Suggested Solutions to Homework #5
Econ 511b (Part I), Spring 2004

1. **Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^{\alpha}$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\overline{k}_1, \overline{k}_2\}$. In addition, assume that z takes on values in the set $\{\overline{z}_1, \overline{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = P\left(z^{'} = \overline{z}_j | z = \overline{z}_i\right)$.**

   (a) **Let** $\overline{z}_1 = 0.9, \overline{z}_2 = 1.1, p_{11} = 0.95, p_{22} = 0.9$. **Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z.**

   Given the transition matrix

$$P = \begin{pmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{pmatrix}$$

   we can calculate the stationary distribution π according to the formula

$$\pi^{'} = \pi^{'} P$$
$$\Rightarrow \begin{cases} \pi_1 = 0.95\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.05\pi_1 + 0.9\pi_2 \end{cases}$$

   The solution to this equation is

$$\pi_1 = 2\pi_2$$

   Imposing the condition that $\pi_1 + \pi_2 = 1$, the solution is

$$\begin{cases} \pi_1 = \frac{2}{3} \\ \pi_2 = \frac{1}{3} \end{cases}$$

   Correspondingly, the long run expected value is

$$E_{\pi} z = \pi^{'} z = \frac{2}{3} \times 0.9 + \frac{1}{3} \times 1.1 = \frac{29}{30}$$

   (b) **Let** β= **0.9**, α= **0.36**, $\overline{k}_1 = 0.95k_{ss}$, **and** $\overline{k}_2 = 1.05k_{ss}$, **where** k_{ss} **is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital (i.e.,** $k_{ss} = (\alpha\beta)^{\frac{1}{1-\alpha}}$**. Using Matlab (if you need to), find the optimal decision rule for capital, i.e., a function mapping pairs of the form** (k, z) **into the optimal choice for capital.**

   Form the dynamic programming problem as

$$v(k_i, z_i) = \max_{k^{'} \in \{k_1, k_2\}} \ln\left(z_i \overline{k}_i^{\alpha} - \overline{k}^{'}\right) + \beta\left(p_{i1} v\left(\overline{k}^{'}, z_1\right) + p_{i2} v\left(\overline{k}^{'}, z_2\right)\right)$$

Since (k_i, z_i) can take on only four values, we can solve for value function $v(k_i, z_i)$ and policy function $g(k_i, z_i)$ as 4×1 vectors. For example, we can solve that using value function iteration on the computer. Using Matlab, we can get the policy function as

$$\begin{aligned} g(k_1, z_1) &= k_1 \\ g(k_2, z_1) &= k_1 \\ g(k_1, z_2) &= k_2 \\ g(k_2, z_2) &= k_2 \end{aligned}$$

This is our policy function.

(c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

Based on policy function $g(k_i, z_i)$ and transition matrix of z, the pair (k, z) follows a Markov process with the transition matrix

	(z_1, k_1)	(z_1, k_2)	(z_2, k_1)	(z_2, k_2)
(z_1, k_1)	0.95	0	0.05	0
(z_1, k_2)	0.95	0	0.05	0
(z_2, k_1)	0	0.1	0	0.9
(z_2, k_2)	0	0.1	0	0.9

Now we can calculate the stationary distribution either on the computer or by hand. The result is

$$\begin{aligned} p(k_1, z_1) &= \frac{19}{30} = 0.63333 \\ p(k_2, z_1) &= \frac{1}{30} = 0.033333 \\ p(k_1, z_2) &= \frac{1}{30} = 0.033333 \\ p(k_2, z_2) &= \frac{3}{10} = 0.3 \end{aligned}$$

Using the stationary distribution, the long-run (or unconditional) expected values of the capital stock and of output are

$$\begin{aligned} Ek &= \frac{2}{3}k_1 + \frac{1}{3}k_2 = 0.1747 \\ Ey &= \frac{2}{3}k_1^\alpha + \frac{1}{3}k_2^\alpha = 0.5149 \end{aligned}$$

(d) In Matlab, use the optimal decision rule, the law of motion for z, and a random number generator to create a simulated time series $\{k_t, y_t\}_{t=0}^T$, given an initial condition (k_0, z_0). Compute $T^{-1}\sum_{t=1}^T k_t$ and $T^{-1}\sum_{t=1}^T y_t$ for a suitably large value of T and confirm that these sample means

are close to the corresponding population means that you computed in part (c).

Depending on the realization of each simulation, the result will differ a little bit. For example, one possible result based on $T = 10000$ is

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} k_t &= 0.1747 \\ \frac{1}{T}\sum_{t=1}^{T} y_t &= 0.4687 \end{aligned}$$

2. **Consider a two-period exchange economy with two (types of) consumers labelled A and B. The two types of consumers have identical preferences given by $u(c_0) + \beta E u(c_1)$, where u is strictly increasing and strictly concave. Each consumer is endowed with one consumption good in period 0. In period 1, each type A consumer is endowed with θy consumption goods and each type B consumer is endowed with $(1-\theta) y$ consumption goods. The random variable θ can be interpreted as the consumer's share of the aggregate endowment y. Let θ equal $1/2 + z$ with probability p and equal $1/2 - z$ with probability $1-p$, where $0 < z < 1/2$. The aggregate endowment y is also random: it equals $1 + x$ with probability one-half and equals $1 - x$ with probability one-half. The random variables y and θ are statistically independent.**

   (a) **Assume that markets are complete: in period 0, consumers can trade a full set of Arrow securities. Express the competitive equilibrium allocations and prices in terms of primitives as explicitly as you can. You might want to start with the case where $p = 1/2$ (so that the consumers are identical in all respects) and then consider the more general case $p \neq 1/2$.**

