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Problem Set 5	: Suggested Solutions	

1. (a) We have

State	Probability	$\overline{\omega}$	θ	ω^1	ω^2
1	$\eta\alpha$	2	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{2}$
2	$\eta(1-\alpha)$	2	$\frac{1}{2}$	1	1
3	$(1-\eta)\,\alpha$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$
4	$(1-\eta)\,(1-\alpha)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

(b) A competitive equilibrium with date-0 trading for this economy is a set of sequences $\{c_t^1(z^t), c_t^2(z^t), p_t(z^t)\}_{t=0}^{\infty}$, such that,

i. Given $\{p_t(z^t)\}_{t=0}^{\infty}$, $\{c_t^i(z^t)\}_{t=0}^{\infty}$ solves the consumer's problem:

$$\begin{aligned}
&\max_{\{c_t^i(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t^i(z^t))\\
&s.t.\\
&\sum_{t=0}^{\infty}\sum_{z^t}p_t(z^t)c_t^i(z^t) = \sum_{t=0}^{\infty}\sum_{z^t}p_t(z^t)\omega_t^i(z^t)\\
&\text{for } i = 1, 2.
\end{aligned}$$

ii. Markets clear:

$$c_t^1(z^t) + c_t^2(z^t) = \omega_t^1(z^t) + \omega_t^2(z^t) \quad \forall t$$

(c) The first order condition for consumer i gives us:

$$\frac{\beta\pi(z^{t+1})u'(c_{t+1}^i(z^{t+1}))}{\pi(z^t)u'(c_t^i(z^t))} = \frac{p_{t+1}(z^{t+1})}{p_t(z^t)}.$$

So we have:

$$\begin{aligned}
\frac{\beta\pi(z^{t+1})u'(c_{t+1}^1(z^{t+1}))}{\pi(z^t)u'(c_t^1(z^t))} &= \frac{\beta\pi(z^{t+1})u'(c_{t+1}^2(z^{t+1}))}{\pi(z^t)u'(c_t^2(z^t))} = \frac{p_{t+1}(z^{t+1})}{p_t(z^t)}\\
\frac{\beta\pi(z^{t+1})(c_{t+1}^1(z^{t+1}))^{-\gamma}}{\pi(z^t)(c_t^1(z^t))^{-\gamma}} &= \frac{\beta\pi(z^{t+1})(c_{t+1}^2(z^{t+1}))^{-\gamma}}{\pi(z^t)(c_t^2(z^t))^{-\gamma}} = \frac{p_{t+1}(z^{t+1})}{p_t(z^t)}\\
\left(\frac{\beta\pi(z^{t+1})}{\pi(z^t)}\right)^{\frac{-1}{\gamma}}\frac{c_{t+1}^1(z^{t+1})}{c_t^1(z^t)} &= \left(\frac{\beta\pi(z^{t+1})}{\pi(z^t)}\right)^{\frac{-1}{\gamma}}\frac{c_{t+1}^2(z^{t+1})}{c_t^2(z^t)} = \left(\frac{p_{t+1}(z^{t+1})}{p_t(z^t)}\right)^{\frac{-1}{\gamma}},
\end{aligned}$$

which can be re-written as:

$$\begin{aligned}\left(\frac{p_{t+1}(z^{t+1})}{p_t(z^t)}\right)^{\frac{-1}{\gamma}} &= \left(\frac{\beta\pi(z^{t+1})}{\pi(z^t)}\right)^{\frac{-1}{\gamma}}\left[\frac{c^1_{t+1}(z^{t+1})+c^2_{t+1}(z^{t+1})}{c^1_t(z^t)+c^2_t(z^t)}\right] \\ &= \left(\frac{\beta\pi(z^{t+1})}{\pi(z^t)}\right)^{\frac{-1}{\gamma}}\left[\frac{\bar{\omega}_{t+1}}{\bar{\omega}_t}\right]\end{aligned}$$

where we have substituted in the market clearing condition in the last line. As we can see, the price ratio depends on the aggregate endowment, not how the endowment is distributed amongst the agents.

