

Econ 510a (second half)  
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## Solutions for Homework #6

### Question 1

a) The problem of the new generation when they're born at time  $t$  can be written as:

$$\max_{\{c_{1t}, c_{2t+1}, s_t\}} u(c_{1t}) + \beta u(c_{2t+1})$$

s.t.

$$\begin{aligned} w_t(1 - \tau) &= c_{1t} + s_t \\ c_{2t+1} &= R_{t+1}s_t + \tau w_{t+1} \end{aligned}$$

We substitute in for  $c_{1t}$  and  $c_{2t+1}$  and take the f.o.c. w.r.t.  $s_t$ :

$$-u'(w_t(1 - \tau) - s_t) + \beta u'(R_{t+1}s_t + \tau w_{t+1})R_{t+1} = 0$$

Assuming that prices are competitively determined, that we have log utility, that  $s_t = k_{t+1}$  and that  $f(k, l) = k^a n^{1-a}$ , we have (after normalizing  $n = 1$ ):

$$\begin{aligned} -\frac{1}{(1 - \tau)(1 - a)k_t^a - k_{t+1}} + \frac{\beta a k_{t+1}^{a-1}}{a k_{t+1}^{a-1} k_{t+1} + \tau(1 - a)k_{t+1}^a} &= 0 \Leftrightarrow \\ \beta a k_{t+1}^{a-1} (1 - \tau)(1 - a)k_t^a - \beta a k_{t+1}^{a-1} k_{t+1} &= (a + \tau(1 - a))k_{t+1}^a \Leftrightarrow \\ \beta a k_{t+1}^{a-1} (1 - \tau)(1 - a)k_t^a - \beta a k_{t+1}^a &= (a + \tau(1 - a))k_{t+1}^a \Leftrightarrow \\ \beta a k_{t+1}^{a-1} (1 - \tau)(1 - a)k_t^a &= (\beta a + a + \tau(1 - a))k_{t+1}^a \Leftrightarrow \\ (\beta a + a + \tau(1 - a))k_{t+1} &= \beta a(1 - \tau)(1 - a)k_t^a \Leftrightarrow \\ k_{t+1} &= \frac{\beta a(1 - \tau)(1 - a)}{\beta a + a + \tau(1 - a)} k_t^a \end{aligned}$$

and thus the steady state level of capital is:

$$\bar{k}^\tau = \left( \frac{\beta a(1 - \tau)(1 - a)}{\beta a + a + \tau(1 - a)} \right)^{\frac{1}{1-a}}$$

b) If there was no social security ( $\tau = 0$ ), then the law of motion would be given by:

$$k_{t+1} = \frac{\beta(1-a)}{1+\beta} k_t^a$$

and the steady state is equal to :

$$\bar{k} = \left( \frac{\beta(1-a)}{1+\beta} \right)^{\frac{1}{1-a}}$$

We need to show that:

$$\begin{aligned} \frac{\beta a(1-\tau)(1-a)}{(\beta a + a + \tau(1-a))} &< \frac{\beta(1-a)}{1+\beta} \Leftrightarrow \\ \frac{a(1-\tau)}{(\beta a + a + \tau(1-a))} &< \frac{1}{1+\beta} \Leftrightarrow \\ a(1-\tau)(1+\beta) &< \beta a + a + \tau(1-a) \Leftrightarrow \\ a + a\beta - \tau a - \tau a\beta &< \beta a + a + \tau - \tau a \Leftrightarrow \\ -\tau a\beta &< \tau \end{aligned}$$

which is true since  $0 < \tau, a, \beta < 1$ .

c) In a world with full depreciation the golden rule level of capital (in the steady state) is given by:

$$\max_{\bar{k}} f(\bar{k}) - \bar{k} \Rightarrow f'(\bar{k}) = 1$$

A steady state is therefore dynamically efficient if:

$$\begin{aligned} f'(\bar{k}) &> 1 \Leftrightarrow \\ a\bar{k}^{a-1} &> 1 \end{aligned}$$

Setting  $\tau = 0$  and substituting in for the steady state level of capital we found in part a), we have:

$$f'(\bar{k}) = a \frac{1+\beta}{(1-a)\beta} = \frac{1}{4} \frac{7/4}{3/4 \cdot 3/4} = \frac{7}{9} < 1$$

and therefore for these parameter values the steady state is not dynamically efficient.

d) We need:

$$\begin{aligned}
\alpha \frac{\beta a + a + \tau(1-a)}{\beta a(1-\tau)(1-a)} &= 1 \Leftrightarrow \\
\frac{1}{4} \frac{(3/4)(1/4) + (1/4) + \tau(3/4)}{(3/4)(1/4)(1-\tau)(3/4)} &= 1 \Leftrightarrow \\
\frac{(3/4) + 1 + 3\tau}{(3/4)(1-\tau)(3/4)} &= 4 \Leftrightarrow \\
\frac{3}{4} + 1 + 3\tau &= \frac{9}{4}(1-\tau) \Leftrightarrow \\
\frac{7}{4} + 3\tau &= \frac{9}{4} - \frac{9}{4}\tau \Leftrightarrow \\
\frac{21}{4}\tau &= \frac{1}{2} \Leftrightarrow \\
\tau &= \frac{2}{21}
\end{aligned}$$

Thus by setting  $\tau = \frac{2}{21}$ , the government can push the economy to the golden rule level of capital, which is dynamically efficient.