   First, let's define states in this economy. From the specification of the problem, there are four states indexed by the pair (θ, y). The probability of each state is

$$\begin{aligned} \pi_1 &= prob\left(\theta = \frac{1}{2} + z, y = 1 + x\right) = \frac{1}{2}p \\ \pi_2 &= prob\left(\theta = \frac{1}{2} + z, y = 1 - x\right) = \frac{1}{2}p \\ \pi_3 &= prob\left(\theta = \frac{1}{2} - z, y = 1 + x\right) = \frac{1}{2}(1-p) \\ \pi_4 &= prob\left(\theta = \frac{1}{2} - z, y = 1 - x\right) = \frac{1}{2}(1-p) \end{aligned}$$

   Now we begin by defining a competitive equilibrium. A competitive equilibrium with complete markets is a pair of consumption, asset holdings, and asset price $\left\{\{c_0^i, c_s^i, a_s^i\}_{i=A,B}, q_s\right\}_{s=1}^{4}$ such that

(a) Consumers solve the problem

$$\max_{\{c_s^i\}_{s=0}^4} u\left(c_0^i\right) + \beta \sum_{s=1}^{4} u\left(c_s^i\right)$$
$$s.t$$
$$c_0^i + \sum_{s=1}^{4} q_s a_s^i = 1$$
$$c_s^i = a_s^i + \omega_s^i$$

where $\omega_s^A = \theta_s y_s$ and $\omega_s^B = (1-\theta_s)\, y_s$.
(b) Product market clearing: $c_0^A + c_0^B = 2$, $c_s^A + c_s^B = y_s$ for $s = 1, ...4$.
(c) Asset market clearing: $a_s^A + a_s^B = 0$ for $s = 1, ..., 4$.

We know that the consumer's choice is determined by F.O.C. and budget constraint, i.e.

$$q_s = \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)}$$
$$c_0^i + \sum_{s=1}^{4} q_s a_s^i = 1$$
$$c_s^i = a_s^i + \omega_s^i$$

Now we start to find the system of equations for equilibrium. Because of Walras's law, we have five redundant equations: we take consumer B's five BC out of our system of equations. Therefore, the necessary and sufficient conditions for the equilibrium are

$$q_s = \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)}$$
$$c_0^A + \sum_{s=1}^{4} q_s a_s^A = 1$$
$$c_s^A = a_s^A + \omega_s^A$$
$$c_0^A + c_0^B = 2$$
$$c_s^A + c_s^B = y_s$$
$$a_s^A + a_s^B = 0$$

which is a system of 22 equations and 22 unknowns $\left\{\{c_0^i, c_s^i, a_s^i\}_{i=A,B}, q_s\right\}_{s=1}^4$. The existence of solution is guaranteed by the standard existence result for Walrasian equilibrium. This system implicitly defines our equilibrium.
For the case $p = \frac{1}{2}$, we can solve for the equilibrium explicitly. Due to the

symmetry when $p = \frac{1}{2}$, we can guess the solution as

$$\begin{aligned}
c_0^A &= c_0^B \\
c_1^A &= c_3^A = c_1^B = c_3^B \\
c_2^A &= c_4^A = c_2^B = c_4^B \\
q_1 &= q_3 \\
q_2 &= q_4
\end{aligned}$$

Plug into equilibrium equations, we have

$$\begin{aligned}
c_0^A &= c_0^B = 1 \\
c_1^A &= c_3^A = c_1^B = c_3^B = \frac{1}{2}(1+x) \\
c_2^A &= c_4^A = c_2^B = c_4^B = \frac{1}{2}(1-x) \\
q_1 &= q_3 = \pi_1 \frac{\beta u'\left(c_1^i\right)}{u'\left(c_0^i\right)} = \frac{1}{4}\frac{\beta u'\left(\frac{1}{2}(1+x)\right)}{u'(1)} \\
q_2 &= q_4 = \pi_2 \frac{\beta u'\left(c_2^i\right)}{u'\left(c_0^i\right)} = \frac{1}{4}\frac{\beta u'\left(\frac{1}{2}(1-x)\right)}{u'(1)}
\end{aligned}$$

and

$$\begin{aligned}
a_1^A &= c_1^A - \omega_1^A = \frac{1}{2}(1+x) - \left(\frac{1}{2}+z\right)(1+x) = -z(1+x) \\
a_2^A &= c_2^A - \omega_2^A = \frac{1}{2}(1-x) - \left(\frac{1}{2}+z\right)(1-x) = -z(1-x) \\
a_3^A &= c_3^A - \omega_3^A = \frac{1}{2}(1+x) - \left(\frac{1}{2}-z\right)(1+x) = z(1+x) \\
a_4^A &= c_4^A - \omega_4^A = \frac{1}{2}(1-x) - \left(\frac{1}{2}-z\right)(1-x) = z(1-x) \\
a_1^B &= -a_1^A = z(1+x) \\
a_2^B &= -a_2^A = z(1-x) \\
a_3^B &= -a_3^A = -z(1+x) \\
a_4^B &= -a_4^A = -z(1-x)
\end{aligned}$$

For the case $p \neq \frac{1}{2}$, we can not find explicit analytical solution. But we can reduce the equation a little bit. The intuition tells us that complete market can insure against idiosyncratic risk. Therefore, we conjecture the following fact:

$$\begin{aligned}
c_1^A &= c_3^A,\ c_2^A = c_4^A \\
c_1^B &= c_3^B,\ c_2^B = c_4^B
\end{aligned}$$

Plug this into equilibrium equation and substitute out q_3, q_4 and a_s^i, it becomes

$$
\begin{aligned}
q_1 &= \pi_1 \frac{\beta u'\left(c_1^i\right)}{u'\left(c_0^i\right)} \\
q_2 &= \pi_2 \frac{\beta u'\left(c_2^i\right)}{u'\left(c_0^i\right)} \\
c_0^A &+ \left(q_1 + \frac{\pi_3}{\pi_1} q_1\right) c_1^A + \left(q_2 + \frac{\pi_4}{\pi_2} q_2\right) c_2^A = 1 + \sum_{s=1}^{4} q_s \omega_s^A \\
c_0^A &+ c_0^B = 2 \\
c_1^A &+ c_1^B = 1 + x \\
c_2^A &+ c_2^B = 1 - x
\end{aligned}
$$

The reduced system contains 8 equations and 8 unknowns $\left\{\left\{c_0^i, c_1^i, c_2^i\right\}_{i=A,B}, q_1, q_2\right\}$. This system implicitly defines our equilibrium. We cannot go further beyond that.

