%). The risk free rate is alittle too high.

3. By definition, with $i = H, L$, we have:

$$p_{2,i}^f = \beta^2 E\left(\left(\frac{c_{t+2}}{c_t}\right)^{-\gamma}\right)$$

which, after some straightforward algebra, gives:

$$p_{2,i}^f = \beta \sum_{j=L,H} \lambda_j^{-\gamma} \phi_{i,j} p_j^r$$

thus:

$$r_2^f = \pi_L \left(\frac{1}{p_{2,L}^f} - 1\right) + \pi_H \left(\frac{1}{p_{2,H}^f} - 1\right)$$

Matlab tells us:

$$r_2^f = 0.055$$

Using the calibration from section (a), Matlab tells us:

$$\bar{r}_2 = 0.1675$$

2. Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^\alpha$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\bar{k}_1, \bar{k}_2\}$. In addition, assume that z takes on values in the set $\{\bar{z}_1, \bar{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = P(z' = \bar{z}_j | z = \bar{z}_i)$.

(a) Let $\bar{z}_1 = 0.9$, $\bar{z}_2 = 1.1$, $p_{11} = 0.95$, and $p_{22} = 0.9$. Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z; that is, compute $E(z) = \pi_1 \bar{z}_1 + \pi_2 \bar{z}_2$, where π_1 and π_2 determine the invariant distribution.

(b) Let $\beta = 0.9$, $\alpha = 0.36$, $\bar{k}_1 = 0.95k_{ss}$, and $\bar{k}_2 = 1.05k_{ss}$, where k_{ss} is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital (i.e., $k_{ss} = (\alpha\beta)^{\frac{1}{(1-\alpha)}}$). Let $g(k, z)$ denote the planner's optimal decision rule. Prove that $g(k_i, z_j) = k_j$ for all i and j.

(c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. (That is, find probabilities $\pi_{ij} = P(k = k_i, z = z_j)$ that "reproduce" themselves: if π_{ij} is the unconditional probability that the economy is in state (k_i, z_j) today, then it is also the unconditional probability that the economy is in this state tomorrow. For a more complete discussion of this concept, see pp. 78 and 79 in the lecture notes by Per Krusell.) Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

(d) In Matlab, use the optimal decision rule, the law of motion for z, and a random number generator to create a simulated time series $\{k_t, y_t\}_{t=0}^T$, given an initial condition (k_0, z_0). Compute $T^{-1}\sum_{t=1}^T k_t$ and $T^{-1}\sum_{t=1}^T y_t$ for a suitably large value of T and confirm that these sample means are close to the corresponding population means that you computed in part (c). (You may find useful the Matlab code by Ljunqvist and Sargent for simulating a Markov chain that I have posted on the course web site.)

Solution (a) Given the transition matrix

$$P = \begin{pmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{pmatrix} \tag{1}$$

we can calculate the stationary distribution π according to the formula

$$\begin{aligned} &\pi' = \pi' P \\ \Rightarrow &\begin{cases} \pi_1 = 0.95\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.05\pi_1 + 0.9\pi_2 \end{cases} \end{aligned}$$

The solution to this equation is

$$\pi_1 = 2\pi_2 \tag{2}$$

Imposing the condition that $\pi_1 + \pi_2 = 1$, the solution is

$$\begin{cases} \pi_1 = \frac{2}{3} \\ \pi_2 = \frac{1}{3} \end{cases} \tag{3}$$

Correspondingly, the long run expected value is

$$E(z) = \pi' z = \frac{2}{3} \times 0.9 + \frac{1}{3} \times 1.1 = \frac{29}{30} \tag{4}$$

(b) Form the dynamic programming problem as

$$v\left(\overline{k}_i, z_i\right) = \max_{\overline{k}' \in \{k_1, k_2\}} \ln\left(z_i \overline{k}_i^\alpha - \overline{k}'\right) + \beta\left(p_{i1} v\left(\overline{k}', z_1\right) + p_{i2} v\left(\overline{k}', z_2\right)\right) \tag{5}$$

Since (k_i, z_i) takes on only 4 values, we assume that the policy function takes the following form.

$$\begin{aligned} g(k_1, z_1) &= k_1 \\ g(k_2, z_1) &= k_1 \\ g(k_1, z_2) &= k_2 \\ g(k_2, z_2) &= k_2 \end{aligned}$$

We need to prove that this is true. The way we are going to proceed is that we are going to calculate the value function values associated with this policy function and then verify that they are indeed maximum.