(d) For our states,1, 2, 3, 4, and log utility our conditions for competitive equilibrium with date-0 trading are:
Foc for each consumer

$$\frac{\beta\pi(z^{t+1})c^i_t(z^t)}{\pi(z^t)c^i_{t+1}(z^{t+1})} = \frac{p_{t+1}(z^{t+1})}{p_t(z^t)}$$

or

$$\begin{aligned}\frac{\beta\pi(z^t)u'(c^1_t(z^t))}{\pi(z_0)u'(c^1_0(z_0))} &= \frac{p_t(z^t)}{p_0(z_0)} \\ \frac{\beta\pi(z^t)u'(c^1_t(z^t))}{\pi(1)u'(c^1_0(1))} &= \frac{p_t(z^t)}{p_0(1)} \\ \frac{\beta\pi(z^t)c^1_0(1)}{\eta\alpha c^1_t(z^t)} &= \frac{p_t(z^t)}{p_0(1)}\end{aligned}$$

and

$$\frac{\beta\pi(z^t)c^2_0(1)}{\eta\alpha c^2_t(z^t)} = \frac{p_t(z^t)}{p_0(1)}$$

which implies

$$\frac{c^1_0(1)}{c^1_t(z^t)} = \frac{c^2_0(1)}{c^2_t(z^t)}.$$

Budget constraints

$$\sum_{t=0}^{\infty}\sum_{z^t} p_t(z^t)c^i_t(z^t) = \sum_{t=0}^{\infty}\sum_{z^t} p_t(z^t)\omega^i_t(z^t)$$

of which we use only one due to Walras' law.
And market clearing condition

$$c^1_t(z^t) + c^2_t(z^t) = \omega^1_t(z^t) + \omega^2_t(z^t).$$

We can guess that the agents will consume the same amount in each period for which the

aggregate endowment is the same (it is straight forward to show that consumption and prices do not depend on history but only on the current state). That is,

$$\begin{aligned} c_t^1(1) &= c_t^1(2) = c^1(H) \\ c_t^1(3) &= c_t^1(4) = c^1(L) \end{aligned}$$

and

$$\begin{aligned} c_t^2(1) &= c_t^2(2) = c^2(H) \\ c_t^2(3) &= c_t^2(4) = c^2(L) \end{aligned}$$

with

$$\begin{aligned} c^1(H) + c^2(H) &= 2 \\ c^1(L) + c^2(L) &= 1 \end{aligned}$$

from market clearing.
Also, we can normalize the price as

$$p_0(1) = 1$$

So from

$$\frac{c_0^1(1)}{c_t^1(z_t)} = \frac{c_0^2(1)}{c_t^2(z_t)}$$

we have

$$\begin{aligned} \frac{c^1(H)}{c^1(L)} &= \frac{c^2(H)}{c^2(L)} \\ \frac{c^1(H)}{c^1(L)} &= \frac{2 - c^1(H)}{1 - c^1(L)} \\ c^1(H) - c^1(H)c^1(L) &= 2c^1(L) - c^1(H)c^1(L) \\ c^1(H) &= 2c^1(L) \\ \frac{c^1(H)}{c^1(L)} &= \frac{c^2(H)}{c^2(L)} = 2. \end{aligned}$$

From

$$\frac{\beta\pi(z^t)c_0^1(1)}{\eta\alpha c_t^1(z^t)} = \frac{p_t(z^t)}{p_0(1)}$$

we have

$$
\begin{aligned}
p_t(z^t) &= \frac{\beta^t \pi(z^t) c_0^1(1)}{\eta\alpha c_t^1(z^t)} \\
&= \begin{cases} \frac{\beta^t \pi(z^{t-1},1) c^1(H)}{\eta\alpha c^1(H)} & \text{for } z_t = 1 \\ \frac{\beta^t \pi(z^{t-1},2) c^1(H)}{\eta\alpha c^1(H)} & \text{for } z_t = 2 \\ \frac{\beta^t \pi(z^{t-1},3) c^1(H)}{\eta\alpha c^1(L)} & \text{for } z_t = 3 \\ \frac{\beta^t \pi(z^{t-1},4) c^1(H)}{\eta\alpha c^1(L)} & \text{for } z_t = 4 \end{cases} \\
&= \begin{cases} \frac{\beta^t \pi(z^{t-1},1)}{\eta\alpha} & \text{for } z_t = 1 \\ \frac{\beta^t \pi(z^{t-1},2)}{\eta\alpha} & \text{for } z_t = 2 \\ 2\frac{\beta^t \pi(z^{t-1},3)}{\eta\alpha} & \text{for } z_t = 3 \\ 2\frac{\beta^t \pi(z^{t-1},4)}{\eta\alpha} & \text{for } z_t = 4 \end{cases}.
\end{aligned}
$$