A little digression (only for nerds!!! Don't spend time on that if you cannot understand it.). *The argument above only goes very loosely. To be rigorous, we have not verified our guess yet. In the above system of 8 equations and 8 unknowns, we have not proved that there is no contradiction for our guess, i.e. we have not shown the existence of solution for that 8-equation system. Now I give a rough sketch of the existence proof about $c_1^i = c_3^i$, $c_2^i = c_4^i$.*
From First Welfare Theorem complete market equilibrium is Pareto optimal. Therefore, it maximizes a social welfare function by the concavity of utility function and Kuhn-Tucker Theorem. In addition, the price q_s is Lagrangian multiplier associated with the resource constraint. From envelope theorem, q_s is indeed the marginal utility of representative agent evaluated at aggregate endowment. Therefore, we know that $q_s = f(w_s)$ for $s = 0, 1, .., 4$ *(here we use the assumption of additive separable utility). From the F.O.C. of individuals $q_s u'\left(c_0^i\right) = \pi_s \beta u'\left(c_s^i\right)$, we know that $c_s^i = g^i(\omega_s)$ (here g depends on i because the marginal utility of i or $u'\left(c_0^i\right)$ depends on i, indeed it is the inverse of the weight we give to social welfare function), which finishes the proof.*

(b) Use the prices of the Arrow securities from part (a) to find the equilibrium period-0 price of a risk-free bond (i.e., an asset that pays one unit of the consumption in all states of the world in period 1). If you are unable to solve for the Arrow prices explicitly, then show how you would use these prices to compute the price of a risk-free bond.

Due to non-arbitrage property, the price of risk-free asset must be

$$
q^{rf} = \sum_{s=1}^{4} q_s
$$

For the case $p = \frac{1}{2}$, we have

$$
q^{rf} = \sum_{s=1}^{4} q_s = \frac{1}{2} \frac{\beta\left[u'\left(\frac{1}{2}(1+x)\right) + u'\left(\frac{1}{2}(1-x)\right)\right]}{u'(1)}
$$

(c) Now suppose that markets are incomplete: in period 0, the only asset that consumers are allowed to trade is a risk-free bond. The net supply of bonds is zero (since the economy is closed). Find the competitive equilibrium allocations and the equilibrium price of the bond as explicitly as you can. Compare your answers to those in parts (a) and (b). Show that eliminating complete markets makes consumers worse off. Does eliminating complete markets increase or decrease the risk-free rate of return (i.e., the inverse of the bond price)? Why? (Again, you might want to start with the case $p = 1/2$ before considering the case $p \neq 1/2$.)

Now we begin by defining a competitive equilibrium with a risk-free asset. A competitive equilibrium with incomplete markets is a pair of consumption, asset holdings, and asset price $\left\{ \{c_0^i, c_s^i, a^i\}_{i=A,B}, q^{rf} \right\}_{s=1}^4$ such that

(a) Consumers solve the problem

$$\begin{aligned}
&\max_{\{c_s^i\}_{s=0}^4} u\left(c_0^i\right) + \beta \sum_{s=1}^4 \pi_s u\left(c_s^i\right) \\
&s.t \\
&c_0^i + q^{rf} a^i = 1 \\
&c_s^i = a^i + \omega_s^i
\end{aligned}$$

where $\omega_s^A = \theta_s y_s$ and $\omega_s^B = (1 - \theta_s) y_s$.

(b) Product market clearing: $c_0^A + c_0^B = 2$, $c_s^A + c_s^B = y_s$ for $s = 1, ...4$.

(c) Asset market clearing: $a^A + a^B = 0$.

We know that the consumer's choice is determined by F.O.C. and budget constraint, i.e.

$$\begin{aligned}
q^{rf} &= \sum_{s=1}^4 \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)} \\
c_0^i &+ q^{rf} a^i = 1 \\
c_s^i &= a^i + \omega_s^i
\end{aligned}$$

Now we start to find the system of equations for equilibrium. Because of Walras law, we have five redundant equations: we take consumer B's BC out of our system of equations. Therefore, the necessary and sufficient conditions for the equilibrium are

$$
\begin{aligned}
q^{rf} &= \sum_{s=1}^{4} \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)} \\
c_0^A + q^{rf} a^A &= 1 \\
c_s^A &= a^A + \omega_s^A \\
c_0^A + c_0^B &= 2 \\
c_s^A + c_s^B &= y_s \\
a^A + a^B &= 0
\end{aligned}
$$

which is a system of 13 equations with 13 unknowns $\left\{\left\{c_0^i, c_s^i, a^i\right\}_{i=A,B}, q^{rf}\right\}_{s=1}^{4}$. This system implicitly defines our equilibrium.

Now we start to discuss the difference of allocation from complete market case. The first fact is that with only one risk-free asset, consumers cannot insure again the idiosyncratic risk. This can been seen from the third line of our equilibrium system. From $c_s^A = a^A + \omega_s^A$, if there is a change in ω_s^A across the state (which is the case here), we must have $c_1^A \neq c_2^A \neq c_3^A \neq c_4^A$. To say more about this issue, we start from the simple case with $p = \frac{1}{2}$.