Using the parameter values given in the problem we get:

$$
\begin{aligned}
k_{ss} &= 0.1719 \\
k_1 &= 0.1633 \\
k_2 &= 0.18047
\end{aligned}
$$

Let's try out all 4 cases and substitute in for the assumed policy function. For $(k_i, z_i) = (k_1, z_1)$ we have:

$$
\begin{aligned}
v\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_1\right) + \beta\left(p_{11} v\left(\overline{k}_1, z_1\right) + p_{12} v\left(\overline{k}_1, z_2\right)\right) \Leftrightarrow \\
v\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.1633\right) + 0.9\left(0.95 v\left(\overline{k}_1, \overline{z}_1\right) + 0.05 v\left(\overline{k}_1, \overline{z}_2\right)\right) \Leftrightarrow \\
v\left(\overline{k}_1, \overline{z}_1\right) &= -1.1861 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \Leftrightarrow \\
v\left(\overline{k}_1, \overline{z}_1\right) &= -8.18 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right)
\end{aligned}
$$

Similarly we get:

$$
\begin{aligned}
v\left(\overline{k}_1, \overline{z}_2\right) &= -0.93543 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \\
v\left(\overline{k}_2, \overline{z}_1\right) &= -1.13134 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \\
v\left(\overline{k}_2, \overline{z}_2\right) &= -0.8833 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \Leftrightarrow \\
v\left(\overline{k}_2, \overline{z}_2\right) &= -0.9249 + 0.8953 v\left(\overline{k}_2, \overline{z}_1\right)
\end{aligned}
$$

Essentially we have a system of 4 equations with 4 unknowns. Plugging in the first equation into the third we get:

$$
v\left(\overline{k}_2, \overline{z}_1\right) = -8.12524 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right) \tag{6}
$$

Similarly, plugging in the last equation into the second one we get:

$$
v\left(\overline{k}_1, \overline{z}_2\right) = 0.89381 + 0.89529 v\left(\overline{k}_2, \overline{z}_1\right) \tag{7}
$$

Solving out the above system of 2 equations and 2 unknowns, we get:

$$
\begin{aligned}
v\left(\overline{k}_2, \overline{z}_1\right) &= -10.8668 \\
v\left(\overline{k}_1, \overline{z}_2\right) &= -8.8351
\end{aligned}
$$

Substituting into equations 1 and 4 from above we get:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_1\right) &= -10.9215 \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -10.6539 \end{aligned}$$

We now need to check that this decision rule is optimal. We will go about checking this changing the decision rule in each of the 4 cases and then calculating the resulting value function. For example, when we have $\left(\overline{k}_1, \overline{z}_1\right)$ we will assume that the planner chooses $\overline{k}_2$ instead of $\overline{k}_1$. In that case we would get:

$$\begin{aligned} v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_2\right) + \beta\left(p_{11} v\left(\overline{k}_2, z_1\right) + p_{12} v\left(\overline{k}_2, z_2\right)\right) \Rightarrow \\ v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.18047\right) + 0.9\left(0.95 * (-10.8668) + 0.05 * (-10.6539)\right) \Leftrightarrow \\ v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= -11.0145 < -10.9215 = v\left(\overline{k}_1, \overline{z}_1\right) \end{aligned}$$

Similarly we get:

$$\begin{aligned} v^{alt}\left(\overline{k}_1, \overline{z}_2\right) &= -10.7126 < -8.8351 = v\left(\overline{k}_1, \overline{z}_2\right) \\ v^{alt}\left(\overline{k}_2, \overline{z}_1\right) &= -11.4859 < -10.8668 = v\left(\overline{k}_2, \overline{z}_1\right) \\ v^{alt}\left(\overline{k}_2, \overline{z}_2\right) &= -10.7581 < -10.6539 = v\left(\overline{k}_2, \overline{z}_2\right) \end{aligned}$$

and we see that in each case the original decision rule performs better. The rational

we use to conclude that our decision rule is indeed optimal is the following: our guess about the decision rule implies a certain value for our value function. On the other hand changing the decision rule with the only other possible alternative for each case would imply a different value function. If this alternative value function was actually closer to the true value function (which is the optimal) then it would give a higher value than our guess. But that is not the case. Therefore our guess is closer to the optimal. You can think of this procedure as a value function iteration. Given however that for each decision rule there is only one other alternative, this implies that out value function is indeed optimal and that our guess is indeed the correct decision rule.