Going back to the budget constraint,

$$
\sum_{t=0}^{\infty} \sum_{z^t} p_t(z^t) c_t^1(z^t) = \sum_{t=0}^{\infty} \sum_{z^t} p_t(z^t) \omega_t^1(z^t)
$$

we can plug in the prices we have found

$$
\begin{aligned}
& \sum_{t=0}^{\infty} \left[\sum_{z^{t-1}} \beta^t \left(\frac{\pi(z^{t-1},1)}{\eta\alpha} + \frac{\pi(z^{t-1},2)}{\eta\alpha} \right) c^1(H) + 2\beta^t \left(\frac{\pi(z^{t-1},3)}{\eta\alpha} + \frac{\pi(z^{t-1},4)}{\eta\alpha} \right) c^1(L) \right] \\
= & \sum_{t=0}^{\infty} \left[\sum_{z^{t-1}} \beta^t \frac{\pi(z^{t-1},1)}{\eta\alpha} \omega^1(1) + \beta^t \frac{\pi(z^{t-1},2)}{\eta\alpha} \omega^1(2) + 2\beta^t \frac{\pi(z^{t-1},3)}{\eta\alpha} \omega^1(2) + 2\beta^t \frac{\pi(z^{t-1},4)}{\eta\alpha} \omega^1(2) \right].
\end{aligned}
$$

Noting that

$$
\pi(z^{t-1},1) = \pi(z^{t-1})\pi(1)
$$

we have

$$
\begin{aligned}
\pi(z^{t-1},1) &= \pi(z^{t-1})\pi(1) \\
&= \pi(z^{t-1})\eta\alpha, \\
\pi(z^{t-1},2) &= \pi(z^{t-1})\eta(1-\alpha), \\
\pi(z^{t-1},3) &= \pi(z^{t-1})\,(1-\eta)\,\alpha, \\
\pi(z^{t-1},4) &= \pi(z^{t-1})\,(1-\eta)\,(1-\alpha).
\end{aligned}
$$

And so the budget constraint becomes

$$\begin{aligned}
& \sum_{t=0}^{\infty}\sum_{z^{t-1}} \beta^t \left(\pi(z^{t-1}) + \frac{\pi(z^{t-1})(1-\alpha)}{\alpha}\right) c^1(H) \\
& +2\beta^t \left(\frac{\pi(z^{t-1})\,(1-\eta)}{\eta} + \frac{\pi(z^{t-1})\,(1-\eta)\,(1-\alpha)}{\eta\alpha}\right) c^1(L) \\
= & \sum_{t=0}^{\infty}\sum_{z^{t-1}} \beta^t \pi(z^{t-1})\omega^1(1) + \beta^t \frac{\pi(z^{t-1})(1-\alpha)}{\alpha}\omega^1(2) \\
& +2\beta^t \frac{\pi(z^{t-1})\,(1-\eta)}{\eta}\omega^1(3) + 2\beta^t \frac{\pi(z^{t-1})\,(1-\eta)\,(1-\alpha)}{\eta\alpha}\omega^1(4).
\end{aligned}$$

Since

$$\sum_{z^{t-1}} \pi(z^{t-1}) = 1$$

the budget constraint simplifies to

$$\begin{aligned}
& \sum_{t=0}^{\infty}\left[\beta^t \left(1 + \frac{(1-\alpha)}{\alpha}\right) c^1(H) + 2\beta^t \left(\frac{(1-\eta)}{\eta} + \frac{(1-\eta)\,(1-\alpha)}{\eta\alpha}\right) c^1(L)\right] \\
= & \sum_{t=0}^{\infty}\left[\beta^t \omega^1(1) + \beta^t \frac{(1-\alpha)}{\alpha}\omega^1(2) + 2\beta^t \frac{(1-\eta)}{\eta}\omega^1(3) + 2\beta^t \frac{(1-\eta)\,(1-\alpha)}{\eta\alpha}\omega^1(4)\right]
\end{aligned}$$

$$\begin{aligned}
& \sum_{t=0}^{\infty}\left[\beta^t \frac{1}{\alpha} c^1(H) + 2\beta^t \frac{(1-\eta)}{\eta\alpha} c^1(L)\right] \\
= & \sum_{t=0}^{\infty}\left[\beta^t \frac{1}{2} + \beta^t \frac{(1-\alpha)}{\alpha} 1 + 2\beta^t \frac{(1-\eta)}{\eta}\frac{1}{4} + 2\beta^t \frac{(1-\eta)\,(1-\alpha)}{\eta\alpha}\frac{1}{2}\right]
\end{aligned}$$