If $p = \frac{1}{2}$, due to the symmetry of the consumers we conjecture the solution as

$$
\begin{aligned}
c_1^A &= c_3^B \\
c_2^A &= c_4^B \\
c_3^A &= c_1^B \\
c_4^A &= c_2^B
\end{aligned}
$$

Combine this fact with $c_s^A = a^A + \omega_s^A$ and $a^A = a^B$, we have the further conjecture

$$
\begin{aligned}
a^A &= a^B = 0 \\
c_s^i &= \omega_s^i
\end{aligned}
$$

Plug this into equilibrium conditions, it satisfies every equation, which verifies our conjecture. (You may wonder why this conjecture cannot hold for $p \neq \frac{1}{2}$. The reason is that the same conjecture holds for every equation except the first two equations in the equilibrium system: with $p \neq \frac{1}{2}$, the first two equations (F.O.C. for A and B) contradict each other.)

We can analyze the equilibrium utility and asset price as follows. Since

$$
\begin{aligned}
& u(1) + \beta \sum_{s=1}^{4} \pi_s u\left(\omega_s^i\right) \\
< \; & u(1) + \beta \left[(\pi_1 + \pi_3) u\left(\frac{1}{2}(1+x)\right) + (\pi_2 + \pi_4) u\left(\frac{1}{2}(1-x)\right) \right]
\end{aligned}
$$

due to the concavity of utility function, we know that both consumers become worse off under incomplete markets. As for the price for risk-free asset of incomplete market

$$
q_{incomplete}^{rf} = \sum_{s=1}^{4} \pi_s \frac{\beta u'\left(\omega_s^i\right)}{u'(1)}
$$

compared to the case of complete market

$$q^{rf}_{complete} = \frac{\beta}{u'(1)}\left[(\pi_1+\pi_3)u'\left(\frac{1}{2}(1+x)\right)+(\pi_2+\pi_4)u'\left(\frac{1}{2}(1-x)\right)\right]$$

We can not determine relative magnitude unless we are willing to make an assumption about marginal utility. If $u''' > 0$, $q^{rf}_{incomplete} > q^{rf}_{complete}$; if $u''' < 0$, we have $q^{rf}_{incomplete} < q^{rf}_{complete}$. The economic explanation is that if $u''' > 0$, there is precautionary saving motive in incomplete market, which leads to higher demand for saving and hence higher asset price.

For the case of $p \neq \frac{1}{2}$, without specification of utility function, it is too complex to give a definite answer. Now I only give a verbal explanation.

For the welfare, the general result is that the transition from incomplete market to complete market could be Pareto-improving or Non-Pareto improving. We can speak two facts. First, there is possibility that it is Non-Pareto improving. The intuition is that the change from incomplete market to complete market will have a general equilibrium effect: the change in the price of existing assets. The change in asset price will bring wealth redistribution between debtors and creditors. Therefore, there is a tradeoff between two effects: the effects of insurance from complete markets and the redistribution effects from the asset price change. If the change in the price of existing asset is dramatic, the redistribution effect will dominate the insurance effect, in this case the transition from incomplete market to complete market won't be pareto-improving.

Second, the transition from incomplete market to complete market won't make everybody worseoff, as long as there is only one good. The logic goes as follows. The insurance effect increase utility, while the sign of redistribution effects must be opposite. Therefore, in the world of our problem at least one agent will become better off from completing the market.

For the effect on interest rate, it is uncertain. This effect can be seen even in the case of $p = \frac{1}{2}$. It depends on the sum of marginal rate of substitution between future and date-0. Therefore, the change in p may change the result. We cannot say more about this.

(d) Introduce a second asset into the economy you studied in part (c). Specifically, in addition to the endowments described above (which can be viewed as "labor income"), let each consumer be endowed with one "Lucas tree" in period 0. Each tree yields a "dividend" of d consumption goods in period 1, where d equals d_H if y equals $1+x$ and equals $d_L < d_H$ if y equals $1-x$. Trees, as well as risk-free bonds, can be bought and sold in competitive markets in period 0. Without doing any explicit calculations, show how you would go about solving for the equilibrium prices of the two assets in this economy.

We begin by defining a competitive equilibrium with two assets. A incomplete market competitive equilibrium with risk-free asset and Lucas tree is a pair of consumption, asset holdings, and asset price $\left\{\left\{c^i_0, c^i_s, a^i_{rf}, a^i_2\right\}_{i=A,B}, \left\{q^{rf}, q_2\right\}\right\}_{s=1}^4$ such that

(a) Consumers solve the problem

$$
\begin{aligned}
&\max_{\{c_s^i\}_{s=0}^4} \; u\left(c_0^i\right) + \beta \sum_{s=1}^{4} \pi_s u\left(c_s^i\right) \\
&s.t \\
&c_0^i + q^{rf} a_{rf}^i + q_2\left(a_2^i - 1\right) = 1 \\
&c_s^i = a^i + \omega_s^i + d_s a_2^i
\end{aligned}
$$

where $\omega_s^A = \theta_s y_s$, $\omega_s^B = (1-\theta_s)\, y_s$, $d_1 = d_3 = d_H$, $d_2 = d_4 = d_L$.
(b) Product market clearing: $c_0^A + c_0^B = 2$, $c_s^A + c_s^B = y_s + 2d_s$ for $s = 1, ...4$.
(c) Asset market clearing: $a_{rf}^A + a_{rf}^B = 0$, $a_2^A + a_2^B = 2$.