(c) Based on policy function $g(k_i, z_i)$ and transition matrix of z, the pair (k, z) follows a Markov process with the transition matrix

	(z_1, k_1)	(z_1, k_2)	(z_2, k_1)	(z_2, k_2)
(z_1, k_1)	0.95	0	0.05	0
(z_1, k_2)	0.95	0	0.05	0
(z_2, k_1)	0	0.1	0	0.9
(z_2, k_2)	0	0.1	0	0.9

(8)

Now we can calculate the stationary distribution either on the computer or by hand. The result is

$$\begin{aligned} p(k_1, z_1) &= \frac{19}{30} = 0.63333 \\ p(k_2, z_1) &= \frac{1}{30} = 0.033333 \\ p(k_1, z_2) &= \frac{1}{30} = 0.033333 \\ p(k_2, z_2) &= \frac{3}{10} = 0.3 \end{aligned}$$

Using the stationary distribution, the long-run (or unconditional) expected values of the capital stock and of output are

$$\begin{aligned} Ek &= p(k_1, z_1)k_1 + p(k_1, z_2)k_1 + p(k_2, z_1)k_2 + p(k_2, z_2)k_2 = \frac{2}{3}k_1 + \frac{1}{3}k_2 = 0.1747 \\ Ey &= p(k_1, z_1) * z_1 k_1^a + p(k_2, z_1) * z_1 k_2^a + p(k_1, z_2) * z_2 k_1^a + p(k_2, z_2) * z_2 k_2^a = 0.51031 \end{aligned}$$

(d) Depending on the realization of each simulation, the result will differ a little bit. For example, one possible result based on $T = 10000$ is

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} k_t &= 0.1747 \\ \frac{1}{T} y_t &= 0.4687 \end{aligned}$$

3. **(a)** Consider a neoclassical growth model with shocks to total factor productivity and valued leisure. Carefully define a sequential competitive equilibrium for this economy in which households own the factors of production and rent them to firms in competitive markets.

(b) Carefully define a recursive competitive equilibrium for the economy in part (a). (Hint: You need *two* functions to describe the aggregate economy, one for aggregate capital and one for aggregate labor supply.)

(c) Carefully describe a procedure (analogous to the one that we will, or did, discuss in lecture on December 5 for a similar economy without valued leisure) for computing a linear approximation to the competitive equilibrium behavior of the economy in parts (a) and (b). You do not need to implement the procedure, but you need to explain it in enough detail that someone who knows nothing about economics, but who can take derivatives and manipulate algebra, could implement it. (Hint: There are *two* unknown decision rules, one for savings and one for labor supply, that must satisfy simultaneously *two* first-order conditions.)

Solution

4. A sequential competitive equilibrium for this economy is a sequence $\{R_t^*, w_t^*, c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0}$ such that

(1)Given $\{R_t^*, w_t^*\}_{t\geq 0}$, $\{c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0}$ solves the consumer's problem:

$$\{c_t^*, k_{t+1}^*, n_t^*\}_{t\geq 0} = \text{argmax} \sum_{t\geq 0} \beta^t u(c_t, L - n_t)$$

$$s.t. \quad c_t + k_{t+1} = k_t R_t^* + w_t^* n_t$$

$$k_0 \quad \text{given}$$

$$n_t \leq L \quad \forall t$$

$$\lim_{t\to\infty} \frac{k_{t+1}}{\prod_{j=0}^t R_t^*}$$

where L is the total endowment of time every period.

(2)$\{k_{t+1}^*, n_t^*\}_{t\geq 0}$ solves the firm's problem:

$$(k_t^*, n_t^*) = \arg\max_{k_t, n_t} F(k_t, n_t) + (1-\delta)k_t - R_t^* k_t - w_t^* n_t \quad \forall t$$

(3)Market clearing:

$$c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta)k_t^* \quad \forall t$$

5. A recursive competitive equilibrium for the neoclassical growth model with valued leisure is a set of functions:

price function:	$r(\bar{k}), w(\bar{k})$
policy function:	$k' = g_k(k, \bar{k}), l = g_l(k, \bar{k})$
value function:	$v(k, \bar{k})$
aggregate function:	$\bar{k}' = G(\bar{k}), \bar{l} = \bar{l}(\bar{k})$

such that:

(1) Given $\bar{k}' = G(\bar{k})$; $k' = g_k\left(k, \bar{k}\right)$, $l = g_l\left(k, \bar{k}\right)$ and $v\left(k, \bar{k}\right)$ solve the consumer's problem:

$$\begin{aligned} v\left(k, \bar{k}\right) &= \max_{\{c,l,k'\}} u\left(c, l\right) + \beta v\left(k', \bar{k}'\right) \\ & s.t. \\ & c + k' = r(\bar{k})k + w(\bar{k})(L - l) \\ & \bar{k}' = G(\bar{k}) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r(\bar{k}) &= F_1(\bar{k}, L - \bar{l}(\bar{k})) + (1-\delta) \\ w(\bar{k}) &= F_2(\bar{k}, L - \bar{l}(\bar{k})) \end{aligned}$$

(3) Consistency:

$$\begin{aligned} G\left(\bar{k}\right) &= g_k\left(\bar{k}, \bar{k}\right) \\ \bar{l}(\bar{k}) &= g_l\left(\bar{k}, \bar{k}\right) \end{aligned}$$

6. We use a linearization method around the steady state.

   (a) Find the two first-order conditions in the usual way:

$$u_1(zF(k, 1-l) + (1-\delta)k - k', l) =$$
$$\beta E[u_1(z'F(k', 1-l') + (1-\delta)k' - k'', l')(z'F_1(k', 1-l') + (1-\delta)|z]$$

$$u_1(zF(k, 1-l) + (1-\delta)k - k', l)F_2(k, 1-l) =$$
$$u_2(zF(k, 1-l+(1-\delta)k') - k', l)$$

   Notice that there is no temporal trade-off for the "optimal labor" equation.

   (b) Find the steady state by setting $k = k' = k'' = \bar{k}$ and $l = l' = \bar{l}$ in the two first-order conditions and then solving for $\bar{k}$ and $\bar{l}$.

   (c) Linearize u_1, u_2, F_1 and F_2 around the steady state, with the help of a 1st-order Taylor expansion. Use the generic notation $\hat{x} = x - \bar{x}$ for each variable z, l and k. Remember that

$$g(x, y) \approx g(\bar{x}, \bar{y}) + (x - \bar{x})g_1(\bar{x}, \bar{y}) + (y - \bar{y})g_2(\bar{x}, \bar{y})$$

   You now have two linearized first-order conditions. For example, after linearizing the intratemporal first-order condition (for labor supply) and combining terms, you will have an equation:

$$\gamma_0 + \gamma_1 k + \gamma_2 k' + \gamma_3 l + \gamma_4 z = 0,$$

   where the γ_i's depend on β, δ, and the derivatives of u and f evaluated at the steady state.

   (d) Assume linear policy functions:

$$\begin{aligned} k' &= a_0 + a_1 k + a_2 z \\ l &= b_0 + b_1 k + b_2 z \end{aligned}$$

and use these to eliminate the choice variables (k', k'', l, and l') from the two linearized first-order conditions. For example, in the linearized first-order condition for labor supply, one obtains:

$$\gamma_0 + \gamma_1 k + \gamma_2(a_0 + a_1 k + a_2 z) + \gamma_3(b_0 + b_1 k + b_2 z) + \gamma_4 z = 0,$$

Combining terms, one obtains:

$$B_0 + B_1 k + B_2 z = 0,$$

where the B_i's depend on the a_i's, the b_i's (which in turn depend on preferences and technology). Similarly, the intertemporal first-order condition for savings ultimately yields an equation

$$A_0 + A_1 k + A_2 z = 0,$$

where the A_i's depend on the a_i's and the b_i's. These last two equations are *identitites*: they must hold for *all* values of k and z. This fact imposes six restrictions on the A_i's and the B_i's:

$$A_0 = A_1 = A_2 = B_0 = B_1 = B_2 = 0.$$

This is a set of six equations in the six unknowns a_0, a_1, a_2, b_0, b_1, and b_2. Solving these equations for the a_i's and b_i's yields the two (linearized) decision rules (or policy functions).

4. Read Sections 1 and 3 of "Finn Kydland and Edward Prescott's Contribution to Dynamic Macroeconomics: The Time Consistency of Economic Policy and the Driving Forces Behind Business Cycles" (available on the course web site).

Solution Do it !