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t c^1(H)\left(\frac{1}{\alpha} + \frac{(1-\eta)}{\eta\alpha}\right) &= \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} + \frac{(1-\alpha)}{\alpha} + \frac{1}{4}\frac{(1-\eta)}{\eta} + \frac{1}{2}\frac{(1-\eta)\,(1-\alpha)}{\eta\alpha}\right) \\
c^1(H)\frac{1}{\eta\alpha} &= \frac{2\eta - \alpha - \alpha\eta + 2}{4\alpha\eta} \\
c^1(H) &= \frac{(2-\alpha)\,(1+\eta)}{4}.
\end{aligned}$$

And so we have

$$\begin{aligned}
c^2(H) &= 2 - c^1(H) \\
&= 2 - \frac{(2-\alpha)(1+\eta)}{4} \\
c^1(L) &= \frac{1}{2}c^1(H) \\
&= \frac{(2-\alpha)(1+\eta)}{8} \\
c^2(L) &= \frac{1}{2}c^2(H) \\
&= 1 - \frac{(2-\alpha)(1+\eta)}{8}.
\end{aligned}$$

(e) From part (d) we have that

$$p_t(z^t) = \begin{cases} \frac{\beta^t \pi(z^{t-1},1)}{\eta\alpha} & \text{for } z_t = 1 \\ \frac{\beta^t \pi(z^{t-1},2)}{\eta\alpha} & \text{for } z_t = 2 \\ 2\frac{\beta^t \pi(z^{t-1},3)}{\eta\alpha} & \text{for } z_t = 3 \\ 2\frac{\beta^t \pi(z^{t-1},4)}{\eta\alpha} & \text{for } z_t = 4 \end{cases}.$$

So the prices for the Arrow securities are

$$\begin{aligned}
q_t(1,1) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{\beta^{t+1}\pi(z^t,1)}{\beta^t \pi(z^{t-1},1)} \\
&= \frac{\beta\pi(z^{t-1},1,1)}{\pi(z^{t-1},1)} \\
&= \beta\frac{\pi(z^{t-1})\pi(1)\pi(1)}{\pi(z^{t-1})\pi(1)} \\
&= \beta\eta\alpha
\end{aligned}$$

$$
\begin{aligned}
q_t(1,2) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{\beta^{t+1}\pi(z^t,2)}{\beta^t\pi(z^{t-1},1)} \\
&= \frac{\beta\pi(z^{t-1},1,2)}{\pi(z^{t-1},1)} \\
&= \beta\frac{\pi(z^{t-1})\pi(1)\pi(2)}{\pi(z^{t-1})\pi(1)} \\
&= \beta\eta(1-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
q_t(1,3) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{2\beta^{t+1}\pi(z^t,3)}{\beta^t\pi(z^{t-1},1)} \\
&= \frac{2\beta\pi(z^{t-1},1,3)}{\pi(z^{t-1},1)} \\
&= \beta\frac{2\pi(z^{t-1})\pi(1)\pi(3)}{\pi(z^{t-1})\pi(1)} \\
&= \beta 2\eta(1-\alpha)
\end{aligned}
$$

Similarly

$$q_t(1,4) = 2\beta\,(1-\eta)\,(1-\alpha).$$

And

$$q_t(2,1) = \beta\eta\alpha, \quad q_t(2,2) = \beta\eta(1-\alpha), \quad q_t(2,3) = 2\beta\,(1-\eta)\,\alpha, \quad q_t(2,4) = 2\beta\,(1-\eta)\,(1-\alpha).$$