It is easy to see that the equilibrium is determined in the same way as in part (c). The necessary and sufficient conditions for the equilibrium are

$$
\begin{aligned}
q^{rf} &= \sum_{s=1}^{4} \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)} \\
q_2 &= \sum_{s=1}^{4} \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)} d_s \\
c_0^A &+ q^{rf} a_{rf}^A + q_2\left(a_2^A - 1\right) = 1 \\
c_s^A &= a^A + \omega_s^A + d_s a_2^A \\
c_0^A &+ c_0^B = 2 \\
c_s^A &+ c_s^B = y_s + 2d_s \\
a_{rf}^A &+ a_{rf}^B = 0 \\
a_2^A &+ a_2^B = 2
\end{aligned}
$$

which is a system of 16 equations with 16 unknowns $\left\{\left\{c_0^i, c_s^i, a_{rf}^i, a_2^i\right\}_{i=A,B}, \left\{q^{rf}, q_2\right\}\right\}_{s=1}^4$. This system implicitly defines our equilibrium.

(e) Determine (as completely as you can) the prices of a risk-free bond and of a Lucas tree under the assumption that consumers can trade a full set of Arrow securities in the economy that you studied in part (d). Compare (if possible) these prices to the corresponding prices in part (d).

If we can trade a full set of Arrow security, again we go back to the complete market economy in part (a). The only difference is that the product market clearing condition in period 1 changes to the equation $c_s^A + c_s^B = y_s + 2d_s$. Since with complete Arrow security, the risk-free asset and Lucas tree become redundant, we can start by finding equilibrium in an economy with only Arrow securities. Then we can price those two assets with Arrow security.

Follow the same steps as in part (a) and use the same notation, we get the system of equilibrium equations as

$$
\begin{aligned}
q_s &= \pi_s \frac{\beta u'\left(c_s^i\right)}{u'\left(c_0^i\right)} \\
c_0^A &+ \sum_{s=1}^{4} q_s a_s^A = 1 \\
c_s^A &= a_s^A + \omega_s^A \\
c_0^A &+ c_0^B = 2 \\
c_s^A &+ c_s^B = y_s + 2d_s \\
a_s^A &+ a_s^B = 0
\end{aligned}
$$

which is a system of 22 equations and 22 unknowns $\left\{\{c_0^i, c_s^i, a_s^i\}_{i=A,B}, q_s\right\}_{s=1}^{4}$. This system implicitly defines our equilibrium.

Using the price of Arrow security, we can find the price of risk-free bond and Lucas tree as

$$
\begin{aligned}
q^{rf} &= \sum_{s=1}^{4} q_s \\
q^{tree} &= \sum_{s=1}^{4} q_s d_s
\end{aligned}
$$

3. **Consider a complete-markets exchange economy populated by identical consumers whose preferences exhibit "habit persistence":**

$$
E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t - \lambda c_{t-1}\right)^{1-\sigma} - 1}{1-\sigma},
$$

where $\sigma > 0, \beta \in (0,1)$, and λ is positive and bounded. Each consumer has the same endowment ω_t in period t. Assume, for simplicity, that ω_t grows deterministically according to: $\omega_{t+1} = g\omega_t$, with $g > 1$.

(a) **Formulate the consumer's consumption-savings problem as a dynamic programming problem: there is one asset, a one-period riskless bond whose price is q.**

To simplify notation, we conjecture that in equilibrium the bond price is constant across time (we will check this conjecture later). Now the recursive formulation of the consumer's problem is

$$
\begin{aligned}
v\left(a_t, c_{t-1}, \omega_t\right) \quad = \quad & \max_{\{c_t, a_{t+1}\}} \frac{\left(c_t - \lambda c_{t-1}\right)^{1-\sigma} - 1}{1-\sigma} + \beta v\left(a_{t+1}, c_t, \omega_{t+1}\right) \\
& s.t. \\
& c_t + q a_{t+1} = a_t + \omega_t
\end{aligned}
$$

or equivalently,

$$\begin{aligned} v\left(a_t, c_{t-1}, \omega_t\right) &= \max_{\{a_{t+1}\}} \frac{\left(a_t+\omega_t-q a_{t+1}-\lambda c_{t-1}\right)^{1-\sigma}-1}{1-\sigma} \\ &\quad +\beta v\left(a_{t+1}, a_t+\omega_t-q a_{t+1}, \omega_{t+1}\right) \end{aligned}$$

Note the choice of aggregate state variable here. In principle we should include the triple aggregate state $(A, \overline{\omega}_{t-1}, \overline{\omega}_t)$ into our state variable. But here we know that $A=0$, since it is a representative agent economy. And as long as we know about one value in the pair $(\overline{\omega}_{t-1}, \overline{\omega}_t)$, we can deduce the other from the constant growth rate g. Therefore, we need only one aggregate endowment (either $\overline{\omega}_{t-1}$ or $\overline{\omega}_t$) as our aggregate state variable. For example, we could choose $\overline{\omega}_{t-1}$ and the bond price would be $q_t = q(\overline{\omega}_{t-1})$. To save notation further, we can even write $q_t = q(\omega_{t-1})$, since this is a representative agent exchange economy ($\omega_{t-1} = \overline{\omega}_{t-1}$) and we cannot change (either individual or aggregate) endowment anyway. Furthermore, due to the special utility funciton here, we can conjecture that the bond price is constant across time and check it later. So after a long chain of reasoning, we choose $q_t = q$ and only include the individual triple state (a_t, c_{t-1}, ω_t) into our recursive formulation.

(b) Derive the Euler equation for the consumer's problem.