## Question 2

a) If we multiply both sides of the government budget constraint by  $p_t$ , we get:

$$p_t \tau_t + p_t q_t b_{t+1} = p_t b_t + p_t g_t$$

which holds for every  $t$ . Therefore if we sum up over all  $t$ 's we have:

$$\sum_{t=0}^{\infty} p_t \tau_t + \sum_{t=0}^{\infty} p_t q_t b_{t+1} = \sum_{t=0}^{\infty} p_t b_t + \sum_{t=0}^{\infty} p_t g_t$$

Using the fact that  $p_t q_t = p_{t+1}$  (no-arbitrage condition), the above equation becomes:

$$\sum_{t=0}^{\infty} p_t \tau_t + \sum_{t=0}^{\infty} p_{t+1} b_{t+1} = \sum_{t=0}^{\infty} p_t b_t + \sum_{t=0}^{\infty} p_t g_t$$

Finally, since  $b_0 = 0$ , it will be the case that  $\sum_{t=0}^{\infty} p_{t+1} b_{t+1} = \sum_{t=0}^{\infty} p_t b_t$  (notice that the only difference between the two sums is that the first sum starts from  $p_1 b_1$ , whereas the second also includes  $p_0 b_0$ . But since  $b_0 = 0$ , the sums are equal). Therefore we get:

$$\sum_{t=0}^{\infty} p_t \tau_t = \sum_{t=0}^{\infty} p_t g_t$$

b) Substituting in the that in equilibrium there is no excess demand for government debt ( $a_t = b_t$  for all  $t$ ), we can write consumer's lifetime budget constraint as:

$$\sum_{t=0}^{\infty} p_t c_t + \sum_{t=0}^{\infty} p_t k_{t+1} + \sum_{t=0}^{\infty} p_t q_t b_{t+1} = \sum_{t=0}^{\infty} p_t (r_t + 1 - \delta) k_t + \sum_{t=0}^{\infty} p_t w_t + \sum_{t=0}^{\infty} p_t b_t + \sum_{t=0}^{\infty} p_t \tau_t$$

Again using the no-arbitrage condition  $p_t q_t = p_{t+1}$  we can write:

$$\sum_{t=0}^{\infty} p_t c_t + \sum_{t=0}^{\infty} p_t k_{t+1} + \sum_{t=0}^{\infty} p_{t+1} b_{t+1} = \sum_{t=0}^{\infty} p_t (r_t + 1 - \delta) k_t + \sum_{t=0}^{\infty} p_t w_t + \sum_{t=0}^{\infty} p_t b_t + \sum_{t=0}^{\infty} p_t \tau_t$$

As we saw however in part a), it is the case that  $\sum_{t=0}^{\infty} p_{t+1} b_{t+1} = \sum_{t=0}^{\infty} p_t b_t$ , so the two cancel out:

$$\sum_{t=0}^{\infty} p_t c_t + \sum_{t=0}^{\infty} p_t k_{t+1} = \sum_{t=0}^{\infty} p_t (r_t + 1 - \delta) k_t + \sum_{t=0}^{\infty} p_t w_t + \sum_{t=0}^{\infty} p_t \tau_t$$

Finally substituting in for  $\sum_{t=0}^{\infty} p_t \tau_t$  from the government consolidated budget constraint which we derived in part a), we have:

$$\sum_{t=0}^{\infty} p_t (c_t + k_{t+1} + g_t) = \sum_{t=0}^{\infty} p_t ((r_t + 1 - \delta) k_t + w_t)$$

Thus the way in which the government finances its expenditure stream is irrelevant to the consumer's optimization problem. This is an important result and it implies that for example a tax cut is unable to boost consumer demand.

Note: The arbitrage condition  $p_t q_t = p_{t+1}$  simply says that the price of the bond of bond today ( $q_t$ ) is equal to what it pays next period relative to today ( $\frac{p_{t+1}}{p_t} \cdot 1$ ). If the price was anything different and the inequality didn't hold, then there would be room for arbitrage, since either the bond would be too cheap compared to what it is really worth, or too expensive.