Also,

$$
\begin{aligned}
q_t(3,1) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{\beta^{t+1}\pi(z^t,1)}{2\beta^t\pi(z^{t-1},3)} \\
&= \frac{\beta\pi(z^{t-1},3,1)}{2\pi(z^{t-1},3)} \\
&= \beta\frac{\pi(z^{t-1})\pi(3)\pi(1)}{2\pi(z^{t-1})\pi(3)} \\
&= \beta\frac{\eta\alpha}{2}
\end{aligned}
$$

$$
\begin{aligned}
q_t(3,2) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{\beta^{t+1}\pi(z^t,2)}{2\beta^t\pi(z^{t-1},3)} \\
&= \frac{\beta\pi(z^{t-1},3,2)}{2\pi(z^{t-1},3)} \\
&= \beta\frac{\pi(z^{t-1})\pi(3)\pi(2)}{2\pi(z^{t-1})\pi(3)} \\
&= \beta\frac{\eta(1-\alpha)}{2}
\end{aligned}
$$

$$
\begin{aligned}
q_t(3,3) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{2\beta^{t+1}\pi(z^t,3)}{2\beta^t\pi(z^{t-1},3)} \\
&= \frac{2\beta\pi(z^{t-1},3,3)}{2\pi(z^{t-1},3)} \\
&= \beta\frac{2\pi(z^{t-1})\pi(3)\pi(3)}{2\pi(z^{t-1})\pi(3)} \\
&= \beta\left(1-\eta\right)\alpha
\end{aligned}
$$

$$
\begin{aligned}
q_t(3,4) &= \frac{p_{t+1}(z^{t+1})}{p_t(z^t)} \\
&= \frac{2\beta^{t+1}\pi(z^t,4)}{2\beta^t\pi(z^{t-1},3)} \\
&= \frac{2\beta\pi(z^{t-1},3,4)}{2\pi(z^{t-1},3)} \\
&= \beta\frac{2\pi(z^{t-1})\pi(3)\pi(4)}{2\pi(z^{t-1})\pi(3)} \\
&= \beta\left(1-\eta\right)\left(1-\alpha\right)
\end{aligned}
$$

And

$$
q_t(4,1) = \beta\frac{\eta\alpha}{2}, \quad q_t(4,2) = \beta\frac{\eta(1-\alpha)}{2}, \quad q_t(4,3) = \beta\left(1-\eta\right)\alpha, \quad q_t(4,4) = \beta\left(1-\eta\right)\left(1-\alpha\right).
$$

The price of a one-period risk free bond:

$$
\begin{aligned}
q(1) &= q(1,1) + q(1,2) + q(1,3) + q(1,4) \\
&= \beta(\eta\alpha + \eta(1-\alpha) + 2\eta(1-\alpha) + 2\left(1-\eta\right)\left(1-\alpha\right))
\end{aligned}
$$

$$
\begin{aligned}
q(2) &= q(2,1)+q(2,2)+q(2,3)+q(2,4) \\
&= \beta(\eta\alpha+\eta(1-\alpha)+2\eta(1-\alpha)+2\,(1-\eta)\,(1-\alpha))
\end{aligned}
$$

$$
\begin{aligned}
q(3) &= q(3,1)+q(3,2)+q(3,3)+q(3,4) \\
&= \beta\left(\frac{\eta\alpha}{2}+\frac{\eta(1-\alpha)}{2}+(1-\eta)\,\alpha+(1-\eta)\,(1-\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
q(4) &= q(4,1)+q(4,2)+q(4,3)+q(4,4) \\
&= \beta\left(\frac{\eta\alpha}{2}+\frac{\eta(1-\alpha)}{2}+(1-\eta)\,\alpha+(1-\eta)\,(1-\alpha)\right).
\end{aligned}
$$

2. (a) Given the transition matrix

$$
P=\begin{pmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{pmatrix}
$$

we can calculate the stationary distribution π according to the formula

$$
\begin{aligned}
&\pi' = \pi' P \\
\Rightarrow\ &\begin{cases} \pi_1 = 0.95\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.05\pi_1 + 0.9\pi_2 \end{cases}
\end{aligned}
$$

The solution to this equation is

$$
\pi_1 = 2\pi_2
$$

Imposing the condition that $\pi_1+\pi_2=1$, the solution is

$$
\begin{cases} \pi_1 = \frac{2}{3} \\ \pi_2 = \frac{1}{3} \end{cases}
$$

Correspondingly, the long run expected value is

$$
E(z) = \pi' z = \frac{2}{3}\times 0.9 + \frac{1}{3}\times 1.1 = \frac{29}{30}
$$

(b) Form the dynamic programming problem as

$$
v\left(\overline{k}_i, z_i\right) = \max_{\overline{k}'\in\{k_1,k_2\}} \ln\left(z_i\overline{k}_i^{\alpha}-\overline{k}'\right)+\beta\left(p_{i1}v\left(\overline{k}', z_1\right)+p_{i2}v\left(\overline{k}', z_2\right)\right)
$$