F.O.C. for this problem is

$$a_{t+1} : \left(u_1\left(c_t, c_{t-1}\right)+\beta v_2\left(t+1\right)\right) q=\beta v_1\left(t+1\right)$$

where $v_1(t+1)$ and $v_2(t+1)$ are partial derivatives of $v(a_{t+1}, c_t, \omega_{t+1})$.
The envelope condition is

$$\begin{aligned} a_t &: v_1(t)=u_1\left(c_t, c_{t-1}\right)+\beta v_2(t+1) \\ c_{t-1} &: v_2(t)=u_2\left(c_t, c_{t-1}\right) \end{aligned}$$

Solve for this, we get

$$\begin{aligned} v_1(t) &= u_1\left(c_t, c_{t-1}\right)+\beta u_2\left(c_{t+1}, c_t\right) \\ v_2(t) &= u_2\left(c_t, c_{t-1}\right) \end{aligned}$$

Iterate forward for one period and plug into F.O.C., we get the Euler Equation

$$\begin{aligned} & \left(u_1\left(c_t, c_{t-1}\right)+\beta u_2\left(c_{t+1}, c_t\right)\right) q_{t+1}=\beta\left(u_1\left(c_{t+1}, c_t\right)+\beta u_2\left(c_{t+2}, c_{t+1}\right)\right) \\ \Rightarrow\ & q=\frac{\beta\left(u_1\left(c_{t+1}, c_t\right)+\beta u_2\left(c_{t+2}, c_{t+1}\right)\right)}{u_1\left(c_t, c_{t-1}\right)+\beta u_2\left(c_{t+1}, c_t\right)} \\ \Rightarrow\ & q=\frac{\beta\left(\left(c_{t+1}-\lambda c_t\right)^{-\sigma}-\beta \lambda\left(c_{t+2}-\lambda c_{t+1}\right)^{-\sigma}\right)}{\left(c_t-\lambda c_{t-1}\right)^{-\sigma}-\beta \lambda\left(c_{t+1}-\lambda c_t\right)^{-\sigma}} \end{aligned}$$

Notice the similarity with normal Euler equation: it is the marginal rate of substitution between consumption c_{t+1} and c_t. The difference is the involvement of two period felicity function, which is due to the "habit persistence".

(c) Find the equilibrium bond price in this model as a function of the structural parameters.

In equilibrium, we must have $c_t = \omega_t$. Plug into the Euler Equation, we get the equilibrium bond price as

$$
\begin{aligned}
q &= \frac{\beta\left(u_1\left(\omega_{t+1},\omega_t\right)+\beta u_2\left(\omega_{t+2},\omega_{t+1}\right)\right)}{u_1\left(\omega_t,\omega_{t-1}\right)+\beta u_2\left(\omega_{t+1},\omega_t\right)} \\
&= \frac{\beta\left(\left(\omega_{t+1}-\lambda\omega_t\right)^{-\sigma}-\beta\lambda\left(\omega_{t+2}-\lambda\omega_{t+1}\right)^{-\sigma}\right)}{\left(\omega_t-\lambda\omega_{t-1}\right)^{-\sigma}-\beta\lambda\left(\omega_{t+1}-\lambda\omega_t\right)^{-\sigma}} \\
&= \frac{\beta\left(\left(g\omega_t-\lambda\omega_t\right)^{-\sigma}-\beta\lambda\left(g^2\omega_t-\lambda g\omega_t\right)^{-\sigma}\right)}{\left(\omega_t-\frac{\lambda}{g}\omega_t\right)^{-\sigma}-\beta\lambda\left(g\omega_t-\lambda\omega_t\right)^{-\sigma}} \\
&= \beta g^{-\sigma}
\end{aligned}
$$

This verifies our conjecture that $q_t = q$. The result is quite intuitive: (a) the more patient ($\beta \uparrow$) the individuals are, the higher the demand for savings, and the higher the asset price will be; (b) the higher of the growth rate of endowment ($g \uparrow$), the less need for saving, the lower the asset price.

We need to be careful about the result. The independence of q on λ only because of this special utility function. From the derivation, we can see that with other functional form, λ may have a direct effect on q.

(d) Suppose now that the endowment grows stochastically: $\omega_{t+1} = g_{t+1}\omega_t$, where the growth rate g_{t+1} is independent across time and takes on the two values $\lambda_1 > 1$ and $\lambda_2 < 1$ with equal probability. Find the prices of the Arrow securities and use them to compute the (long-run) average rate of return on a riskless bond in this model. If you cannot find explicit solutions for the prices of the Arrow securities, then show what conditions they must satisfy (i.e., find a set of equations that determine these prices) and explain how you would use them to compute the average rate of return on a riskless bond in this model.

Define the price and holdings of Arrow security at time t as $\{q_{1t}, q_{2t}\}$ and $\{a_{1,t+1}, a_{2,t+1}\}$, respectively. Now the recursive formulation of the consumer's decision problem becomes

$$
\begin{aligned}
v\left(a_{i,t}, c_{t-1}, \omega_{i,t}, \lambda_{i,t}\right) = & \max_{\{c_t, a_{1,t+1}, a_{2,t+1}\}} \frac{\left(c_{i,t}-\lambda c_{t-1}\right)^{1-\sigma}-1}{1-\sigma} + \\
& \beta E v\left(a_{j,t+1}, c_{i,t}, \omega_{j,t+1}, \lambda_{j,t+1}\right) \\
& s.t. \\
& c_{i,t} + q_1(\omega_{it}, \lambda_{i,t})a_{1,t+1} + q_2(\omega_{it}, \lambda_{i,t})a_{2,t+1} = a_{i,t} + \omega_{i,t}
\end{aligned}
$$

where $a_{i,t}$ means the asset holdings in state i. Again notice the choice of state variables here. Different from the deterministic case, here we need to include both $(\overline{\omega}_{t-1}, \overline{\omega}_t)$ in our aggregate state variable. (or equivalently, here we use the pair $(\omega_{i,t}, \lambda_{i,t})$ as our aggregate state variable). Refer to the comments given in part (a) for the chain of logic.