Since (k_i, z_i) takes on only 4 values, we assume that the policy function takes the following form.

$$
\begin{aligned}
g(k_1,z_1) &= k_1 \\
g(k_2,z_1) &= k_1 \\
g(k_1,z_2) &= k_2 \\
g(k_2,z_2) &= k_2
\end{aligned}
$$

We need to prove that this is true. The way we are going to proceed is that we are going to calculate the value function values associated with this policy function and then verify that they are indeed maximum.
Using the parameter values given in the problem we get:

$$\begin{aligned} k_{ss} &= 0.1719 \\ k_1 &= 0.1633 \\ k_2 &= 0.18047 \end{aligned}$$

Let's try out all 4 cases and substitute in for the assumed policy function.

For $(k_i, z_i) = (k_1, z_1)$ we have:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_1\right) + \beta\left(p_{11} v\left(\overline{k}_1, z_1\right) + p_{12} v\left(\overline{k}_1, z_2\right)\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.1633\right) + 0.9\left(0.95 v\left(\overline{k}_1, \overline{z}_1\right) + 0.05 v\left(\overline{k}_1, \overline{z}_2\right)\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= -1.1861 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z}_1\right) &= -8.18 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right) \end{aligned}$$

Similarly we get:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_2\right) &= -0.93543 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \\ v\left(\overline{k}_2, \overline{z}_1\right) &= -1.13134 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -0.8833 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \Leftrightarrow \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -0.9249 + 0.8953 v\left(\overline{k}_2, \overline{z}_1\right) \end{aligned}$$

Essentially we have a system of 4 equations with 4 unknowns. Plugging in the first equation into the third we get:

$$v\left(\overline{k}_2, \overline{z}_1\right) = -8.12524 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right)$$

Similarly, plugging in the last equation into the second one we get:

$$v\left(\overline{k}_1, \overline{z}_2\right) = 0.89381 + 0.89529 v\left(\overline{k}_2, \overline{z}_1\right)$$

Solving out the above system of 2 equations and 2 unknowns, we get:

$$\begin{aligned} v\left(\overline{k}_2, \overline{z}_1\right) &= -10.8668 \\ v\left(\overline{k}_1, \overline{z}_2\right) &= -8.8351 \end{aligned}$$

Substituting into equations 1 and 4 from above we get:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_1\right) &= -10.9215 \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -10.6539 \end{aligned}$$

We now need to check that this decision rule is optimal. We will go about checking this changing the decision rule in each of the 4 cases and then calculating the resulting value function. For example, when we have $\left(\overline{k}_1, \overline{z}_1\right)$ we will assume that the planner chooses $\overline{k}_2$ instead of $\overline{k}_1$. In that case we would get:

$$\begin{aligned}
v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_2\right) + \beta\left(p_{11} v\left(\overline{k}_2, z_1\right) + p_{12} v\left(\overline{k}_2, z_2\right)\right) \Rightarrow \\
v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.18047\right) + 0.9\left(0.95 * (-10.8668) + 0.05 * (-10.6539)\right) \Leftrightarrow \\
v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= -11.0145 < -10.9215 = v\left(\overline{k}_1, \overline{z}_1\right)
\end{aligned}$$

Similarly we get:

$$\begin{aligned}
v^{alt}\left(\overline{k}_1, \overline{z}_2\right) &= -10.7126 < -8.8351 = v\left(\overline{k}_1, \overline{z}_2\right) \\
v^{alt}\left(\overline{k}_2, \overline{z}_1\right) &= -11.4859 < -10.8668 = v\left(\overline{k}_2, \overline{z}_1\right) \\
v^{alt}\left(\overline{k}_2, \overline{z}_2\right) &= -10.7581 < -10.6539 = v\left(\overline{k}_2, \overline{z}_2\right)
\end{aligned}$$

and we see that in each case the original decision rule performs better. The rational we use to conclude that our decision rule is indeed optimal is the following: our guess about the decision rule implies a certain value for our value function. On the other hand changing the decision rule with the only other possible alternative for each case would imply a different value function. If this alternative value function was actually closer to the true value function (which is the optimal) then it would give a higher value than our guess. But that is not the case. Therefore our guess is closer to the optimal. You can think of this procedure as a value function iteration. Given however that for each decision rule there is only one other alternative, this implies that out value function is indeed optimal and that our guess is indeed the correct decision rule.