Plug BC into objective function, we have

$$\begin{aligned} v\left(a_{i,t}, c_{t-1}, \omega_{i,t}, \lambda_{i,t}\right) &= \max_{\{a_{1,t+1}, a_{2,t+1}\}} \frac{\left(a_{i,t} + \omega_{it} - q_1 a_{1,t+1} - q_2 a_{2,t+1} - \lambda c_{t-1}\right)^{1-\sigma} - 1}{1-\sigma} \\ &\quad + \beta E v\left(a_{j,t+1}, a_{i,t} + \omega_{it} - q_1 a_{1,t+1} - q_2 a_{2,t+1}, \omega_{j,t+1}, \lambda_{j,t+1}\right) \end{aligned}$$

Now we follow the same steps as in part (b) to derive the Euler equation. F.O.C. for this problem is

$$\begin{aligned} a_{1,t+1} &: \left(u_1\left(c_t, c_{t-1}\right) + \beta E v_2\left(t+1\right)\right) q_1(\omega_{i,t}, \lambda_{i,t}) = \frac{1}{2}\beta v_1\left(1, t+1\right) \\ a_{2,t+1} &: \left(u_1\left(c_t, c_{t-1}\right) + \beta E v_2\left(t+1\right)\right) q_2(\omega_{i,t}, \lambda_{i,t}) = \frac{1}{2}\beta v_1\left(2, t+1\right) \end{aligned}$$

where $v_1\left(i, t+1\right)$ and $v_2\left(i, t+1\right)$ are partial derivatives of $v\left(a_{i,t+1}, c_{it}, \omega_{i,t+1}, \lambda_{i,t+1}\right)$. The envelope condition is

$$\begin{aligned} a_{i,t} &: v_1\left(i, t\right) = u_1\left(c_{i,t}, c_{t-1}\right) + \beta E v_2\left(t+1\right) \\ c_{t-1} &: v_2\left(i, t\right) = u_2\left(c_{i,t}, c_{t-1}\right) \end{aligned}$$

Solve for this, we get

$$\begin{aligned} v_1\left(i, t\right) &= u_1\left(c_{i,t}, c_{t-1}\right) + \beta E u_2\left(c_{j,t+1}, c_{i,t}\right) \\ v_2\left(i, t\right) &= u_2\left(c_{i,t}, c_{t-1}\right) \end{aligned}$$

Iterate forward for one period and plug into F.O.C., we get the Stochastic Euler Equation

$$\begin{aligned} &\left(u_1\left(c_{i,t}, c_{t-1}\right) + \beta E\left(u_2\left(c_{j,t+1}, c_{i,t}\right)\right)\right) q_1(\omega_{i,t}, \lambda_{i,t}) = \frac{1}{2}\beta \begin{pmatrix} u_1\left(c_{1,t+1}, c_{i,t}\right) \\ +\beta E u_2\left(c_{j,t+2}, c_{1,t+1}\right) \end{pmatrix} \\ \Rightarrow\ & q_1(\omega_{i,t}, \lambda_{i,t}) = \frac{\frac{1}{2}\beta\left(u_1\left(c_{1,t+1}, c_{i,t}\right) + \beta E u_2\left(c_{j,t+2}, c_{1,t+1}\right)\right)}{u_1\left(c_{i,t}, c_{t-1}\right) + \beta E\left(u_2\left(c_{j,t+1}, c_{i,t}\right)\right)} \\ \Rightarrow\ & q_1(\omega_{i,t}, \lambda_{i,t}) = \frac{\frac{1}{2}\beta\left(\left(c_{1,t+1} - \lambda c_{i,t}\right)^{-\sigma} - \lambda\beta E\left(c_{j,t+2} - \lambda c_{1,t+1}\right)^{-\sigma}\right)}{\left(c_{i,t} - \lambda c_{t-1}\right)^{-\sigma} - \lambda\beta E\left(\left(c_{j,t+1} - \lambda c_{i,t}\right)^{-\sigma}\right)} \end{aligned}$$

and

$$\begin{aligned} &\left(u_1\left(c_{i,t}, c_{t-1}\right) + \beta E\left(u_2\left(c_{j,t+1}, c_{i,t}\right)\right)\right) q_2(\overline{\omega}_{t-1}, \lambda_{i,t}) = \frac{1}{2}\beta \begin{pmatrix} u_1\left(c_{2,t+1}, c_{i,t}\right) \\ +\beta E u_2\left(c_{j,t+2}, c_{2,t+1}\right) \end{pmatrix} \\ \Rightarrow\ & q_2(\omega_{i,t}, \lambda_{i,t}) = \frac{\frac{1}{2}\beta\left(u_1\left(c_{2,t+1}, c_{i,t}\right) + \beta E u_2\left(c_{j,t+2}, c_{2,t+1}\right)\right)}{u_1\left(c_{i,t}, c_{t-1}\right) + \beta E\left(u_2\left(c_{j,t+1}, c_{i,t}\right)\right)} \\ \Rightarrow\ & q_2(\omega_{i,t}, \lambda_{i,t}) = \frac{\frac{1}{2}\beta\left(\left(c_{2,t+1} - \lambda c_{i,t}\right)^{-\sigma} - \lambda\beta E\left(c_{j,t+2} - \lambda c_{2,t+1}\right)^{-\sigma}\right)}{\left(c_{i,t} - \lambda c_{t-1}\right)^{-\sigma} - \lambda\beta E\left(\left(c_{j,t+1} - \lambda c_{i,t}\right)^{-\sigma}\right)} \end{aligned}$$

Plug equilibrium condition $c_{i,t} = \omega_{i,t}$ into S.E.E., we have

$$\begin{aligned}
q_1(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\omega_{1,t+1} - \lambda\omega_{i,t})^{-\sigma} - \lambda\beta E\left(\omega_{j,t+2} - \lambda\omega_{1,t+1}\right)^{-\sigma}\right)}{(\omega_{i,t} - \lambda\omega_{t-1})^{-\sigma} - \lambda\beta E\left((\omega_{j,t+1} - \lambda\omega_{i,t})^{-\sigma}\right)} \\
\Rightarrow \quad q_1(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\lambda_1\omega_{i,t} - \lambda\omega_{i,t})^{-\sigma} - \lambda\beta E\left(\lambda_j\lambda_1\omega_{i,t} - \lambda\lambda_1\omega_{i,t}\right)^{-\sigma}\right)}{\left(\omega_{i,t} - \frac{\lambda}{\lambda_i}\omega_{i,t}\right)^{-\sigma} - \lambda\beta E\left((\lambda_j\omega_{i,t} - \lambda\omega_{i,t})^{-\sigma}\right)} \\
\Rightarrow \quad q_1(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\lambda_1 - \lambda)^{-\sigma} - \lambda\beta\lambda_1^{-\sigma} E\left(\lambda_j - \lambda\right)^{-\sigma}\right)}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)}
\end{aligned}$$

and

$$\begin{aligned}
q_2(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\omega_{2,t+1} - \lambda\omega_{i,t})^{-\sigma} - \lambda\beta E\left(\omega_{j,t+2} - \lambda\omega_{2,t+1}\right)^{-\sigma}\right)}{(\omega_{i,t} - \lambda\omega_{t-1})^{-\sigma} - \lambda\beta E\left((\omega_{j,t+1} - \lambda\omega_{i,t})^{-\sigma}\right)} \\
\Rightarrow \quad q_2(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\lambda_2\omega_{i,t} - \lambda\omega_{i,t})^{-\sigma} - \lambda\beta E\left(\lambda_j\lambda_2\omega_{i,t} - \lambda\lambda_2\omega_{i,t}\right)^{-\sigma}\right)}{\left(\omega_{i,t} - \frac{\lambda}{\lambda_i}\omega_{i,t}\right)^{-\sigma} - \lambda\beta E\left((\lambda_j\omega_{i,t} - \lambda\omega_{i,t})^{-\sigma}\right)} \\
\Rightarrow \quad q_2(\omega_{i,t}, \lambda_{i,t}) &= \frac{\frac{1}{2}\beta\left((\lambda_2 - \lambda)^{-\sigma} - \lambda\beta\lambda_2^{-\sigma} E\left(\lambda_j - \lambda\right)^{-\sigma}\right)}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)}
\end{aligned}$$

where $E\left(\lambda_j - \lambda\right)^{-\sigma} = \frac{1}{2}\left(\lambda_1 - \lambda\right)^{-\sigma} + \frac{1}{2}\left(\lambda_2 - \lambda\right)^{-\sigma}$. Notice that the price of Arrow securities does not depend on the state variable $\omega_{i,t}$, which is due to the assumptions on the preference and endowment process. Now we can calculate the price of risk-free bond as

$$\begin{aligned}
q^{rf}(\omega_{i,t}, \lambda_{i,t}) &= q_1(\omega_{i,t},, \lambda_{i,t}) + q_2(\omega_{i,t},, \lambda_{i,t}) \\
&= \frac{\frac{1}{2}\beta\left((\lambda_1 - \lambda)^{-\sigma} - \lambda\beta\lambda_1^{-\sigma} E\left(\lambda_j - \lambda\right)^{-\sigma}\right)}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)} + \\
&\quad \frac{\frac{1}{2}\beta\left((\lambda_2 - \lambda)^{-\sigma} - \lambda\beta\lambda_2^{-\sigma} E\left(\lambda_j - \lambda\right)^{-\sigma}\right)}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)} \\
&= \frac{\beta}{2}\frac{\left((\lambda_1 - \lambda)^{-\sigma} + (\lambda_2 - \lambda)^{-\sigma} - \lambda\beta\left(\lambda_1^{-\sigma} + \lambda_2^{-\sigma}\right) E\left(\lambda_j - \lambda\right)^{-\sigma}\right)}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)} \\
&= \frac{\beta}{2}\frac{\left[(\lambda_1 - \lambda)^{-\sigma} + (\lambda_2 - \lambda)^{-\sigma}\right]\left[1 - \frac{1}{2}\lambda\beta\left(\lambda_1^{-\sigma} + \lambda_2^{-\sigma}\right)\right]}{(\lambda_i - \lambda)^{-\sigma}\lambda_i^{\sigma} - \lambda\beta E\left((\lambda_j - \lambda)^{-\sigma}\right)}
\end{aligned}$$

and the long-run average return as

$$q^{rf} = \frac{1}{2}\left(q^{rf}(\omega_{i,t},, \lambda_1) + q^{rf}(\omega_{i,t},, \lambda_2)\right)$$

Here we can do a little more exercise to check the relationship with deterministic case and the case without "habit formation". First, let $\lambda_1 = \lambda_2 = g$. Then we get $q^{rf}(\omega_{i,t}, \lambda_{i,t}) = \beta g^{-\sigma}$ and $q^{rf} = \beta g^{-\sigma}$, which is the same as in the deterministic case. Second, let $\lambda = 0$, we get

$$q^{rf}(\omega_{i,t}, \lambda_{i,t}) = \frac{1}{2}\beta\left(\lambda_1^{-\sigma} + \lambda_2^{-\sigma}\right)$$

which is the same as the case of without "habit formation".