(c) Based on policy function $g\left(k_i, z_i\right)$ and transition matrix of z, the pair (k, z) follows a Markov process with the transition matrix

	(z_1, k_1)	(z_1, k_2)	(z_2, k_1)	(z_2, k_2)
(z_1, k_1)	0.95	0	0.05	0
(z_1, k_2)	0.95	0	0.05	0
(z_2, k_1)	0	0.1	0	0.9
(z_2, k_2)	0	0.1	0	0.9

Now we can calculate the stationary distribution either on the computer or by hand. The

result is

$$
\begin{aligned}
p(k_1, z_1) &= \frac{19}{30} = 0.63333 \\
p(k_2, z_1) &= \frac{1}{30} = 0.033333 \\
p(k_1, z_2) &= \frac{1}{30} = 0.033333 \\
p(k_2, z_2) &= \frac{3}{10} = 0.3
\end{aligned}
$$

Using the stationary distribution, the long-run (or unconditional) expected values of the capital stock and of output are

$$
\begin{aligned}
Ek &= p(k_1, z_1)k_1 + p(k_1, z_2)k_1 + p(k_2, z_1)k_2 + p(k_2, z_2)k_2 = \frac{2}{3}k_1 + \frac{1}{3}k_2 = 0.1747 \\
Ey &= p(k_1, z_1) * z_1 k_1^a + p(k_2, z_1) * z_1 k_2^a + p(k_1, z_2) * z_2 k_1^a + p(k_2, z_2) * z_2 k_2^a = 0.51031
\end{aligned}
$$

3. The parameter values are:

$$
\begin{aligned}
\beta &= 0.96 \\
\mu &= 0.018 \\
\delta &= 0.036 \\
\phi_{11} &= \phi_{22} = \phi = 0.43 \\
\phi_{12} &= \phi_{21} = 1 - \phi = 0.57 \\
\bar{\phi}_1 &= \bar{\phi}_2 = 0.5.
\end{aligned}
$$

The equity premium is

$$r^e - r^f = 6\%$$

where

$$r^f = \bar{\phi}_1 r_1^f + \bar{\phi}_2 r_2^f$$

$$
\begin{aligned}
r_1^f &= \frac{1}{\left(\phi_{11}\beta\lambda_1^{-\gamma} + \phi_{12}\beta\lambda_2^{-\gamma}\right)} - 1 \\
r_2^f &= \frac{1}{\left(\phi_{21}\beta\lambda_1^{-\gamma} + \phi_{22}\beta\lambda_2^{-\gamma}\right)} - 1
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1 &= 1 + \mu + \delta \\
\lambda_2 &= 1 + \mu - \delta
\end{aligned}
$$

and

$$r^e = \bar{\phi}_1 r_1^e + \bar{\phi}_2 r_2^e$$

$$\begin{aligned} r_1^e &= \phi_{11} r_{11}^e + \phi_{12} r_{12}^e \\ r_2^e &= \phi_{21} r_{21}^e + \phi_{22} r_{22}^e \end{aligned}$$

$$\begin{aligned} r_{11}^e &= \frac{\lambda_1 (\omega_1 + 1)}{\omega_1} - 1 \\ r_{12}^e &= \frac{\lambda_2 (\omega_2 + 1)}{\omega_1} - 1 \\ r_{21}^e &= \frac{\lambda_1 (\omega_1 + 1)}{\omega_2} - 1 \\ r_{22}^e &= \frac{\lambda_2 (\omega_2 + 1)}{\omega_2} - 1 \end{aligned}$$

with

$$\begin{aligned} \omega_1 &= \beta \phi_{11} \lambda_1^{-\gamma} (\omega_1 \lambda_1 + \lambda_1) + \beta \phi_{12} \lambda_2^{-\gamma} (\omega_2 \lambda_2 + \lambda_2) \\ \omega_2 &= \beta \phi_{21} \lambda_1^{-\gamma} (\omega_1 \lambda_1 + \lambda_1) + \beta \phi_{22} \lambda_2^{-\gamma} (\omega_2 \lambda_2 + \lambda_2) . \end{aligned}$$

When we plug in all the parameter values, we find

$$\gamma \approx 17$$

with

$$r^f \approx 18\%,$$

much greater than its historical average of 1%.